Quasigroups and Related Systems **18** (2010), 229 – 240

Parallelograms in quadratical quasigroups

Vladimir Volenec and Ružica Kolar–Šuper

Abstract. The "geometric" concept of parallelogram is introduced and investigated in a general quadratical quasigroup and geometrical interpretation in the quadratical quasigroup $\mathbb{C}(\frac{1+i}{2})$ is given. Some statements about relationships between the parallelograms and some other "geometric" structures in a general quadratical quasigroup will be also considered.

A grupoid $(Q,\cdot)$ is said to be *quadratical* if the identity

$$ab\cdot a = ca\cdot bc \tag{1}$$

holds and the equation $ax=b$ has a unique solution $x\in Q$ for all $a,b\in Q$ i.e., $(Q,\cdot)$ is a right quasigroup. In [16] it is proved that $(Q,\cdot)$ is then a quasigroup. $(Q,\cdot)$ is satisfying the following identitites

$$aa = a, \tag{2}$$

$$ab\cdot cd = ac\cdot bd, \tag{3}$$

$$ab\cdot a = a\cdot ba, \tag{4}$$

$$ab\cdot a = ba\cdot b, \tag{5}$$

$$a\cdot bc = ab\cdot ac, \tag{6}$$

$$ab\cdot c = ac\cdot bc \tag{7}$$

and the equivalencies

$$ab = cd \Leftrightarrow bc = da, \tag{8}$$

$$ax = b \Leftrightarrow x = (b\cdot ba)\cdot(b\cdot ba)(ba\cdot a), \tag{9}$$

$$xa = b \Leftrightarrow x = (a\cdot ab)(ab\cdot b)\cdot(ab\cdot b). \tag{10}$$

2010 Mathematics Subject Classification: 20N05
Keywords: Quadratical quasigroup, parallelogram, square

Let $(\mathbb{C}, +, \cdot)$ be the field of complex numbers and $*$ the operation on $\mathbb{C}$ defined by

$$a * b = (1 - q)a + qb \tag{11}$$

where $q = \frac{1+i}{2}$. It can be proved that $(\mathbb{C}, *)$ is a quadratical quasigroup. This quasigroup has a nice geometric interpretation which motivates the study of quadratical quasigroup. Let us regard the complex numbers as points of the Euclidean plane. For any point a we obviously have $a * a = a$, and for two different points a, b the equality (11) can be written in the form

$$\frac{a * b - a}{b - a} = \frac{q - 0}{1 - 0},$$

which means that the points $a, b, a * b$ are the vertices of a triangle directly similar to the triangle with the vertices $0, 1, q$ (Figure 1). We can say that $a * b$ is the centre of a square with two adjacent vertices a and b, which justifies the name "quadratical quasigroup". We shall denote this quasigroup by $\mathbb{C}(\frac{1+i}{2})$ because we have $a * b = \frac{1+i}{2}$ if $a = 0$ and $b = 1$.

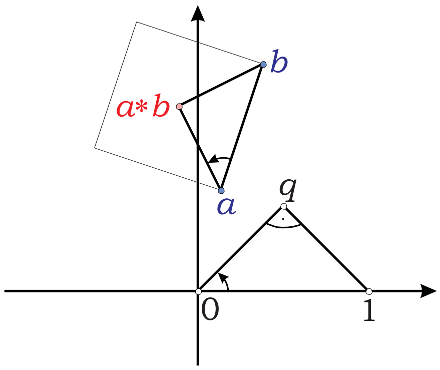

Figure 1.

The figures in the quasigroup $\mathbb{C}(\frac{1+i}{2})$ can be used as the illustrations of "geometric" relations in any quadratical quasigroup $(Q, \cdot)$. For example, the left side of the identity (1) is obviously the midpoint of the points a and b and this identity is illustrated in Figure 2 (here and in all other figures in the article we shall use the sign $\cdot$ instead of the sign $*$).

