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$\mathcal{N}$-fuzzy quasi-ideals in ordered semigroups

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Abstract. In this paper, we introduce the concept of $\mathcal{N}$-fuzzy quasi-ideals in ordered semigroups and investigate the basic theorem of quasi-ideals of ordered semigroups in terms of $\mathcal{N}$-fuzzy quasi-ideals. We characterize left (resp. right) regular and completely regular ordered semigroups in terms of $\mathcal{N}$-fuzzy quasi-ideals. We define semiprime $\mathcal{N}$-fuzzy quasi-ideals and characterize completely regular ordered semigroups in terms of semiprime $\mathcal{N}$-fuzzy quasi-ideals. We provide characterizations of some semilattices of left and right simple semigroups in terms of $\mathcal{N}$-fuzzy quasi-ideals.

1. Introduction

A fuzzy subset f of a given set S is described, as an arbitrary function $f : S \longrightarrow [0,1]$, where $[0,1]$ is the usual closed interval of real numbers. This fundamental concept of a fuzzy set was first introduced by Zadeh in his pioneering paper [26] of 1965, provides a natural frame-work for the generalizations of some basic notions of algebra, e.g. logic, set theory, group theory, ring theory, groupoids, real analysis, topology, and differential equations etc. Rosenfeld (see [21]) was the first who considered the case when S is a groupoid. He gave the definition of a fuzzy subgroupoid and the fuzzy left (right, two-sided) ideals of S and justified these definitions by showing that a subset A of a groupoid S is is a subgroupoid or left (right, or two-sided) ideal of S if and only if the characteristic mapping $f_A : S \longrightarrow \{0,1\}$ of A defined by

$$x \longmapsto f_A(x) := \begin{cases} 1 \text{ if } x \in A \\ 0 \text{ if } x \notin A, \end{cases}$$

is respectively, a fuzzy subgroupoid or a fuzzy left (right or two-sided) ideal of S. The concept of a fuzzy ideal in semigroups was first developed by

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Kuroki (see [13-17]). He studied fuzzy ideals, fuzzy bi-ideals, fuzzy quasi-ideals and semiprime fuzzy ideals of semigroups. Fuzzy ideals and Green's relations in semigroups were studied by McLean and Kummer in [18]. Ahsan et. al in [1] characterized semigroups in terms of fuzzy quasi-ideals. A systematic exposition of fuzzy semigroups was given by Mordeson, Malik and Kuroki appeared in [20], where one can find theoretical results on fuzzy semigroups and theoretical results on fuzzy semigroups and their use in fuzzy coding, fuzzy finite state machines and fuzzy languages. The monograph given by Mordeson and Malik [19] deals with the applications of fuzzy approach to the concepts of automata and formal languages. Fuzzy sets in ordered semigroups/ordered groupoids were first introduced by Kehayopulu and Tsingelis in [8]. They also introduced the concepts of fuzzy bi-ideals and fuzzy quasi-ideals in ordered semigroups in (see [9]).

The quasi-ideals in rings and semigroups were studied by Stienfeld in [25] and Kehayopulu extended this concept in ordered semigroups and defined a quasi-ideal of an ordered semigroup S as a non-empty subset Q of S such that

$$(1)\ (QS] \cap (SQ] \subseteq Q \quad \text{and} \quad (2)\ \text{If } a \in Q \text{ and } S \ni b \leqslant a \text{ then } b \in Q.$$

