Congruences on an inverse $AG^{**}$-groupoid via the natural partial order

Petar V. Protić

In memory of Nebojša Stevanović (1962–2009), my colleague and dear friend.

Abstract. In this paper we first describe natural partial order on an inverse $AG^{**}$-groupoid. With it we introduce a notion of pseudo normal congruence pair and normal congruence pair and describe congruences.

1. Introduction

A groupoid $S$ on which the following is true

$$(\forall a, b, c \in S) \quad ab \cdot c = cb \cdot a,$$

is called an Abel-Grassmann’s groupoid ($AG$-groupoid) [8] (or in some papers Left almost semigroups ($LA$-semigroups)) [3]. It is easy to verify that in every $AG$-groupoid medial law $ab \cdot cd = ac \cdot bd$ holds. Thus, $AG$-groupoids belong to the wider class of medial groupoids.

We denote the set of all idempotents of $S$ by $E(S)$.

Abel-Grassmann’s groupoid $S$ satisfying

$$(\forall a, b, c \in S) \quad a \cdot bc = b \cdot ac$$

is an $AG^{**}$-groupoid. It is obvious that in $AG^{**}$-groupoid for $a, b, c, d \in S$

$$ab \cdot cd = c(ab \cdot d) = c(db \cdot a) = db \cdot ca.$$
If $AG$-groupoid $S$ has the left identity $e$, then
\[ a \cdot bc = ea \cdot bc = eb \cdot ac = b \cdot ac, \]
so $S$ is an $AG^{**}$-groupoid.

In [5] an $AG$-groupoid $S$ is called an inverse $AG$-groupoid if for every $a \in S$ there exists $a' \in S$ such that $a = aa' \cdot a$ and $a' = a'a \cdot a'$. Then $a'$ is an inverse element of $a$, and by $V(a)$ we shall mean the set of all inverses of $a$. It is easy to prove that if $a' \in V(a)$, $b' \in V(b)$, then $a'b' \in V(ab)$ and that $aa'$ or $a'a$ are not necessarily idempotents.

**Remark 1.** In [1] it is proved that in an $AG^{**}$-groupoid $S$ the set $E(S)$ is a semilattice (Remark 2). Also, in [1] it is proved that in an inverse $AG^{**}$-groupoid for $a \in S$, by Remark 3, we have $|V(a)| = 1$. If $a^{-1}$ is a unique inverse for $a$, then by Lemma 1 $aa^{-1}, a^{-1}a \in E(S)$ if and only if $aa^{-1} = a^{-1}a$.

The following proposition is trivially true.

**Proposition 1.** Let $S$ be an inverse $AG^{**}$-groupoid and $\rho$ congruence relation on $S$. Then $S/\rho$ is an inverse $AG^{**}$-groupoid. Also, if $a, b \in S$ then $a\rho b$ if and only if $a^{-1}\rho b^{-1}$. \[\square\]

## 2. Natural partial order

In this section we define a natural partial relation on inverse $AG^{**}$-groupoid $S$ and prove some of its properties.

**Theorem 1.** If $S$ is an inverse $AG^{**}$-groupoid, then the relation
\[ a \preceq b \iff a = aa^{-1} \cdot b \quad (1) \]
on $S$ is a natural partial order relation and it is compatible.

**Proof.** The proof that $\preceq$ is reflexive is obvious. For antisymmetry let us suppose that $a \preceq b$ and $b \preceq a$. Then $a = aa^{-1} \cdot b$ and $b = bb^{-1} \cdot a$, and
\[ a = aa^{-1} \cdot b = aa^{-1} \cdot (bb^{-1} \cdot a) = bb^{-1} \cdot (aa^{-1} \cdot a) = bb^{-1} \cdot a = b, \]
imply antisymmetry.
Let us now suppose that \(a \leq b\) and \(b \leq c\). Then \(a = aa^{-1} \cdot b = bb^{-1} \cdot c\), and
\[
a = aa^{-1} \cdot b = aa^{-1}(bb^{-1} \cdot c) = ((aa^{-1} \cdot a)a^{-1})(bb^{-1} \cdot c)
\]
\[
= (a^{-1}a \cdot aa^{-1})(bb^{-1} \cdot c) = (a^{-1}a \cdot bb^{-1})(aa^{-1} \cdot c)
\]
\[
= b(a^{-1}a \cdot b^{-1}) \cdot (aa^{-1} \cdot c) = b(aa^{-1} \cdot b^{-1}) \cdot (aa^{-1} \cdot c)
\]
\[
= ba^{-1} \cdot (aa^{-1} \cdot c) = ca^{-1} \cdot (aa^{-1} \cdot b) = ca^{-1} \cdot a = aa^{-1} \cdot c,
\]

imply that \(a \leq c\). Hence transitivity holds and \(\leq\) is a partial order on \(S\).

