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Intersection graphs of normal subgroups of groups

Sayyed Heidar Jafari and Nader Jafari Rad

Abstract. We give characterizations of groups whose intersection graphs of normal subgroups are connected, complete, forests, or bipartite.

1. Introduction

Let $F = \{S_i : i \in I\}$ be an arbitrary family of sets. The *intersection graph* $G(F)$ of F is the graph whose vertices are S_i, $i \in I$ and in which the vertices S_i and S_j $(i, j \in I)$ are adjacent if and only if $S_i \neq S_j$ and $S_i \cap S_j \neq \emptyset$. It is known that every simple graph is an intersection graph, ([4]).

It is interesting to study the intersection graphs $G(F)$ when the members of F have an algebraic structure. Bosak [1] in 1964 studied graphs of semigroups. Then Csákány and Pollák [2] in 1969 studied the graphs of subgroups of a finite group. Zelinka [6] in 1975 continued the work on intersection graphs of nontrivial subgroups of finite abelian groups.

Recall that a subgroup H of a group G is *normal* if $g^{-1}Hg = H$ for every $g \in G$.

In this paper, we consider the intersection graph of normal subgroups of a group. For a group G, the *intersection graph of normal subgroups of* G, denoted by $\Gamma(G)$, is the graph whose vertices are in a one-to-one correspondence with proper nontrivial normal subgroups of G and two distinct vertices are adjacent if and only if the corresponding normal subgroups of G have a nontrivial (nonzero) intersection. Clearly $\Gamma(G)$ does not exist if and only if G is simple. Note that the intersection graph of a simple group G is not defined, since a graph can not have an empty vertex set.

The graph theory and group theory notation terminology follow from [5] and [3], respectively.

Throughout the paper, to simplify, for a normal subgroup N in a group

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G we use "the vertex N" instead of "the vertex in $\Gamma(G)$ corresponded to N". Also we use 0 as the trivial subgroup.

2. Connected and complete graphs

In this section we characterize all groups whose intersection graphs are connected or complete. We first some graph theory and group theory definitions. A graph G is *complete* if there is an edge between every pair of the vertices. We denote the complete graph on n vertices by K_n. A *path* of length n in a graph G is an ordered list of distinct vertices $v_0, v_1, \ldots, v_n$ such that v_i is adjacent to v_{i+1} for $i = 1, 2, \ldots, n-1$. We denote by $v_0 - v_1 - \ldots - v_n$ to such a path. A (u,v)-path is a path with endpoints u and v. For vertices x and y of G, let $d(x,y)$ be the length (the number of edges) of a shortest path from x to y ($d(x,x) = 0$, and $d(x,y) = \infty$ if there is no path between x and y). A graph G is *connected* if it has a (u,v)-path for each pair $u, v \in V(G)$.

Recall that a *chain* $0 = G_0 \subset G_1 \subset \ldots \subset G_n = G$ of subgroup of a group G is a *composition seriers* if $G_i \trianglelefteq G_{i+1}$ and $\frac{G_{i+1}}{G_i}$ is simple for $i = 0, 1, \ldots, n$. The *length* of the chain is n. If G has a composition series, then any two compositione series of G have the same length, denoted by $lc(G)$.

Lemma 2.1. *Let $G = A_1 \times A_2$. If $N_i \trianglelefteq A_i$ for $i = 1, 2$, then $N_1 \times N_2 \trianglelefteq G$.*

The *complement* $\overline{G}$ of G is the graph with vertex set $V(\overline{G}) = V(G)$, and $E(\overline{G}) = \{uv : uv \notin E(G)\}$. The complement of a complete graph is the *null graph.*

Lemma 2.2. *Let $G = N_1 \times N_2$, where N_1, N_2 are simple. Then $\Gamma(G)$ is null.*

Proof. Since N_1 and N_2 are simple, then $lc(G) = 2$. Then any normal nontrivial proper subgroup of G is both maximal and minimal. This completes the proof. $\square$

Recall that a group G is a *direct sum* of two normal subgroups N_1 and N_2 if $N_1 \cap N_2 = 0$ and $N_1 N_2 = G$, where $N_1 N_2 = \{xy : x \in N_1, y \in N_2\}$.