In the sequel let $(Q, \cdot)$ be any quadratical quasigroup. The elements of Q are said to be *points*.

If $\bullet$ is an operation in the set Q defined by

$$a \bullet b = a \cdot ba = ab \cdot a = ca \cdot bc, \tag{12}$$

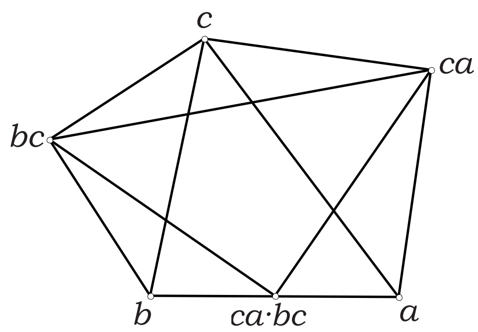

Figure 2.

then (cf. [16]) $(Q, \bullet)$ is an idempotent medial commutative quasigroup, i.e., the identities

$$a \bullet a = a, \tag{13}$$

$$(a \bullet b) \bullet (c \bullet d) = (a \bullet c) \bullet (b \bullet d), \tag{14}$$

$$a \bullet b = b \bullet a \tag{15}$$

hold. The point $a \bullet b$ is said to be a *midpoint* of the pair $\{a, b\}$ of points.

In [15] the notion of a parallelogram is defined in any medial quasigroup and because of mediality (3) we can apply this definition in our quadratical quasigroup $(Q, \cdot)$. According to [15, Cor.1] the points a, b, c, d are said to be the vertices of a *parallelogram* and we write $Par(a, b, c, d)$ if there are two points p and q such that $ap = bq$, $dp = cq$. In [15] it is proved that (Q, Par) is a *parallelogram space*, i.e., we have the properties:

(P1) For any $a, b, c \in Q$ there is an unique point d such that $Par(a, b, c, d)$ holds.

(P2) If (e, f, g, h) is any cyclical permutation of (a, b, c, d) or of (d, c, b, a), then $Par(a,\ b, c, d)$ implies $Par(e, f, g, h)$.

(P3) $Par(a, b, c, d)$, $Par(c, d, e, f) \Rightarrow Par(a, b, f, e)$.

But, the parallelogram can be defined directly, using the midpoints, as we have:

Theorem 1. $Par(a, b, c, d) \Leftrightarrow a \bullet c = b \bullet d$.

Proof. Let $ap = bq$. We must prove the equivalence of the equalities $dp = cq$ and $a \bullet c = b \bullet d$. We obtain successively

$$(a \bullet c)(pq \cdot p) \overset{(12)}{=} (ac \cdot a)(pq \cdot p) \overset{(3)}{=} (ac \cdot pq) \cdot ap \overset{(3)}{=} (ap \cdot cq) \cdot ap = (bq \cdot cq) \cdot bq,$$
$$(b \bullet d)(pq \cdot p) \overset{(12)}{=} (bd \cdot b)(pq \cdot p) \overset{(5)}{=} (bd \cdot b) \cdot (qp \cdot q) \overset{(3)}{=} (bd \cdot qp) \cdot bq \overset{(3)}{=} (bq \cdot dp) \cdot bq,$$

wherefrom it follows the mentioned equivalence. □

Corollary 1. $Par(a, c, b, c) \Leftrightarrow a \bullet b = c.$

If we use the equivalence $Par(a, b, c, d) \Leftrightarrow a \bullet c = b \bullet d$ as the definition for parallelograms, then the properties (P1)–(P3) can be proved simply by the properties of the quasigroup $(Q, \bullet)$. The properties (P1) and (P2) are obvious. For the proof of (P3) we must prove that $a \bullet c = b \bullet d$ and $c \bullet e = d \bullet f$ imply $a \bullet f = b \bullet e$. We obtain

$$\begin{aligned}(a \bullet f) \bullet (c \bullet d) &\overset{(14)}{=} (a \bullet c) \bullet (f \bullet d) \overset{(15)}{=} (a \bullet c) \bullet (d \bullet f) = (b \bullet d) \bullet (c \bullet e) \\ &\overset{(15)}{=} (b \bullet d) \bullet (e \bullet c) \overset{(14)}{=} (b \bullet e) \bullet (d \bullet c) \overset{(15)}{=} (b \bullet e) \bullet (c \bullet d)\end{aligned}$$

and therefore $a \bullet f = b \bullet e$.

