Quasigroups and Related Systems **19** (2011), 123 – 132

Centrally nilpotent finite loops

Markku Niemenmaa and *Miikka Rytty*

Abstract. We discuss the connection between centrally nilpotent finite loops and their multiplication groups and inner mapping groups.

1. Introduction

The purpose of this survey is to explore the connection between centrally nilpotent finite loops and their multiplication groups and inner mapping groups. In section 2, we describe the classical approach by Bruck [3] from 1946 and we show that if Q is a centrally nilpotent finite loop, then $M(Q)$ (and $I(Q)$) is a solvable group. In section 3 we start from groups and show that if $I(Q)$ is abelian, then Q is centrally nilpotent (we give a modernized and simplified version of the original proof by Kepka and Niemenmaa [17] from 1994). Loops with abelian inner mapping groups have drawn a lot of attention during the last five years - main reason for this being the invention by Csörgő [5]: there exist loops whose inner mapping groups are abelian and whose nilpotency class is higher than two. We try to cover some of the most interesting results by Csörgő, Drápal, Kinyon, Nagy and Vojtěchovský. One of the interesting recent results is the following: if $I(Q)$ is nilpotent, then a finite loop Q is centrally nilpotent [15]. We shall discuss loops with nilpotent inner mapping groups in section 4 based on the results by Mazur, Niemenmaa and Rytty. Finally, in section 5 we introduce an example by Vesanen: the inner mapping group of a centrally nilpotent loop is not necessarily nilpotent although it has to be solvable.

2010 Mathematics Subject Classification: 20D10, 20N05
Keywords: loop, group, connected transversals.

2. From loops to groups

This section is for the most part based on the article by Bruck [3, pp. 278 – 283]. At some points we have tried to modernize the presentation – we hope that the readers will accept this.

If Q is a loop, then the multiplication group $M(Q)$ is the permutation group of Q generated by all left and right translations. The inner mapping group $I(Q)$ is the stabilizer of the neutral element in Q. The centre $Z(Q)$ consists of all elements a such that the equations $(ax)y = a(xy)$, $(xa)y = x(ay)$, $(xy)a = x(ya)$ and $ax = xa$ are satisfied for all $x, y \in Q$. As the inner mapping group $I(Q)$ is generated by the mappings $R_{yx}^{-1}R_xR_y$, $L_{xy}^{-1}L_xL_y$ and $L_x^{-1}R_x$ $(x, y \in Q)$, we conclude that $a \in Z(Q)$ if and only if $U(a) = a$ for every $U \in I(Q)$. The classical result by Albert [1] is

Lemma 2.1. *If Q is a loop, then $Z(Q) \cong Z(M(Q))$.* $\square$

If we write $Z_0 = 1$, $Z_1 = Z(Q)$ and $Z_{i+1}/Z_i = Z(Q/Z_i)$, then we get a series of normal subloops of Q. If Z_{n-1} is a proper subloop of Q and $Z_n = Q$, then we say that Q is centrally nilpotent of class n.

For every normal subloop N of Q, the mapping $f : I(Q) \to I(Q/N)$ defined by

$$f(P)(xN) = P(x)N$$

is a surjective homomorphism. We shall write

$$K(N) = \mathrm{Ker}(f) = \{P \in I(Q) \mid P(x)N = xN \text{ for every } x \in Q\}.$$

Now we have the isomorphism

$$I(Q)/K(Z(Q)) \cong I(Q/Z(Q)).$$

If $U \in K(Z(Q))$, then $U(x) = xz$, where $z \in Z(Q)$. If $y = xc$, where $c \in Z(Q)$, then $U(y) = U(xc) = U(x)c = xz.c = xc.z = yz$. If $Q = Z(Q) \cup x_1Z(Q) \cup \ldots \cup x_{r-1}Z(Q)$, then U is uniquely determined by the elements $U(x_1), \ldots, U(x_{r-1})$. It follows that the mapping $g : K(Z(Q)) \to Z(Q) \times \ldots \times Z(Q)$ (with $r - 1$ components), $g(U) = (c_1, \ldots, c_{r-1})$, where $U(x_i) = x_ic_i$, is a homomorphism. Since $\mathrm{Ker}(g) = 1$, we have the following