The purpose of this paper, is to initiate and study the new sort of fuzzy quasi-ideals called $\mathcal{N}$-fuzzy quasi-ideals in ordered semigroups. We characterize regular, left and right simple ordered semigroups and completely regular ordered semigroups in terms of $\mathcal{N}$-fuzzy quasi-ideals. In this respect, we prove that: An ordered semigroup S is regular, left and right simple if and only if every $\mathcal{N}$-fuzzy quasi-ideal of S is a constant $\mathcal{N}$-function. We also prove that S is completely regular if and only if for every $\mathcal{N}$-fuzzy quasi-ideal f of S we have $f(a) = f(a^2)$ for every $a \in S$. We define semiprime $\mathcal{N}$-fuzzy quasi-ideal in ordered semigroups and prove that an ordered semigroup S is completely regular if and only if every $\mathcal{N}$-fuzzy quasi-ideal f of S is semiprime. Next, we characterize semilattices of left and right simple ordered semigroup in terms of $\mathcal{N}$-fuzzy quasi-ideals. We prove that an ordered semigroup S is a semilattice of left and right simple if and only if for every $\mathcal{N}$-fuzzy quasi-ideal f of S, we have $f(a) = f(a^2)$ and $f(ab) = f(ba)$ for every $a, b \in S$. In the last of this paper, we discuss ordered semigroups having the property $a \leqslant a^2$ for all $a \in S$ and prove that an ordered semigroup S (having the property $a \leqslant a^2$ for all $a \in S$) is a semilattice of left and right simple ordered semigroup if and only if for every $\mathcal{N}$-fuzzy quasi-ideal f of S we have $f(ab) = f(ba)$ for all $a, b \in S$.

2. Some basic definitions and results

By an *ordered semigroup* (or *po-semigroup*) we mean a *structure* $(S, \cdot, \leqslant)$ in which

$(OS1)$ $(S, \cdot)$ is a *semigroup*,

$(OS2)$ $(S, \leqslant)$ is a *poset*,

$(OS3)$ $(\forall a, b, x \in S)(a \leqslant b \Longrightarrow ax \leqslant bx$ and $xa \leqslant xb)$.

Let $(S, \cdot, \leqslant)$ be an ordered semigroup. For $A \subseteq S$, we denote

$$(A] := \{t \in S \mid t \leqslant h \text{ for some } h \in A\} \quad \text{and} \quad AB := \{ab \mid a \in A, b \in B\}.$$

A non-empty subset Q of S is called a *quasi-ideal* (see [8]) of S if:

(1) $(QS] \cap (SQ] \subseteq Q$ and (2) If $a \in Q$ and $S \ni b \leqslant a$ then $b \in Q$.

Let $A, B \subseteq S$. Then $A \subseteq (A]$, $(A](B] \subseteq (AB]$, $((A]] = (A]$ and $((A](B]] \subseteq (AB]$ (see [8]).

$\emptyset = A \subseteq S$ is called a *subsemigroup* of S if $A^2 \subseteq A$, and a *right* (resp. *left*) *ideal* of S if (1) $AS \subseteq A$ (resp. $SA \subseteq A$) and (2) $a \in A$, $S \ni b \leqslant a$ imply $b \in A$. If A is both a right and a left ideal of S, then it is called an *ideal.* A subsemigroup B of S is called a *bi-ideal* of S if: (1) $BSB \subseteq B$ and (2) $a \in B$, $S \ni b \leqslant a$ imply $b \in B$.

By a *negative fuzzy subset* (briefly, an $\mathcal{N}$*-fuzzy subset*) of S we mean a *function* $f : S \longrightarrow [-1, 0]$. An $\mathcal{N}$-fuzzy subset f of S is called an $\mathcal{N}$*-fuzzy left* (resp. *right*) *ideal* of S if:

(1) $x \leqslant y \Longrightarrow f(x) \leqslant f(y)$ and (2) $f(xy) \leqslant f(y)$ (resp. $f(xy) \leqslant f(x)$) for all $x, y \in S$.

If f is both an $\mathcal{N}$-fuzzy left and an $\mathcal{N}$-fuzzy right ideal of S. Then it is called an $\mathcal{N}$*-fuzzy ideal* of S.