Theorem 1 enables us to define the centre of a parallelogram. We say that (a, b, c, d) is a parallelogram with a *centre* o and we write $Par_o(a, b, c, d)$ if $a \bullet c = b \bullet d = o$.

The parallelogram can be defined explicitly in the quasigroup $(Q, \cdot)$ (Figure 3), without the auxiliary points, because of the following theorem.

Theorem 2. *The statement* $Par(a, b, c, d)$ *is equivalent with the equality*

$$d = [b(bc \cdot c) \cdot (bc \cdot c)c][a(a \cdot ab) \cdot (a \cdot ab)b] \tag{16}$$

Proof. According to (P1) it is sufficient only to prove that (16) implies $Par(a, b, c, d)$. Let

$$p = b(bc \cdot c) \cdot (bc \cdot c)c, \tag{17}$$

$$q = a(a \cdot ab) \cdot (a \cdot ab)b. \tag{18}$$

By (16) we have $d = pq$. According to (6) and (3) the equality (17) can be written in the form

$$p = (b \cdot bc)(bc) \cdot (bc \cdot c)c = (b \cdot bc)(bc \cdot c) \cdot (bc \cdot c)$$

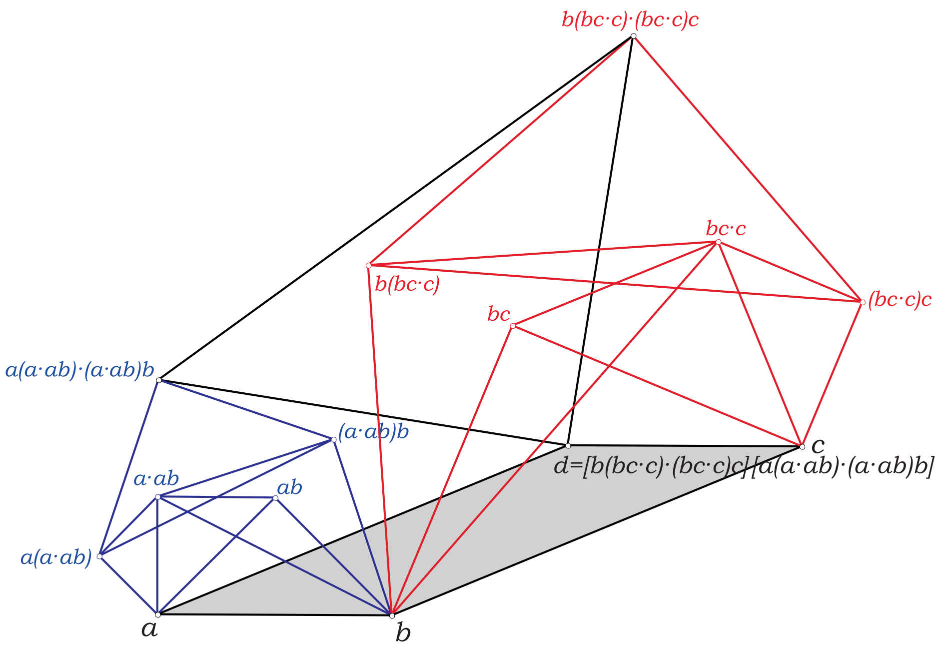

Figure 3.