Let \(a \leq b\) and \(c \in S\). Then
\[
ca = c(aa^{-1} \cdot b) = (cc^{-1} \cdot c)(aa^{-1} \cdot b) = (cc^{-1} \cdot aa^{-1}) \cdot cb
\]
\[
= (ca \cdot c^{-1}a^{-1}) \cdot cb = (ca \cdot (ca)^{-1}) \cdot cb,
\]
and so the relation \(\leq\) is left compatible. Also, since
\[
ac = (aa^{-1} \cdot b)c = (aa^{-1} \cdot b)(cc^{-1} \cdot c) = (aa^{-1} \cdot cc^{-1}) \cdot bc
\]
\[
= (ac \cdot a^{-1}c^{-1}) \cdot bc = (ac \cdot (ac)^{-1}) \cdot bc,
\]
therefore the relation \(\leq\) is right compatible. Hence, \(\leq\) is compatible.

**Corollary 1.** Let \(S\) be an inverse \(AG^{**}\)-groupoid and \(a, b \in S\). Then
\[
a \leq b \iff aa^{-1} = ba^{-1}.
\]

**Proof.** If \(a \leq b\) then by (1) we have
\[
aa^{-1} = (aa^{-1} \cdot b)a^{-1} = a^{-1}b \cdot aa^{-1} = a^{-1}a \cdot ba^{-1} = b(a^{-1}a \cdot a^{-1}) = ba^{-1}.
\]

Conversely, for \(a, b \in S\), \(aa^{-1} = ba^{-1}\) implies that
\[
a = aa^{-1} \cdot a = ba^{-1} \cdot a = aa^{-1} \cdot b.
\]

So, by (1), \(a \leq b\). \(\square\)

### 3. Normal congruence pair

In this section by \(S\) we mean an inverse \(AG^{**}\)-groupoid in which for each \(a \in S\) we have \(aa^{-1} = a^{-1}a\) or equivalently \(aa^{-1}, a^{-1}a \in E(S)\).

First, we prove the following consequence of Theorem 1.
Corollary 2. Let $a, b \in S$. Then
\[ a \leq b \iff (\exists e \in E(S)) \, a = eb. \]

Proof. Let $a, b \in S$. Then $a \leq b$ if and only if $a = (aa^{-1})b$. Since $aa^{-1} \in E(S)$, therefore if $e = aa^{-1}$ implies that $a = eb$.

Conversely, let $a, b \in S$ be such that $e \in E(S)$ and $a = eb$. Because $aa^{-1} = a^{-1}a \in E(S)$ and $E(S)$ is a semilattice, we have
\[ aa^{-1} \cdot b = (eb \cdot eb^{-1})b = (bb^{-1} \cdot e)b = (bb^{-1} \cdot e)(bb^{-1} \cdot b) = (bb^{-1} \cdot bb^{-1}) \cdot eb = bb^{-1} \cdot eb = e(bb^{-1} \cdot b) = eb = a \]
and so $a \leq b$. \[ \square \]

Let $\rho$ be a congruence on $S$. The restriction $\rho|_{E(S)}$ is the trace of $\rho$ and it is denoted by $\rho_{tr}$. Also, kernel $\rho$ is $ker \rho = \{ a \in S \mid (\exists e \in E(S)) \, ape \}$.

If $\rho$ is a congruence relation on $S$, then $ker \rho$ is a subgroupoid of $S$ and $E(S) \subseteq ker \rho$ it is, $ker \rho$ is a full subgroupoid of $S$. Also, $tr \rho$ is a congruence on semilattice $E(S)$.

Definition 1. Let $K$ be a full subgroupoid of $S$ and $\tau$ a congruence on $E(S)$ satisfying the following condition:
\[ (i) \text{ For all } a \in S, b \in K, \, b \leq a \text{ and } aa^{-1} \tau bb^{-1} \text{ imply } a \in K. \]
We call $(K, \tau)$ a pseudo normal congruence pair for $S$. If, in addition,
\[ (ii) \text{ For every } a \in K, \text{ there exists } b \in S \text{ with } b \leq a, \, aa^{-1} \tau bb^{-1} \text{ and } b^{-1} \in K, \]
then $(K, \tau)$ is called a normal congruence pair for $S$.