Theorem 2.2. *Let Q be a finite loop with centre $Z(Q)$ and let $|Q/Z(Q)| = r$. Then $K(Z(Q))$ is an abelian normal subgroup of $I(Q)$ and $K(Z(Q))$ is isomorphic to a subgroup of $Z(Q) \times \ldots \times Z(Q)$, where the direct product has $r - 1$ components.* $\square$

We write $L_i = K(Z_i)$ and thus $L_{i+1}/L_i = K(Z_{i+1}/Z_i) = K(Z(Q/Z_i))$. If Q is a nilpotent loop with central series

$$1 = Z_0 \subset Z_1 \subset \ldots \subset Z_n = Q,$$

then we get a series $1 < L_1 < \ldots < L_{n-1} = I(Q)$ with the following properties:

1. L_i is a normal subgroup of $I(Q)$,
2. $I(Q)/L_i \cong I(Q/Z_i)$,
3. $L_{i+1}/L_i \cong D \leqslant Z_{i+1}/Z_i \times \ldots \times Z_{i+1}/Z_i$, where the direct product has $|Q/Z_{i+1}| - 1$ components.

As $L_1 = K(Z_1)$ is abelian and all factor groups L_{i+1}/L_i are abelian, we have the following

Theorem 2.3. *If Q is a finite nilpotent loop, then $I(Q)$ is a solvable group.* □

By looking at the orders of the factor groups L_{i+1}/L_i we have

Corollary 2.4. *If $|Q| = n$, then $|I(Q)|$ and $|M(Q)|$ divide some power of n.* □

Corollary 2.5. *If p is a prime number such that p divides $|I(Q)|$, then p divides $|Q|$.* □

Corollary 2.6. *If $|Q| = p^m$, then $I(Q)$ and $M(Q)$ are both p-groups.* □

We shall now recall another nilpotency criterion given by Bruck [3, p. 281]. We first write $I_0 = I(Q)$ and $I_i = N_{M(Q)}(I_{i-1})$ for each $i \geqslant 1$. Then

$$I_i = \{R_x U \mid x \in Z_i \text{ and } U \in I(Q)\}.$$

Theorem 2.7. *A necessary and sufficient condition that Q be centrally nilpotent of class n is that $I_n = M(Q)$ but $I_{n-1} \neq M(Q)$.* □

Corollary 2.8. *If $M(Q)$ is a nilpotent group of class n, then Q is centrally nilpotent of class at most n.* □

By Theorem 2.3, $I(Q)$ is solvable and as the factor groups I_{i+1}/I_i are all abelian, we get

Theorem 2.9. *If Q is a finite centrally nilpotent loop, then $M(Q)$ is a solvable group.* □

Corollary 2.10. *Let $|Q| = p^m$, where p is a prime number. Now Q is centrally nilpotent if and only if $M(Q)$ is a p-group.* □

If Q is centrally nilpotent of class 2, then $N_{M(Q)}(I(Q)) = I(Q) \times Z(M(Q))$ is normal in $M(Q)$, hence $I(Q)$ is abelian group. The converse of this result naturally holds in group theory but, as we can see in the following section, it does not hold in loop theory.

3. From groups to loops: abelian case

The multiplication group of a loop can be characterized in purely group theoretic terms by using the notion of connected transversals. Let G be a group, $H \leqslant G$ and let A, B be two left transversals to H in G. If the commutator $a^{-1}b^{-1}ab \in H$ for every $a \in A$ and every $b \in B$, then we say that A and B are H-connected transversals in G. Kepka and Niemenmaa [16] proved the following

Theorem 3.1. *A group G is isomorphic to the multiplication group of a loop if and only if there exist a subgroup H of G satisfying $H_G = 1$ and H-connected transversals A and B such that $G = \langle A, B\rangle$.* □

In the Theorem H corresponds to the inner mapping group of the loop and A and B are the sets of left and right transversals respectively. By H_G we denote the largest normal subgroup of G contained in H. A thorough exposition of connected transversals can be found in [18].

In the following, we shall assume that $H \leqslant G$ and there exist H-connected transversals A and B in G. In this section we need the following two results on connected transversals.

Lemma 3.2. *If $C \subseteq A \cup B$ and $K = \langle H, C\rangle$, then $C \subseteq K_G$.* □

Lemma 3.3. *Let $H \cap H^a = 1$ for some $a \in A$. If $G = \langle A, B\rangle$, then G is an abelian group and $H = 1$.* □

The proof of Lemma 3.2 can be found in [16] whereas Lemma 3.3 is a result which most probably appears for the first time in this article. The kind reader is encouraged to construct the proof.