equivalent with $pb = c$ because of (10). Owing to (7) and (3) the equality (18) can be written in the form

$$q = a(a \cdot ab) \cdot (ab)(ab \cdot b) = (a \cdot ab) \cdot (a \cdot ab)(ab \cdot b)$$

equivalent with $bq = a$ because of (9). This equality can be written as $aa = bq$ by (2). On the other hand we obtain

$$da = pq \cdot bq \overset{(7)}{=} pb \cdot q = cq.$$

The equalities $aa = bq$ and $da = cq$ prove the statement $Par(a, b, c, d)$. □

Corollary 2. *$Par(a, b, c, d)$ holds if and only if there are two points p and q such that $pb = c$, $bq = a$, $pq = d$.*

Figure 4 shows how the equalities $pb = c$, $bq = a$, $pq = d$ imply $Par(a, b, c, d)$ in the quasigroup $\mathbb{C}(\frac{1+i}{2})$.

Using Theorem 1 let us prove some new properties of the relation Par in any idempotent medial commutative quasigroup $(Q, \bullet)$.

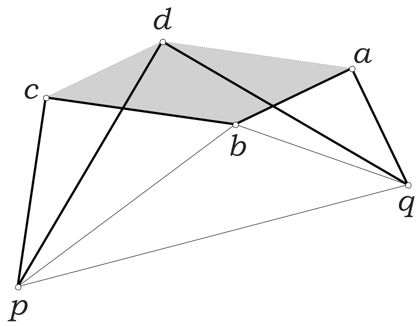

Figure 4.

Theorem 3. *Let* $Par_{o'}(a', b', c', d')$. *The statements* $Par_o(a, b, c, d)$ *and* $Par_{o \bullet o'}(a \bullet a', b \bullet b', c \bullet c', d \bullet d')$ *are equivalent.*

Proof. It is sufficient to prove the equivalence of the equalities $a \bullet c = o$ and $(a \bullet a') \bullet (c \bullet c') = o \bullet o'$ if we have the equality $a' \bullet c' = o'$. But, this is obvious because of

$$(a \bullet c) \bullet o' = (a \bullet c) \bullet (a' \bullet c') \overset{(14)}{=} (a \bullet a') \bullet (c \bullet c'). \quad \square$$

For any $p \in Q$ we have $Par_p(p, p, p, p)$ because of (13). Therefore, we obtain:

Corollary 3. $Par_o(a, b, c, d) \Rightarrow Par_{p \bullet o}(p \bullet a, p \bullet b, p \bullet c, p \bullet d)$.

$Par_o(a, b, c, d)$ implies $Par_o(b, c, d, a)$ and we obtain:

Corollary 4. $Par_o(a, b, c, d) \Rightarrow Par_o(a \bullet b, b \bullet c, c \bullet d, d \bullet a)$.

But, we have more generally:

Theorem 4. *For any points* a, b, c, d *the statement* $Par(a \bullet b, b \bullet c, c \bullet d, d \bullet a)$ *holds.*

Proof. We obtain

$$(a \bullet b) \bullet (c \bullet d) \overset{(15)}{=} (a \bullet b) \bullet (d \bullet c) \overset{(14)}{=} (a \bullet d) \bullet (b \bullet c) \overset{(15)}{=} (b \bullet c) \bullet (d \bullet a). \quad \square$$

Corollary 5. *It holds* $Par(a \bullet b, b \bullet c, c \bullet a, a)$ *for any points* a, b, c.

A concept of a square is defined in [17]. We say that (a, b, c, d) is a square with the centre o and we write $S_o(a, b, c, d)$ or simply $S(a, b, c, d)$ if $ab = bc = cd = da = o$. Then we have the equalities $ac = d$, $bd = a$, $ca = b$, $db = c$ too. Any two of these four equalities imply $S(a, b, c, d)$. In [17, Th. 2] it is proved that $S_o(a, b, c, d)$ implies $o = a \bullet c = b \bullet d$, i.e., we have:

Theorem 5. *$S_o(a,b,c,d) \Rightarrow Par_o(a,b,c,d)$, i.e., every square is a parallelogram with the same centre.*

The following theorem generalizes Theorem 5 in [17].