Theorem 2.3. *Let G be a group. Then $\Gamma(G)$ is disconnected if and only if $G = N_1 \oplus N_2$, where N_1 and N_2 are simple normal subgroups of G.*

Proof. Let $\Gamma(G)$ be disconnected. Then $\Gamma(G)$ has at least two components. Let N_1 and N_2 be two normal subgroups of G and the corresponding vertices included in two different components of $\Gamma(G)$. Thus, $N_1 \cap N_2 = 0$. Since $N_1 \cup N_2 \subseteq N_1N_2$, we obtain $N_1N_2 = G$. We conclude that $G = N_1 \oplus N_2$. Now we show that N_1 and N_2 are simple. If N_1 is not simple, then N_1 has a proper nontrivial subgroup N. Then by Lemma 2.1, $N \trianglelefteq G$. Now NN_2 is adjacent to both N_1 and N_2, a contradiction. Thus N_1 is simple. Similarly, N_2 is simple.

The converse follows from Lemma 2.2. □

The *center* $Z(G)$ of a group G is the set of all elements x which $xy = yx$ for every $y \in G$. A chain $G_0 = 0 \subseteq G_2 \subseteq \ldots \subseteq G_t = G$ is a *central series* of G if $\frac{G_i}{G_{i-1}} \subseteq Z(\frac{G}{G_{i-1}})$ for $i = 1, 2, \ldots, t$. A group G is *nilpotent* if G has a *central series*.

Corollary 2.4. *If G is nilpotent, then $\Gamma(G)$ is disconnected if and only if $G \cong \mathbb{Z}_p \times \mathbb{Z}_q$, where p, q are two non necessarily distinct primes.*

Proof. Notice that any nilpotent simple group is in the form $\mathbb{Z}_p$, where p is a prime. □

The next theorem provides a characterization for all groups whose intersection graphs are complete.

Note that a group G satisfies the minimal condition on normal subgroups if any non-empty subset of normal subgroups of G contains a minimal element.

Theorem 2.5. *Let G be a non-simple group that satisfies the minimal condition on normal subgroups. Then $\Gamma(G)$ is complete if and only if G has a unique minimal normal subgroup.*

Proof. Let G be a non-simple group and G satisfies the minimal condition on normal subgroups. Let $\Gamma(G)$ be complete. Then G has at least one minimal normal subgroup. Let N be a minimal normal subgroup of G. If N_1 is a minimal normal subgroup different from N, then $N \cap N_1 = 0$, since $0 \leq N \cap N_1 \subsetneqq N$ and $N \cap N_1 \trianglelefteq G$. This implies N_1 and N are not adjacent in $\Gamma(G)$. This is a contradiction, since $\Gamma(G)$ is complete. We deduce that N is the unique minimal normal subgroup of G.

Conversely, suppose that G has a unique minimal normal subgroup say N. Let K and L be two nontrivial normal subgroups of G. Since G satisfies the minimal condition on normal subgroups, K and L each contain a minimal normal subgroup. By assumption $N \subseteq K \cap L$, and so $K \cap L \neq 0$. Thus the vertices K and L are adjacent in $\Gamma(G)$. This means that $\Gamma(G)$ is complete. $\square$

Corollary 2.6. *For $n > 1$, $\Gamma(\mathbb{Z}_{p^n})$ is K_{n-1}.* $\square$

Example 2.7. The intersection graph of the generalized quaternion group Q_n, (of order $4n$) is complete. Note that Q_n has a unique minimal normal subgroup of order 2.

Example 2.8. For any prime p, the intersection graph of $\mathbb{Z}_{p^\infty} = \{\frac{m}{n} + \mathbb{Z} : m, n \in \mathbb{Z},\, n = p^t$ for some $t \in \mathbb{N} \cup \{0\}\}$ is an infinite complete graph. To see this notice that all proper nontrivial normal subgroups of $\mathbb{Z}_{p^\infty}$ are in the form $\langle \frac{1}{p^i} + \mathbb{Z} \rangle$, where $i \geqslant 1$. However, the only minimal normal subgroup of $\mathbb{Z}_{p^\infty}$ is $\langle \frac{1}{p} + \mathbb{Z} \rangle$.

Corollary 2.9. *For a finite nilpotent group G, $\Gamma(G)$ is complete if and only if G is a p-group and $Z(G)$ is cyclic.*

Proof. Note that any subgroup of $Z(G)$ of prime order is a minimal normal subgroup of G, and a prime p is a prime factor of $|G|$ if and only if p is a prime factor of $Z(G)$. $\square$

Example 2.10. If n is a power of 2, then the intersection graph of the dihedral group D_n is complete. Notice that D_n is a 2-group and the center of this group is of order 2.