We shall now deal with the case where H (and, in fact, $I(Q)$) is abelian.

Theorem 3.4. *If $G = \langle A, B\rangle$ and H is abelian, then H is subnormal in G.*

Proof. We can assume that G is not an abelian group. If $H_G > 1$, then we use induction and it follows that H/H_G is subnormal in G/H_G and the claim follows.

Assume that H is a maximal subgroup of G. By Lemma 3.3, $L = H \cap H^a > 1$ for some $1 \neq a \in A$. Then $C_G(L) \geqslant \langle H, H^a\rangle = G$, hence $L \leqslant Z(G)$ and $H_G > 1$.

Thus we may assume that $H < T < G$, where H is a maximal subgroup of T. By Lemma 3.2, $T_G > 1$. We can use induction again and get that T/T_G is subnormal in G/T_G, hence T is subnormal in G. If H is maximal in $E < G$, then E is subnormal in G. If $E \neq T$, then $H = E \cap T$ is subnormal in G. Therefore we may assume that $E = T$. Assume that T is a proper normal subgroup of $F \leqslant G$. If P is the Sylow p-subgroup of H and also a Sylow p-subgroup of T, then by Frattini lemma, $F = TN_F(P)$ and thus $T \leqslant N_F(P)$. If Q is the Sylow q-subgroup of H but Q is not a Sylow q-subgroup of T, then $N_T(Q) = T$. We may conclude that $T \leqslant N_G(H)$ and this means that H is subnormal in G. □

Corollary 3.5. *If Q is a finite loop and $I(Q)$ is an abelian group, then Q is centrally nilpotent.*

Proof. By previous theorem, $Z(M(Q)) > 1$ and $Z(Q) > 1$. Now $I(Q/Z(Q))$ is isomorphic with $I(Q)/K(Z_1)$ (where $K(Z_1)$ is as in Theorem 2.2). Again $Z(Q/Z(Q)) > 1$ and it follows that Q is centrally nilpotent. □

How about the nilpotency class of Q in the case that $I(Q)$ is abelian? The results by Csörgő, Kepka and Niemenmaa [4, 17] indicate that if $I(Q) \cong C_p \times C_p$ or $I(Q) \cong C_p \times C_p \times C_p$, then Q is centrally nilpotent of class two. In 2007 Csörgő [5] constructed an example of a finite group G of order 2^{13} with an elementary abelian subgroup H of order 2^6 and with H-connected transversals A and B such that $G = \langle A, B\rangle$ and the derived subgroup G' is not a subgroup of the normalizer $N_G(H)$. Thus this construction gives us a loop Q of order 2^7 and with nilpotency class of 3. Drápal, Kinyon, Nagy and Vojtěchovský have given more explicit constructions of other Csörgő-type loops [7, 8, 13]. All the initial examples had elementary abelian 2-groups as inner mapping groups. Drápal and Vojtěchovský [9] have constructed Csörgő-type loops with $C_4 \times C_4 \times C_2 \times C_2$ as the inner mapping group.

On the other hand, we know that certain classes of loops may not contain loops of Csörgő-type. Nagy and Vojtěchovský have showed that if Q is a

Moufang loop of odd order and with abelian $I(Q)$, then Q has nilpotency class at most two (this is not true for Moufang loops of even order, see [13]). Also left conjugacy closed loops with commuting inner mappings may not be of nilpotency class greater than two by the result of Csörgő and Drápal [6].

4. From groups to loops: nilpotent case

The previous section showed that Q is centrally nilpotent provided that $I(Q)$ is abelian. In this section we explore the case where $I(Q)$ is nilpotent. Before starting to work with the general case, we first assume that $I(Q)$ is a dihedral 2-group. We shall need the following lemma from [14].