Theorem 6. *$Par_o(a,b,c,d) \Leftrightarrow S_o(ba,cb,dc,ad)$.*

Proof. We obtain

$$a \bullet c \overset{(12)}{=} ba \cdot cb$$

and the equalities $a \bullet c = o$ and $ba \cdot cb = o$ are equivalent. Analogously, we have

$$b \bullet d = o \Leftrightarrow cb \cdot dc = o,$$

$$c \bullet a = o \Leftrightarrow dc \cdot ad = o,$$

$$d \bullet b = o \Leftrightarrow ad \cdot ba = o.$$

□

In the quasigroup $\mathbb{C}(\frac{1+i}{2})$ Theorem 6 proves a well–known statement (cf. [13], [2], [3], [9], [7], [10], [12], [11]):

If we construct positively oriented squares on the sides of a given oriented quadrangle, then the centers of these squares form a negatively oriented square if and only if the given quadrangle is a parallelogram.

In [5] and [1, p. 241] a statement is proved, which is illustrated in Figure 5 in the quasigroup $\mathbb{C}(\frac{1+i}{2})$ and can be formulated as the following theorem.

Theorem 7. *If*

$$S_{a'}(b,c,a_1,a_2),\ S_{b'}(c,a,b_1,b_2),\ S_{c'}(a,b,c_1,c_2) \tag{19}$$

and if $\widehat{a}$, $\widehat{b}$, $\widehat{c}$ are points such that

$$Par(b_1,a,c_2,\widehat{a}),\ Par(c_1,b,a_2,\widehat{b}),\ Par(a_1,c,b_2,\widehat{c}) \tag{20}$$

then we have the equalities

$$\widehat{c}\widehat{b} = a, \qquad \widehat{a}\,\widehat{c} = b, \qquad \widehat{b}\,\widehat{a} = c, \tag{21}$$

$$\widehat{b} \bullet \widehat{c} = a', \qquad \widehat{c} \bullet \widehat{a} = b', \qquad \widehat{a} \bullet \widehat{b} = c', \tag{22}$$

$$a\widehat{c} = \widehat{b}a = a', \qquad b\widehat{a} = \widehat{c}b = b', \qquad c\widehat{b} = \widehat{a}c = c'.$$

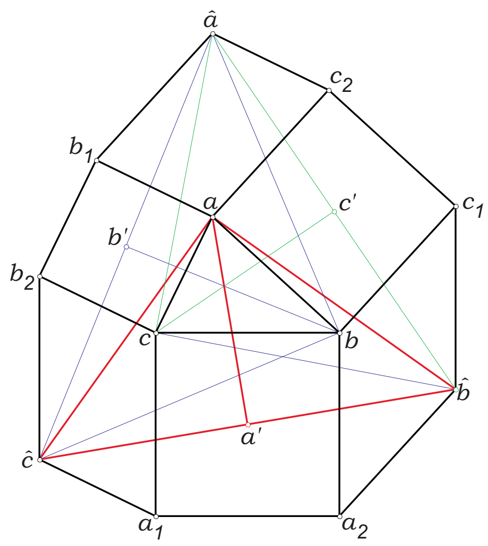

Figure 5.