For a non-empty *family* of $\mathcal{N}$-fuzzy subsets $\{f_i\}_{i \in I}$, of an ordered semigroup S, the $\mathcal{N}$-fuzzy subsets $\bigwedge\limits_{i \in I} f_i$ and $\bigvee\limits_{i \in I} f_i$ of S are defined as follows:

$$\Big(\bigwedge_{i \in I} f_i\Big)(x) := \inf_{i \in I}\{f_i(x)\}, \qquad \Big(\bigvee_{i \in I} f_i\Big)(x) := \sup_{i \in I}\{f_i(x)\}.$$

The *characterisitic* $\mathcal{N}$*-function* κ_A of $\emptyset \neq A \subseteq S$ is given by:

$$\kappa_A(x) := \begin{cases} -1 \text{ if } x \in A, \\ \ \ 0 \text{ if } x \notin A. \end{cases}$$

For $\mathcal{N}$-fuzzy subsets f and g of S we define the N-composition fNg by

$$fNg(x) := \begin{cases} \bigwedge\limits_{(y,z) \in A_x} \max\{f(y), g(z)\} & \text{if } A_x \neq \emptyset, \\ 0 & \text{if } A_x = \emptyset, \end{cases}$$

where

$$A_x := \{(y,z) \in S \times S \mid x \leqslant yz\}.$$

The set $NF(S)$ of all $\mathcal{N}$-fuzzy subsets of $S$with such defined N-composition and the relation

$$f \preceq g \text{ if and only if } f(x) \leqslant g(x) \text{ for all } x \in S$$

is an ordered semigroup denoted by $(NF(S), N, \preceq)$. The fuzzy subsets $\beta(x) = 0$ and $\alpha(x) = -1$ (for all $x \in S$) the *greatest* and *least element* of $(NF(S), \preceq)$. The fuzzy subset β is the *zero element* of $(NF(S), N, \preceq)$ (that is, $fN\beta = \beta Nf = \beta$ and $f \preceq \beta$ for every $f \in NF(S)$). Obviously, $f_S = \alpha$ and $f_\emptyset = \beta$.

Definition 2.1. (cf. [11]). Let S be an ordered semigroup. An $\mathcal{N}$-fuzzy subset f of S is called an *$\mathcal{N}$-fuzzy subsemigroup* of S if

$$f(xy) \leqslant \max\{f(x), f(y)\} \text{ for all } x, y \in S.$$

Definition 2.2. (cf. [11]). Let S be an ordered semigroup. An $\mathcal{N}$-fuzzy subsemigroup f of S is called an *$\mathcal{N}$-fuzzy bi-ideal* of S if:

(1) $x \leqslant y$ implies $f(x) \leqslant f(y)$.

(2) $f(xay) \leqslant \max\{f(x), f(y)\}$ for all $x, a, y \in S$.

Proposition 2.3. (cf. [11]). *Let S be an ordered semigroup and $A, B \subseteq S$. Then*

(a) *$B \subseteq A$ if and only if $\kappa_B \preceq \kappa_A$.*

(b) *$\kappa_A \vee \kappa_B = \kappa_{A\cap B}$.*

(c) *$\kappa_A N \kappa_B = \kappa_{(AB]}$.*

Lemma 2.4. (cf. [22]). *Let S be an ordered semigroup. Then every quasi-ideal of S is subsemigroup of S.*

Lemma 2.5. (cf. [24]). *An ordered semigroup $(S, \cdot, \leqslant)$ is a semilattice of left and right simple semigroups if and only if for all quasi-ideals A, B of S, we have*

$$(A^2] = A \text{ and } (AB] = (BA].$$

An ordered semigroup S is called *regular* (see [5]) if for every $a \in S$ there exists $x \in S$ such that $a \leqslant axa$ or equivalently, (1) $(\forall a \in S)(a \in (aSa])$ and (2) $(\forall A \subseteq S)(A \subseteq (ASA])$.

An ordered semigroup S is called *left* (resp. *right*) *simple* (see [9]) if for every left (resp. right) ideal A of S, we have $A = S$.

Lemma 2.6. (cf. [9, Lemma 3]). *An ordered semigroup S is left (resp. right) simple if and only if $(Sa] = S$ (resp. $(aS] = S$) for every $a \in S$.*

An ordered semigroup S is called *left* (resp. *right) regular* (see [4]) if for every $a \in S$ there exists $x \in S$ such that $a \leqslant xa^2$ (resp. $a \leqslant a^2x$) or equivalently, (1) $(\forall a \in S)(a \in (Sa^2])$ and (2) $(\forall A \subseteq S)(A \subseteq (SA^2])$. An ordered semigroup S is called *completely regular* if it is regular, left regular and right regular [6].