For pseudo normal congruence pair $(K, \tau)$, we define a relation
\[ a\rho_{(K, \tau)}b \iff ab^{-1}, \tau a^{-1}b, ba^{-1}, b^{-1}a \in K, \, aa^{-1} \cdot b^{-1}b \tau aa^{-1} \tau bb^{-1}. \]

Lemma 1. Let $(K, \tau)$ be a pseudo normal congruence pair of $S$, $a, b \in S$. If $a\rho_{(K, \tau)}b$ and $b \in K$, then $a \in K$.

Proof. From $a\rho_{(K, \tau)}b$ we have $ab^{-1} \in K$ and $aa^{-1} \cdot bb^{-1} \tau aa^{-1} \tau bb^{-1}$. Since $b \in K$, it follows that $ab^{-1} \cdot b = bb^{-1} \cdot a \in K$.

We prove that $ab^{-1} \cdot b \leq a$. Here
\[
((ab^{-1} \cdot b)(ab^{-1} \cdot b)^{-1})a = ((ab^{-1} \cdot b)(a^{-1}b \cdot b^{-1}))a = ((bb^{-1} \cdot a)(b^{-1}b \cdot a^{-1}))a \\
= ((bb^{-1} \cdot b^{-1}b)a a^{-1})a = (bb^{-1} \cdot a a^{-1})a \\
= (aa^{-1} \cdot bb^{-1})a = (aa^{-1} \cdot bb^{-1})(aa^{-1} \cdot a) \\
= (aa^{-1} \cdot aa^{-1})(bb^{-1} \cdot a) = aa^{-1}(bb^{-1} \cdot a) \\
= bb^{-1}(aa^{-1} \cdot a) = bb^{-1} \cdot a = ab^{-1} \cdot b.
\]
Hence, by (1), it follows that $ab^{-1} \cdot b \leq a$.

Also

$$(ab^{-1} \cdot b)(ab^{-1} \cdot b)^{-1} = (ab^{-1} \cdot b)(a^{-1}b^{-1})$$

$$= (ab^{-1} \cdot a^{-1}b^{-1}bb^{-1} = (aa^{-1} \cdot b^{-1}bb^{-1})$$

$$= (bb^{-1} \cdot b^{-1}b) \cdot aa^{-1} = bb^{-1} \cdot aa^{-1} \cdot \tau aa^{-1},$$

whence by Definition 1 (i) it follows that $a \in K$.

**Theorem 2.** If $(K, \tau)$ is a pseudo normal congruence pair for $S$, then $\rho_{(K, \tau)}$ is a congruence on $S$ with

$$\ker \rho_{(K, \tau)} = \{a \in K \mid (\exists b \in S), a \geq b, aa^{-1} \tau bb^{-1}, b^{-1} \in K\} \quad (2)$$

and the trace is equal to $\tau$. Moreover, if $(K_1, \tau_1)$ and $(K_2, \tau_2)$ are pseudo congruence pairs for $S$ with $K_1 \subseteq K_2$ and $\tau_1 \subseteq \tau_2$, then $\rho_{(K_1, \tau_1)} \subseteq \rho_{(K_2, \tau_2)}$.

**Proof.** Let $(K, \tau)$ be a pseudo normal congruence pair for $S$ and $\rho = \rho_{(K, \tau)}$. Since $K$ is full it follows that $\rho$ is reflexive. Obviously, $\rho$ is symmetric. We verify that $\rho$ is transitive after we prove that $\rho$ is compatible.

Assume now that $a \rho b$ and let $c \in S$. Then

$$(ac^{-1} \cdot bc, bc \cdot (ac)^{-1}, (bc)^{-1} \cdot ac) \in K \cdot E(S) \subseteq K.$$ 

Similarly,

$$(ac)^{-1} \cdot bc, bc \cdot (ac)^{-1}, (bc)^{-1} \cdot ac \in K.$$ 

Next we have

$$(ac \cdot (ac)^{-1})(bc^{-1} \cdot bc) = (ac \cdot (bc)^{-1})(ac)^{-1} \cdot bc$$

$$= (ac \cdot b^{-1}c^{-1})(a^{-1}c^{-1} \cdot bc)$$

$$= (ab^{-1} \cdot c^{-1})(a^{-1}b^{-1}c^{-1}c)$$

$$= (ab^{-1} \cdot a^{-1}b)(cc^{-1} \cdot cc^{-1})$$

$$= (aa^{-1} \cdot b^{-1}b)cc^{-1} \cdot cc^{-1}$$

$$= ac \cdot a^{-1}c^{-1} = ac \cdot (ac)^{-1}.$$

By symmetry, it follows that

$$(ac \cdot (ac)^{-1})(bc^{-1} \cdot bc) \tau bc \cdot (bc)^{-1},$$

whence $ac \rho bc$. Thus $\rho$ is right compatible. Analogously, $\rho$ is left compatible. Hence, $\rho$ is compatible.
Now, suppose that $a \rho b$ and $b \rho c$. Then by right compatibility $a^{-1} \rho bc^{-1}$ and $bc^{-1} \rho cc^{-1}$. Since $cc^{-1} \in E(S) \subseteq K$ and $bc^{-1} \rho cc^{-1}$, we have $bc^{-1} \in K$ by Lemma 1, and subsequently $ac^{-1} \in K$. Similarly, $aa^{-1} \rho ba^{-1}, ba^{-1} \rho ca^{-1}$ yield $ca^{-1} \in K$ by Lemma 1.