3. Forests and bipartite graphs

In this section we characterize all groups whose intersection graphs are forests or bipartite. We recall that a graph G is called *bipartite* if its vertex set can be partitioned into two independent subsets X and Y such that every edge of G has one endpoint in X and other endpoint in Y. We denote by C_n the *cycle* with vertex set $\{v_0, v_1, \ldots, v_n\}$ and edge set $\{v_i v_{i+1} : i = 1, 2, \ldots, n-1\} \cup \{v_1 v_n\}$.

Lemma 3.1. *Let $G = N_1 \times N_2$, where N_1, N_2 are normal subgroups of G. Then $\Gamma(G)$ has a cycle C_3 if and only if N_1 or N_2 is not simple.*

Proof. ($\Longrightarrow$) follows by Lemma 2.2.
($\Longleftarrow$) Assume that N_1 is not simple. Let N be a nontrivial proper normal subgroup of N_1. Then $N \times N_2 - N \times 0 - N_1 \times 0 - N \times N_2$ is a cycle on three vertices. $\square$

Lemma 3.2. *If G is an indecomposable group of length 2, then $\Gamma(G)$ is K_1.*

Proof. Since $lc(G) = 2$, G has at least one proper nontrivial normal subgroup. By assumption any proper nontrivial normal subgroup of G is both minimal and maximal. We show that G has exactly one proper nontrivial normal subgroup. Suppose to the contrary that N_1, N_2 are two distinct proper nontrivial normal subgroups of G. Then $N_1 \cap N_2 = 0$, and $G \cong N_1N_2$, a contradiction. $\square$

A group G is indecomposable if it is not isomorphic to direct product of two nontrivial groups.

Lemma 3.3. *Let G be an indecomposable group with $lc(G) = 3$. If G has a unique maximal normal subgroup, then $\Gamma(G)$ is a forest.*

Proof. By assumption any normal subgroup of G is either minimal or maximal. Let N be the unique maximal normal subgroup of G. If there are two distinct normal subgroups K_1, K_2 of G different from N, then K_1 and K_2 are minimal, and so $K_1 \cap K_2 = 0$. This completes the proof. $\square$

We are now ready to characterize all groups whose intersection graphs are forest.

Theorem 3.4. *The intersection graph of a group G is a forest if and only if one of the following holds:*

(i) $lc(G) = 2$,

(ii) $lc(G) = 3$, *and G is an indecomposable group with a unique maximal normal subgroup,*

(iii) $G \cong M_1 \times M_2$, *where M_1, M_2 are simple groups.*

Proof. ($\Leftarrow$) follows from Lemmas 3.3, 3.2, and 3.1.
($\Rightarrow$): Let $\Gamma(G)$ be a forest. We first show that G is a direct product of at most two groups. Let $G = M_1 \times M_2 \times \ldots \times M_k$, where M_i is a group for $i = 1, 2, \ldots, k$. If $k \geqslant 3$, then $H = M_2 \times M_3 \times \ldots \times M_k$ has at least one normal proper nontrivial subgroup $M_2 \times 0 \times \ldots \times 0$, and by Lemma 3.1 $\Gamma(G)$ contains a cycle. This contradiction implies that $k \leqslant 2$. If $k = 2$,

then Lemma 3.1 implies (iii). Thus we may assume that $k = 1$. So G is indecomposable.

We show that $lc(G) \leqslant 3$. Suppose the contrary that $lc(G) \geqslant 4$. There are three proper nontrivial normal subgroups N_1, N_2, N_3 such that $N_1 \subset N_2 \subset N_3$. Then N_1, N_2 and N_3 form a cycle, a contradiction. So $lc(G) \leqslant 3$. If $lc(G) = 2$, then (i) holds. So we suppose that $lc(G) = 3$. We prove that G has a unique normal maximal subgroup. Since $lc(G) < \infty$, G has a maximal normal subgroup N. If N_1 is another maximal normal subgroup of G, then $NN_1 = G$. Since G is indecomposable, $N \cap N_1 \neq 0$. Then $N - N \cap N_1 - N_1 - N$ forms a cycle in $\Gamma(G)$. This contradiction implies that N is the unique maximal normal subgroup of G. $\square$

Next we characterize all groups whose intersection graphs are bipartite. In view of the proof of Theorem 3.4 any produced cycle has three vertices. Also it is known that a graph G is bipartite if and only if any cycle of G has even number of vertices. These lead to the following.

Corollary 3.5. *The intersection graph* $\Gamma(G)$ *of a group* G *is bipartite if and only if* $\Gamma(G)$ *is a forest.*

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Department of Mathematics, Shahrood University of Technology Shahrood, Iran
E-mails: shjafari55@gmail.com (S.H.Jafari), n.jafarirad@shahroodut.ac.ir (N.J.Rad)