If S is an ordered semigroup and $\emptyset \neq A \subseteq S$, then the set $(A \cup (AS \cap SA)]$ is the quasi-ideal of S generated by A. If $A = \{x\}$ $(x \in S)$, we write $(x \cup (xS \cap Sx)]$ instead of $(\{x\} \cup (\{x\}S \cap S\{x\})]$.

Lemma 2.7. (cf. [6]). *An ordered semigroup S is completely regular if and only if $A \subseteq (A^2SA^2]$ for every $A \subseteq S$. Equivalently, if $a \in (a^2Sa^2]$ for every $a \in S$.*

Lemma 2.8. (cf. [24]). *An ordered semigroup $(S, \cdot, \leqslant)$ is a semilattice of left and right simple semigroups if and only if for every right ideal A and every left ideal B of S, we have $A \cap B = (AB]$.*

Lemma 2.9. *Let $(S, \cdot, \leqslant)$ be an ordered semigroup and let A, B be quasi-ideals of S. Then $(AB]$ is a bi-ideal of S.*

3. $\mathcal{N}$-fuzzy quasi-ideals

In this section we prove the basic theorem which characterizes ordered semigroup in terms of $\mathcal{N}$-fuzzy quasi-ideals.

Definition 3.1. Let $(S, \cdot, \leqslant)$ be an ordered semigroup. An $\mathcal{N}$-fuzzy subset f of S is called a *$\mathcal{N}$-fuzzy quasi-ideal* of S if

(1) $(fN\alpha) \vee (\alpha N f) \succeq f$.

(2) $x \leqslant y$, then $f(x) \leqslant f(y)$ for all $x, y \in S$.

The set
$$L(f;t) := \{x \in S | f(x) \leqslant t\}$$
is called a *level subset* of f.

Theorem 3.2. (cf. [11]). *Let S be an ordered semigroup and f an $\mathcal{N}$-fuzzy subset of S. Then $\forall t \in [-1, 0)$, $L(f;t) (\neq \emptyset)$ is a bi-ideal if and only if f is an $\mathcal{N}$-fuzzy bi-ideal.*

Theorem 3.3. *Let $(S, \cdot, \leqslant)$ be an ordered semigroups and $\emptyset \neq A \subseteq S$. Then A is a quasi-ideal of S if and only if the characteristic $\mathcal{N}$-function κ_A of A is an $\mathcal{N}$-fuzzy quasi-ideal of S.*

Proof. Suppose that A is a quasi-ideal of S. Then $(\kappa_A N\alpha)\vee(\alpha N\kappa_A)\succeq\kappa_A$. Indeed:

$$(\kappa_A N\alpha)\vee(\alpha N\kappa_A)=(\kappa_A N\kappa_S)\vee(\kappa_S N\kappa_A)=\kappa_{(AS]}\vee\kappa_{(SA]}=\kappa_{(AS]\cap(SA]}$$

by Proposition 2.3.

Since $(AS]\cap(SA]\subseteq A$, then by Proposition 2.3, we have $\kappa_{(AS]\cap(SA]}\succeq\kappa_A$. Thus $(\kappa_A N\alpha)\vee(\alpha N\kappa_A)\succeq\kappa_A$. Let $x,y\in S$ be such that $x\leqslant y$. If $\kappa_A(y)=0$. Then $\kappa_A(x)\leqslant\kappa_A(y)$, because $\kappa_A(x)\leqslant 0\ \forall x\in S$. If $\kappa_A(y)=-1$, then $y\in A$. Since $S\ni x\leqslant y\in A$, we have $x\in A$, then $\kappa_A(x)=-1$ and hence $\kappa_A(x)\leqslant\kappa_A(y)$.