Similarly, by left compatibility, from $a \rho b$ and $b \rho c$ we have $a^{-1} \rho a^{-1} b, a^{-1} b \rho a^{-1} c, c^{-1} \rho a^{-1} b$ and $c^{-1} \rho b^{-1} c$. So by Lemma 1 it follows that $a^{-1} c, c^{-1} a \in K$.

Also $a \rho b, b \rho c$ yields

$$a^{-1} a \cdot bb^{-1} \tau aa^{-1} \tau bb^{-1}, b^{-1} b \cdot cc^{-1} \tau bb^{-1} \tau cc^{-1}$$

and by transitivity it follows that $aa^{-1} \tau cc^{-1}$. Moreover,

$$(bb^{-1} \cdot cc^{-1})(aa^{-1} \cdot cc^{-1}) = (bb^{-1} \cdot aa^{-1})cc^{-1} \tau aa^{-1} \cdot cc^{-1},$$

$$(bb^{-1} \cdot cc^{-1})(aa^{-1} \cdot cc^{-1}) = (bb^{-1} \cdot aa^{-1})cc^{-1} \tau bb^{-1} \cdot cc^{-1} \tau cc^{-1},$$

whence $aa^{-1} \cdot cc^{-1} \tau cc^{-1}$.

Now, $ac^{-1}, a^{-1} c, ca^{-1}, e^{-1} a \in K, aa^{-1} \cdot cc^{-1} \tau aa^{-1} \tau cc^{-1}$ is equivalent to $a \rho c$. Hence, $\rho$ is a transitive relation and so is a congruence.

It is apparent that for $e, f \in E(S), e \rho f$ if and only if $e \tau f$ whence $tr \rho = \tau$.

We let

$$H = \{ a \in K \mid (\exists b \in S) a \geq b, b^{-1} \in K, aa^{-1} \tau bb^{-1} \}$$

and we show that $ker \rho = H$.

Let $a \in H$, then there exists $b \in K$ such that $b \leq a, b^{-1} \in H$ and $aa^{-1} \tau bb^{-1}$. By (1) $b \leq a$ it implies that $b = bb^{-1} \cdot a$. We next prove that $apbb^{-1}$ that is

$$bb^{-1} \cdot a^{-1}, a^{-1} \cdot bb^{-1}, bb^{-1} \cdot a, bb^{-1} \cdot aa^{-1} \tau aa^{-1} \tau bb^{-1}.$$ 

Now $b = bb^{-1} \cdot a \in K$ and $b^{-1} = bb^{-1} \cdot a^{-1} \in K$. Also we have $a \cdot bb^{-1} \in K : E(S) \subseteq K$ and

$$a^{-1} \cdot bb^{-1} = (a^{-1} a \cdot a^{-1}) bb^{-1} = (bb^{-1} \cdot a^{-1}) a^{-1} a \in K : E(S) \subseteq K.$$ 

Conversely, let $a \in ker \rho$. Then $ape$ for some $e \in E(S)$. If $b = ea$, then $b \leq a$ by Corollary 2 and $b = ea \in E(S) \cdot K \subseteq K$. From $ape$ it follows that $aa^{-1} = ea^{-1} = b^{-1}$ and since $aa^{-1} \in K$ we have by Lemma 1 that $b^{-1} \in K$. Because $b, b^{-1} \in K$ we have $bb^{-1} = b^{-1} b \in K$ and so $bpbb^{-1}$. Now

$$bb^{-1} \rho b^{-1} b^{-1} = ea^{-1} \cdot ea^{-1} \rho aa^{-1} \cdot ea^{-1}$$

$$= e(a^{-1} a \cdot a^{-1}) = ea^{-1} \rho aa^{-1}.$$ 

Thus $a \in H$ implies that $ker \rho \subseteq H$, that is $H = ker \rho$. 