Proof. Let $\widehat{a}$, $\widehat{b}$, $\widehat{c}$ be points such that $\widehat{a}c = c'$, $\widehat{b}a = a'$, $\widehat{c}b = b'$. According to (19) we have the equalities $b_1c = a$, $ca = b'$, $bc = a'$, $c_2a = c'$ (among others). The equalities $b_1c = a = aa$ and $\widehat{a}c = c' = c_2a$ prove the first statement (20) and analogously the other two statements (20) can be proved. According to (8) from $ca = b' = \widehat{c}b$ it follows $a\widehat{c} = bc$, i.e., $a\widehat{c} = a'$. Therefore we have $a\widehat{c} = \widehat{b}a$ and by (8) it follows $\widehat{c}\widehat{b} = aa$, i.e., the first equality (21). Finally, we obtain the first equality (22): $\widehat{b} \bullet \widehat{c} \overset{(15)}{=} \widehat{c} \bullet \widehat{b} \overset{(12)}{=} a\widehat{c} \cdot \widehat{b}a = a'a' \overset{(2)}{=} a'$. □

A point o is said to be the *center of the square on the segment* (a, b) if $S_o(a, b, c, d)$ holds for some points c and d, i.e., if $ab = o$. A *rotation* for a (positively oriented) right angle about a point o is the mapping $a \mapsto b$ such that $ab = o$.

Theorem 8. *If* a_1, a_2, a_3, a_4 *are any points and* b_{ij} *is the center of the square on the segment* (a_i, a_j) *for any* $i, j \in \{1, 2, 3, 4\}$ $(i \neq j)$*, then we have the statements* $Par(b_{12}, b_{32}, b_{34}, b_{14})$ *and* $Par(b_{21}, b_{23}, b_{43}, b_{41})$*. The rotation for a right angle about the point* $a_1 \bullet a_3$ *maps* $Par(b_{23}, b_{21}, b_{41}, b_{43})$ *onto* $Par(b_{12}, b_{32}, b_{34}, b_{14})$ *and the rotation for a right angle about the point* $a_2 \bullet a_4$ *maps* $Par(b_{12}, b_{32}, b_{34}, b_{14})$ *onto* $Par(b_{41}, b_{43}, b_{23}, b_{21})$ *(Figure 6).*

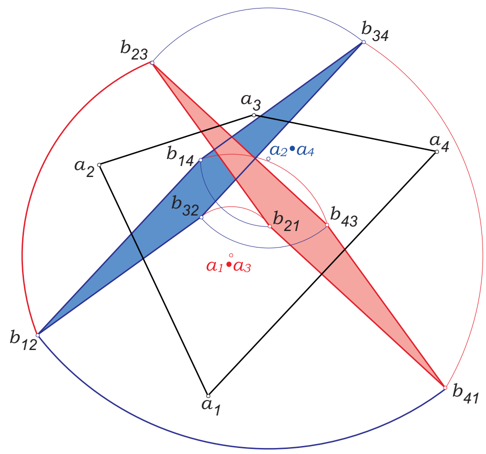

Figure 6.

Proof. According to [15, Th. 28] we have the statement $Par(a_1a_2,\ a_3a_2,\ a_3a_4, a_1a_4)$ and $Par(a_2a_1,\ a_2a_3, a_4a_3, a_4a_1)$ and for any $i, j \in \{1, 2, 3, 4\}$ $(i \neq j)$ we have the equality $a_ia_j = b_{ij}$. The rotation for a right angle about the point $a_1 \bullet a_3$ maps the points b_{23}, b_{21}, b_{41}, b_{43} onto the points b_{12}, b_{32}, b_{34}, b_{14} because of the equalities

$$b_{23}b_{12} = a_2a_3 \cdot a_1a_2 \overset{(12)}{=} a_3 \bullet a_1 \overset{(15)}{=} a_1 \bullet a_3 = a_2a_1 \cdot a_3a_2 = b_{21}b_{32},$$

$$b_{41}b_{34} = a_4a_1 \cdot a_3a_4 \overset{(12)}{=} a_1 \bullet a_3 \overset{(15)}{=} a_3 \bullet a_1 = a_4a_3 \cdot a_1a_4 = b_{43}b_{14}. \qquad \square$$

In the case of the quasigroup $\mathbb{C}(\frac{1+i}{2})$ Theorem 8 proves some statements from [14] and [8].