Conversely, assume that κ_A is an $\mathcal{N}$-fuzzy quasi-ideal of S. Then A is a quasi-ideal of S since $(\kappa_A N\alpha)\vee(\alpha N\kappa_A)\succeq\kappa_A$ implies $(\kappa_A N\kappa_S)\vee(\kappa_S N\kappa_A)\succeq\kappa_A$. By Proposition 2.3, $\kappa_S N\kappa_A=\kappa_{(SA]}$ and $\kappa_A N\kappa_S=\kappa_{(AS]}$, then $\kappa_{(AS]}\vee\kappa_{(SA]}=\kappa_{(AS]\cap(SA]}$ and we have $\kappa_{(AS]\cap(SA]}\succeq\kappa_A$. Thus $(AS]\cap(SA]\subseteq A$.

If $x\in A$ and $S\ni y\leqslant x$. Since $y\leqslant x$ and κ_A is an $\mathcal{N}$-fuzzy quasi-ideal of S, we have $\kappa_A(y)\leqslant\kappa_A(x)$. Since $x\in A$ then $\kappa_A(x)=-1$, and hence $\kappa_A(y)=-1$, i.e., $y\in A$. □

Theorem 3.4. *Let S be an ordered semigroup and f an $\mathcal{N}$-fuzzy subset of S. Then $\forall t\in[-1,0)$, $L(f;t)(\neq\emptyset)$ is a quasi-ideal if and only if f is an $\mathcal{N}$-fuzzy quasi-ideal.*

Proof. Assume that f is an $\mathcal{N}$-fuzzy quasi-ideal of S. Let $x,y\in S$ be such that $x\leqslant y$. If $y\in L(f;t)$, then $f(y)\leqslant t$. Since $x\leqslant y$ and $f(x)\leqslant f(y)\leqslant t$ then $x\in L(f;t)$. Let $a\in S$ be such that $a\in(L(f;t)S]\cap(SL(f;t)]$ then $a\in(L(f;t)S]$ and $a\in(SL(f;t)]$. Then $a\leqslant xy$ and $a\leqslant x'y'$ for some $x,y'\in L(f;t)$ and $x',y\in S$ and so $(x,y)\in A_a$ and $(x',y')\in A_a$. Since $A_a\neq\emptyset$, we have

$$\begin{aligned}
f(a)&\leqslant((fN\alpha)\vee(\alpha Nf))(a)=\max\left[(fN\alpha)(a),(\alpha Nf)(a)\right]\\
&=\max\Big[\bigwedge_{(p,q)\in A_a}\max\{f(p),\alpha(q)\},\ \bigwedge_{(p',q')\in A_a}\max\{\alpha(p'),f(q')\}\Big]\\
&\leqslant\max\left[\max\{f(a),\alpha(b)\},\max\{\alpha(a'),f(b')\}\right]\\
&=\max\left[\max\{f(x),-1\},\max\{-1,f(y')\}\right]=\max\left[f(x),f(y')\right].
\end{aligned}$$

Since $x,y'\in L(f;t)$ we have $f(x)\leqslant t$ and $f(y')\leqslant t$. Then

$$f(a)\leqslant\max\left[f(x),f(y')\right]\leqslant t,$$

so $a \in L(f;t)$. Thus $(L(f;t)S] \cap (SL(f;t)] \subseteq L(f;t)$.

Conversely. Assume that for all $t \in [-1,0], L(f;t) \neq \emptyset$ is a quasi-ideal of S. Let $x, y \in S$ be such that $x \leqslant y$. If $f(x) > f(y)$, then $\exists t \in [-1,0]$ such that $f(x) > t \geqslant f(y)$ then $y \in L(f;t)$ but $x \notin L(f;t)$. This is a contradiction. Thus $f(x) \leqslant f(y)$ for all $x \leqslant y$.

Let $x \in S$ be such that $f(x) > ((fN\alpha) \vee (\alpha Nf))(x)$, then $\exists t \in [-1,0]$ such that $f(x) > t \geqslant ((fN\alpha) \vee (\alpha Nf))(x) = \max[(fN\alpha)(x), (\alpha Nf)(x)]$.