Lemma 4.1. *Let $G = \mathrm{PSL}(2,q)$, where $q = 2^n \pm 1 \geqslant 17$ is a prime. If H is a maximal subgroup of G and H is a dihedral 2-group of order $q \pm 1$, then there exist no H-connected transversals in G.* □

Then let G be a finite group, $H \leqslant G$ a dihedral 2-group and assume that there exist H-connected transversals in G. If G is not solvable, then by Thompson [20] we have two possibilities. If G is simple, then $G \cong \mathrm{PSL}(2,q)$, where $q = |H| \pm 1 \geqslant 17$ is a prime. By Lemma 4.1, we have a contradiction. If G is not simple but nonsolvable, then $G = NH$, where $[G:N] = 2$, $N \cap H$ is a dihedral 2-group and $N \cong \mathrm{PSL}(2,q)$, where q is a prime number or $p = 9$. After somewhat technical calculations we again reach a contradiction and we conclude that G is solvable. We thus have

Theorem 4.2. *Let G be a finite group and $H \leqslant G$ a dihedral 2-group. If there exist H-connected transversals in G, then G is solvable.* □

If we further assume that G is generated by the connected transversals, then H is subnormal in G and we get

Theorem 4.3. *If Q is a finite loop such that $I(Q)$ is a dihedral 2-group, then Q is centrally nilpotent.* □

We shall now extend the results of Theorems 4.2 and 4.3 to the case where H is nilpotent ($I(Q)$ is nilpotent).

Theorem 4.4 (Mazur [11]). *Let H be a nilpotent subgroup of a finite group G which has H-connected transversals. Then G is solvable.* □

The proof follows the usual strategy in finite group theory. Mazur starts with the assumption that G is a minimal counterexample. Thus G is a non-solvable finite group with a nilpotent subgroup H and H-connected transversals. It then follows that H is a maximal subgroup of G and $G = DH$, where D is a minimal normal subgroup of G. Furthermore, H is a Sylow 2-subgroup of G and D is a nonabelian simple group. Then Mazur uses the classification of finite simple groups and concludes that $D \cong \mathrm{PSL}(2, q)$, where $q = 2^n \pm 1 \geqslant 7$ is a prime number (at this point we wish to remark that Baumann [2] has described the structure of nonsolvable finite groups with nilpotent maximal subgroups without the classification of finite simple groups and Baumann's result gives us directly the structure of D). After this somewhat technical calculations (as in the proof of Theorem 4.2) lead us to the conclusion that G has to be solvable.

It remains to show that H is subnormal provided that G is generated by H-connected transversals. Recall that a loop Q is solvable if it has a series $1 = Q_0 \subset \ldots \subset Q_n = Q$, where Q_{i-1} is a normal subloop of Q_i and Q_i/Q_{i-1} is an abelian group. In 1996 Vesanen [21] proved the following result: if Q is a finite loop and $M(Q)$ is solvable group, then Q is a solvable loop.

Theorem 4.5. *Let H be a nilpotent subgroup of a finite group G and let A, B be H-connected transversals in G. If $G = \langle A, B\rangle$, then H is subnormal in G.*

Proof. Let G be a minimal counterexample. We may conclude that $H_G = 1$ and thus $G \cong M(Q)$ and $H \cong I(Q)$ for some loop Q. By Theorem 4.4, $M(Q)$ is a solvable group and by using Vesanen's result we see that Q is a solvable loop. If $H \cong I(Q)$ is a maximal subgroup of $G \cong M(Q)$, then Q may not have a nontrivial normal subloop (otherwise $M(Q)$ is imprimitive on Q and $I(Q)$ is not a maximal subgroup of $M(Q)$). It follows that Q is both simple and solvable, hence a cyclic group of prime order and $I(Q) = 1$.

Thus we may assume that H is not maximal in G. But then we have such a proper subgroup T of G that H is a maximal subgroup of T and we can continue as in the proof of Theorem 3.4. □

Corollary 4.6. *If Q is a finite loop and $I(Q)$ is nilpotent, then Q is a centrally nilpotent loop.* □

The connection between the nilpotency class of Q and the structure of $I(Q)$ is mainly unknown, but by a result from Niemenmaa and Rytty [19] the question can be reduced to p-groups.

Theorem 4.7. *Let H_1 and H_2 be finite nilpotent groups,* $\gcd(|H_1|, |H_2|) = 1$ *and assume the following: if Q is a finite loop with $I(Q) \cong H_i$, then Q is centrally nilpotent of class at most n_i (for $i = 1, 2$). Now if Q is a finite loop with $I(Q) \cong H_1 \times H_2$, then Q is centrally nilpotent of class at most* $\max(n_1, n_2)$. $\square$