Theorem 9. *If*

$$S_o(p, a, u, b), \qquad S_{o'}(p, a', u', b'), \tag{23}$$

$$Par(a', p, b, c), \qquad Par(a, p, b', c') \tag{24}$$

holds, then the rotation for a right angle about the point o *maps* $Par(p, b, c, a')$ *onto* $Par(a, p, b', c')$ *and the rotation for a right angle about the point* o' *maps* $Par(a, p, b', c')$ *onto* $Par(c, a', p, b)$ *(Figure 7).*

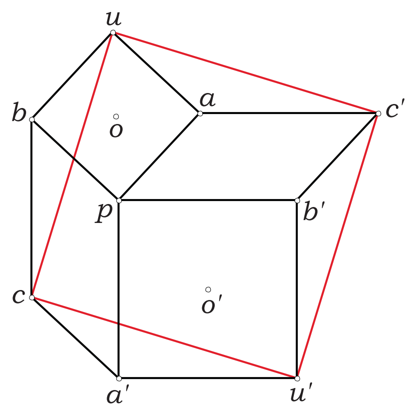

Figure 7.

Proof. Let the statements (23) hold and let c, c' be the points such that $cb' = o$, $c'b = o'$. The equalities

$$pa = o = cb', \quad pa' = o' = c'b$$

imply by (8) the equalities

$$ac = b'p = o', \quad a'c' = bp = o.$$

Now, the equalities

$$a'b' = p = pp,\ cb' = o = bp \ \text{resp.}\ ab = p = pp,\ c'b = o' = b'p$$

prove the statements (24). The last two statements of theorem are the consequences of the equalities
$pa{=}o$, $bp{=}o$, $cb'{=}o$, $a'c'{=}o$ resp. $ac{=}o'$, $pa'{=}o'$, $b'p{=}o'$, $c'b{=}o'$. □

In the case of the quasigroup $\mathbb{C}(\frac{1+i}{2})$ Theorem 9 proves some statements from [4]. The fact that the rotation for a right angle about the points o maps the segment (b, a') onto the segment (p, c') proves that the median from the vertex p of the triangle (p, b', a) is orthogonal to the side (b, a') of the triangle (p, b, a') and equal to the half of this side and a similar fact holds for the median from the vertex p of the triangle $(p,\ b,\ a')$ and the segment (b', a) (cf. [18]).

Theorem 10. *With the hypotheses of Theorem 9 it holds* $S(u, c, u', c')$ (*Figure* 7).

Proof. According to Corollary 2 we observe the implications

$$Par(b,p,a',c),\ u'p=a',\ pu=b\ \Rightarrow u'u=c,$$
$$Par(b',p,a,c'),\ up=a,\ pu'=b'\ \Rightarrow uu'=c',$$

and the equalities $u'u=c,\ uu'=c'$ imply $S(u,c,u',c')$. □

Theorem 11. *The statements* $S(b,c,a_1,a_2),\ S(c,a,b_1,b_2),\ S(a,b,c_1,c_2)$ *and the equalities* $a_o=c_1b_2,\ b_o=a_1c_2,\ c_o=b_1a_2$ *imply*

$$Par(c,a,b,a_o),\ Par(a,b,c,b_o),\ Par(b,c,a,c_o) \tag{25}$$

$b_o\bullet c_o=a,\ \ c_o\bullet a_o=b,\ \ a_o\bullet b_o=c_o$ (*Figure* 8).

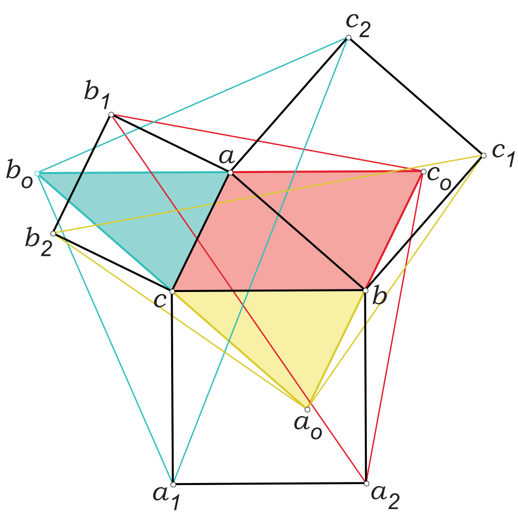

Figure 8.