Then $(fN\alpha)(x) \leqslant t$ and $(\alpha Nf)(x) \leqslant t$ and hence $x \in (L(f;t)S]$ and $x \in (SL(f;t)]$, so $x \in (L(f;t)S] \cap (SL(f;t)]$. But $(L(f;t)S] \cap (SL(f;t)] \subseteq L(f;t)$, hence $x \in L(f;t)$, i.e., $f(x) \leqslant t$. This is a contradiction. Thus $f(x) \leqslant ((fN\alpha) \vee (\alpha Nf))(x)$. □

Example 3.5. Let $S = \{a, b, c, d, f\}$ be an ordered semigroup with the following multiplication

$\cdot$	a	b	c	d	f
a	a	a	a	a	a
b	a	b	a	d	a
c	a	f	c	c	f
d	a	b	d	d	b
f	a	f	a	c	a

and let $\leqslant := \{(a,a), (a,b), (a,c), (a,d), (a,f), (b,b), (c,c), (d,d), (f,f)\}$.

The quasi-ideals of S are: $\{a\}$, $\{a,b\}$, $\{a,c\}$, $\{a,d\}$, $\{a,f\}$, $\{a,b,d\}$, $\{a,c,d\}$, $\{a,b,f\}, \{a,c,f\}$ and S (see [5]). Define $f : S \longrightarrow [-1,0]$ by $f(a) = -0.8$, $f(b) = -0.6$, $f(d) = -0.5$, $f(c) = f(f) = -0.3$. Then

$$L(f;t) := \begin{cases} S & \text{if} \quad t \in [-0.3, 0], \\ \{a,b,d\} & \text{if} \quad t \in [-0.5, -0.3), \\ \{a,b\} & \text{if} \quad t \in [-0.6, -0.5), \\ \{a\} & \text{if} \quad t \in [-0.8, -0.6), \\ \emptyset & \text{if} \quad t \in [-1, -0.8). \end{cases}$$

is a quasi-ideal and by Theorem 3.4, f is an $\mathcal{N}$-fuzzy quasi-ideal of S.

Lemma 3.6. *Every $\mathcal{N}$-fuzzy quasi-ideal is an $\mathcal{N}$-fuzzy bi-ideal.*

Proof. . Let $x, y, z \in S$. Then $xyz = x(yz) = (xy)z$ and $(x, yz) \in A_{xyz}$ and $(xy, z) \in A_{xyz}$. Since $A_{xyz} \neq \emptyset$ then

$$\begin{aligned} f(xyz) &\leqslant [(fN\alpha) \vee (\alpha Nf)](xyz) \\ &= \max\Big[\bigwedge_{(p,q)\in A_{xyz}} \max\{f(p), \alpha(q)\}, \bigwedge_{(p_1,q_1)\in A_{xyz}} \max\{\alpha(p_1), f(q_1)\}\Big] \end{aligned}$$

$$\leqslant \max[\max\{f(x), \alpha(yz)\}, \max\{\alpha(xy), f(z)\}]$$
$$= \max[\max\{f(x), -1\}, \max\{-1, f(z)\}] = \max[f(x), f(z)].$$

If $x, y \in S$, then $xy = x(y)$ and hence $(x, y) \in A_{xy}$. Since $A_{xy} \neq \emptyset$, we have

$$\begin{aligned} f(xy) &\leqslant [(fN\alpha) \vee (\alpha Nf)](xy) \\ &= \max\Big[\bigwedge_{(p,q)\in A_{xy}} \max\{f(p), \alpha(q)\}, \bigwedge_{(p,q)\in A_{xy}} \max\{\alpha(p), f(q)\}\Big] \\ &\leqslant \max[\max\{f(x), \alpha(y)\}, \max\{\alpha(x), f(y)\}] \\ &= \max[\max\{f(x), -1\}, \max\{-1, f(y)\}] = \max[f(x), f(y)]. \end{aligned}$$

Let $x, y \in S$ be such that $x \leqslant y$. Then $f(x) \leqslant f(y)$, because f is an $\mathcal{N}$-fuzzy quasi-ideal of S. Thus f is an $\mathcal{N}$-fuzzy bi-ideal of S. □

The converse of above Lemma is not true, in general.