5. An example by Vesanen

Let C be a loop of order six given by the table

1	2	3	4	5	6
2	1	4	3	6	5
3	4	5	6	1	2
4	3	6	5	2	1
5	6	1	2	4	3
6	5	2	1	3	4

The loop C is commutative and centrally nilpotent of class two. Let $(Q, \cdot) = (C \times \mathbb{Z}_3, \cdot)$, where $(\mathbb{Z}_3, +)$ is the group of residue classes modulo three (with elements 0, 1 and 2) and

$$(x, a) \cdot (y, b) = \begin{cases} (xy, a + b + 1), & \text{if } x \neq 1, y \neq 1 \text{ and } xy \neq 1; \\ (xy, a + b), & \text{if } x = 1 \text{ or } y = 1 \text{ or } xy = 1. \end{cases}$$

Now Q is a commutative loop of order 18 and Q is centrally nilpotent of class three. Furthermore, $I(Q) \cong C_3 \times C_3 \times C_3 \times S_3 \times S_3$ is not a nilpotent group (the computations were checked by using GAP [10] and its package LOOPS [12]).

References

[1] **A. A. Albert**, *Quasigroups I*, Trans. Amer. Math. Soc. **54** (1943), 507 – 519.

[2] **B. Baumann**, *Endliche nichtauflösbare Gruppen mit einer nilpotenten maximalen Untergruppe*, J. Algebra **38** (1976), 119 – 135.

[3] **R. Bruck**, *Contributions to the theory of loops*, Trans. Amer. Math. Soc. **60** (1946), 245 – 354.

[4] **P. Csörgő**, *On connected transversals to abelian subgroups and loop theoretical consequences.* Arch. Math. (Basel) **86** (2006), 499 – 516.

[5] **P. Csörgő**, *Abelian inner mappings and nilpotency class greater than two*, European J. Combin. **28** (2007), 858 – 867.

[6] **P. Csörgő and A. Drápal**, *Left conjugacy closed loops of nilpotency class two*, Results Math. **47** (2005), 242 – 265.

[7] **A. Drápal and M. Kinyon**, *Buchsteiner loops: Associators and constructions*, preprint at http://arxiv.org/abs/0812.0412, (2008).

[8] **A. Drápal and P. Vojtěchovský**, *Explicit constructions of loops with commuting inner mappings*, European J. Combin. **29** (2008), 1662 – 1681.

[9] **A. Drápal and P. Vojtěchovský**, *Small loops of nilpotency class three with commutative inner mapping groups*, to appear in J. Group Theory.

[10] **The GAP Group**, *GAP* (*Groups, Algorithms, and Programming*), version 4.4.12 (2008), http://www.gap-system.org.

[11] **M. Mazur**, *Connected transversals to nilpotent groups*, J. Group Theory **10** (2007), 195 – 203.

[12] **G. Nagy and P. Vojtěchovský**, *LOOPS :Computing with quasigroups and loops in GAP*, version 2.0.0 (2008), http://www.math.du.edu/loops/.

[13] **G. Nagy and P. Vojtěchovský**, *Moufang loops with commuting inner mappings*, J. Pure Appl. Algebra **213** (2009), 2177 – 2186.

[14] **M. Niemenmaa**, *On loops which have dihedral 2-groups as inner mapping groups*, Bull. Aust. Math. Soc. **52** (1995), 153 – 160.

[15] **M. Niemenmaa**, *Finite loops with nilpotent inner mapping groups are centrally nilpotent*, Bull. Aust. Math. Soc. **79** (2009), 109 – 114.

[16] **M. Niemenmaa and T. Kepka**, *On multiplication groups of loops*, J. Algebra **135** (1990), 112 – 122.

[17] **M. Niemenmaa and T. Kepka**, *On connected transversals to abelian subgroups*, Bull. Austral. Math. Soc. **49** (1994), 121 – 128.

[18] **M. Niemenmaa and M. Rytty**, *Connected transversals and multiplication groups of loops*, Quasigroups Related Systems **15** (2007), 95 – 107.

[19] **M. Niemenmaa and M. Rytty**, *On finite loops with nilpotent inner mapping groups*, Arch. Math. (Basel) **95** (2010), 319 – 324.

[20] **J.G. Thompson**, *A special class of non-solvable groups*, Math. Z. **72** (1960), 458 – 462.

[21] **A. Vesanen**, *Solvable groups and loops*, J. Algebra **180** (1996), 862 – 876.

Received April 27, 2011

Department of Mathematical Sciences
University of Oulu
Pentti Kaiteran katu 1
PO Box 3000
90014 University of Oulu
Finland
E-mail: Markku.Niemenmaa@oulu.fi