Proof. We have the equalities $c_1a=b,\ ab_2=c,\ c_1b_2=a_o$ and according to Corollary 2 it follows $Par(c,a,b,a_o)$. Analogously we can prove other statements (25). From $Par(b_o,a,b,c)$ and $Par(b,c,a,c_o)$ by (P3) we obtain $Par(b_o,a,c_o,a)$, i.e., $b_o\bullet c_o=a$. □

References

[1] **P. Baptist**, *Die Entwicklung der neueren Dreiecksgeometrie*, Lehrbücher und Monographien Didaktik Math., Mannheim, 1992.

[2] **A. Barlotti**, *Intorno ad una generalizzazione un noto teorema relativo al triangolo*, Boll. Unione Mat. Ital., III Ser. **7** (1952), 182 – 185.

[3] **A. Barlotti**, *Una propietà degli n-agoni si ottengono transformando di una affinità un n-agono regolare*, Boll. Unione Mat. Ital., III Ser. **10** (1955), 96 – 98.

[4] **V. G. Boltjanskij**, *On a tesselation*, (Rssian), Mat. v škole, 1984, No. 1, 65 – 66.

[5] **H. Demir**, *Problem E* 2124, Amer. Math. Monthly **75** (1968), 899; **76** (1969), 938.

[6] **W. A. Dudek**, *Quadratical quasigroups*, Quasigroups and Related Systems **4** (1997), 9 – 13.

[7] **L. Gerber**, *Napoleon's theorem and the parallelogram inequality for affine-regular polygons*, Amer. Math. Monthly **87** (1980), 644 – 648.

[8] **M. Goljberg**, *Problems* 3275 *and* 3276, Mat. v škole (1989), No. 4, 109 – 110.

[9] **D. I. Han**, *On solving geometrical problems using vectors*, (Russian), Mat. v škole 1974, No. 1, 22 – 25.

[10] **M. Jeger**, *Komplexe Zahlen in der Elementargeometrie*, Elem. Math. **37** (1982), 136 – 147.

[11] **D. Kahle**, *Eine Bemerkung zum Satz von Napoleon–Barlotti für das Parallelogram*, Didaktik Math. **22** (1994), 217 – 218.

[12] **J. Kratz**, *Vom regulären Fünfeck zum Satz von Napoleon–Barlotti*, Didaktik Math. **20** (1992), 261 – 270.

[13] **V. Thébault**, *Problem* 169, Nat. Math. Mag. **12** (1937/38), 55.

[14] **V. Thébault**, *Quadrangle bordé de triangles isoscèles semblables*, Ann. Soc. Sci. Bruxelles **60** (1940/46), 64 – 70.

[15] **V. Volenec**, *Geometry of medial quasigroups*, Yugoslav Academy of Science and ART **421** (1986), 79 – 91.

[16] **V. Volenec**, *Quadratical groupoids*, Note di Mat. **13** (1993), 107 – 115.

[17] **V. Volenec**, *Squares in quadratical quasigroups*, Quasigroups and Related Systems **7** (2000), 37 – 44.

[18] **I. Warburton**, *Brides chair revisited again*, Math. Gaz. **80** (1996), 557 – 558.

Received July 29, 2010

V. Volenec
Department of Mathematics, University of Zagreb, Bijenička c. 30, 10 000 Zagreb, Croatia, *E-mail*: volenec@math.hr

R. Kolar–Šuper
Faculty of Teacher Education, University of Osijek, Lorenza Jägera 9, 31 000 Osijek, Croatia, *E-mail*: rkolar@ufos.hr