$\Box$
Theorem 3. If \((K, \tau)\) is a normal congruence pair for \(S\), then \(\rho_{(K, \tau)}\) is a congruence on \(S\) with kernel \(K\) and trace \(\tau\). Conversely, if \(\rho\) is a congruence on \(S\), then \((\ker \rho, \tr \rho)\) is a normal congruence pair for \(S\) and \(\rho = \rho_{(\ker \rho, \tr \rho)}\).

Proof. Let \((K, \tau)\) be a normal congruence pair and let \(\rho = \rho_{(K, \tau)}\). Then by Theorem 2, \(\rho\) is a congruence with trace equal to \(\tau\) and \(\ker \rho\) as in (2). Thus \(\ker \rho \subseteq K\). Now let \(a \in K\). Then by Definition 1 (ii) there exist \(b \in S\), \(b \leq a\), \(b^{-1} \in K\) and \(bb^{-1} \tau aa^{-1}\) such that \(a \in \ker \rho\) due to Theorem 2. Thus \(K = \ker \rho\).

Conversely, let \(\rho\) be a congruence on \(S\) and let \(K = \ker \rho\), \(\tau = \tr \rho\). Then \(K\) is a full subgroupoid of \(S\) and \(\tau\) is a congruence on \(E(S)\).

Let \(a \in S\), \(b \in K\) and \(a \geq b\). Suppose that \(aa^{-1} \rho bb^{-1}\). Then \(b = bb^{-1} \cdot a\) (by (1)). From \(aa^{-1} \rho bb^{-1}\) it follows that \(a \rho (bb^{-1} a)\) and by above argument we have \(a \rho b\). Hence \(a \in \rho \subseteq \ker \rho = K\). Thus \((i)\) from the Definition 1 holds for \((K, \tau)\) and that it is a pseudo congruence pair for \(S\).

Let \(a \in K\). Then there exists \(e \in E(S)\) with \(a e \rho e\). If \(b = ea\), then \(b \leq a\) by Corollary 2. From \(a e \rho e\) it follows that \(a e \rho e\) whence \(b e \rho e\) and so \(a \rho b\).

Now \(a^{-1} \rho b^{-1}\) by Proposition 1 and so \(aa^{-1} \rho bb^{-1}\). Moreover, from \(a e \rho e\) follows that \(aa^{-1} \rho ea^{-1} = (ea)^{-1} = b^{-1}\), that is \(b^{-1} \in K\). Hence, \((K, \tau)\) is a congruence pair for \(S\).

It remains to prove that \(\rho = \rho_{(K, \tau)}\). Let \(a \rho b\). Then

\[ab^{-1} \rho bb^{-1}, b^{-1} \rho b^{-1} b, aa^{-1} \rho ba^{-1}, a^{-1} \rho a^{-1} b\]

and so \(ab^{-1}, b^{-1} a, ba^{-1}, a^{-1} b \in \ker \rho = K\). Also

\[aa^{-1} \cdot bb^{-1} \rho a^{-1} b \cdot bb^{-1} = (bb^{-1} \cdot b) a^{-1} = ba^{-1} \rho aa^{-1},\]

\[aa^{-1} \cdot bb^{-1} \rho aa^{-1} \cdot ba^{-1} = b(aa^{-1} \cdot a) = ba^{-1} \rho b^{-1} b = bb^{-1},\]

whence it follows that \(a \rho_{(K, \tau)} b\) and so \(\rho \subseteq a \rho_{(K, \tau)}\).

Let \(a \rho_{(K, \tau)} b\). Then \(ab^{-1}, a^{-1} b, ba^{-1}, b^{-1} a \in K\), \(aa^{-1} \cdot bb^{-1} \tau aa^{-1} \tau bb^{-1}\), imply that \(ab^{-1} \rho e, ba^{-1} \rho f\) for some \(e, f \in E(S)\). From \(aa^{-1} \rho bb^{-1}\), it follows that

\[a \rho bb^{-1} \cdot a = ab^{-1} \cdot b \rho eb\]

and

\[b \rho aa^{-1} \cdot b = ba^{-1} \cdot a \rho fa.\]

Also

\[a \rho eb \rho e \cdot f a \rho e(f \cdot eb) = e(e \cdot fb) = ee(e \cdot fb)\]

\[= (fb \cdot e)ee = (fb \cdot e)e = ee \cdot fb = e \cdot fb = f \cdot eb \rho fa \rho b\]

imply that \(a \rho b\), that is \(\rho_{(K, \tau)} \subseteq \rho\). Then \(\rho_{(K, \tau)} = \rho\). \(\square\)
References


Received September 4, 2009

Faculty of Civil Engineering
University of Niš
Aleksandra Medvedeva 14
18000 Niš
Serbia
e-mail: pvprotic@yahoo.mail.com