Example 3.7. Consider the semigroup $S = \{a, b, c, d\}$

$\cdot$	a	b	c	d
a	a	a	a	a
b	a	a	a	a
c	a	a	b	a
d	a	a	b	b

with the order $\leqslant := \{(a, a), (b, b), (c, c), (d, d), (a, b)\}$.

Then $\{a, d\}$ is a bi-ideal but not a quasi-ideal of S. For an $\mathcal{N}$-fuzzy set f defined by $f(a) = f(d) = -0.3$, $f(b) = f(c) = -0.7$ we have

$$L(f;t) := \begin{cases} S & \text{if} \quad t \in [-0.3, 0], \\ \{a, d\} & \text{if} \quad t \in [-0.7, -0.3), \\ \emptyset & \text{if} \quad t \in [-0.7, -1). \end{cases}$$

Then $L(f;t)$ is a bi-ideal of S and by Theorem 3.4, f is an $\mathcal{N}$-fuzzy bi-ideal of S. Moreover, $L(f;t)$ is a bi-ideal of S but not a quasi-ideal for all $t \in [-0.7, -0.3)$ and by Theorem 3.4, f is not an $\mathcal{N}$-fuzzy quasi-ideal of S.

Proposition 3.8. *If $(S, \cdot, \leqslant)$ is an ordered semigroup and f_1, f_2, g_1, g_2 are $\mathcal{N}$-fuzzy subsets of S such that $g_1 \preceq f_1$ and $g_2 \preceq f_2$, then $g_1Ng_2 \preceq f_1Nf_2$.*

Proof. Let $a \in S$. If $A_a = \emptyset$ then $f_1Nf_2(a) = 0 \geqslant g_1Ng_2(a)$. If $A_a \neq \emptyset$ then

$$f_1Nf_2(a) = \bigwedge_{(y,z)\in A_a} \max\{f_1(y), f_2(z)\} \geqslant \bigwedge_{(y,z)\in A_a} \max\{g_1(y), g_2(z)\} = g_1Ng_2(a),$$

which completes the proof. □

From the above Proposition we see that the set of all $\mathcal{N}$-fuzzy subsets of an ordered semigroup is a complete lattice.

4. Characterizations of regular ordered semigroups

In this section, we prove that an ordered semigroup S is regular, left and right simple if and only if every $\mathcal{N}$-fuzzy quasi-ideal f of S is a constant $\mathcal{N}$-function. We define semiprime $\mathcal{N}$-fuzzy quasi-ideals of ordered semigroups and prove that an ordered semigroup $(S, \cdot, \leqslant)$ is completely regular if and only if every $\mathcal{N}$-fuzzy quasi-ideal f of S is a semiprime $\mathcal{N}$-fuzzy quasi-ideal of S.

Theorem 4.1. *For an ordered semigroup S the following are equivalent:*
(i) *S is regular, left and right simple.*
(ii) *Every $\mathcal{N}$-fuzzy quasi-ideal of S is a constant $\mathcal{N}$-function.*

Proof. (i) $\Rightarrow$ (ii). Let S be regular, left and right simple ordered semigroup. Let f be an $\mathcal{N}$-fuzzy quasi-ideal of S. We consider the set $E_\Omega := \{e \in S | e^2 \geqslant e\}$. Then E_Ω is non-empty. In fact, i $a \in S$, since S is regular, then there exists $x \in S$ such that $a \leqslant axa$. We consider the element ax of S. Then $(ax)^2 = (axa)x \geqslant ax$, and so $ax \in E_\Omega$.

(A) We first prove that f is a constant $\mathcal{N}$-function on E_Ω. That is, $f(e) = f(t)$ for every $t \in E_\Omega$. In fact: Since S is left and right simple, we have $(St] = S$ and $(tS] = S$. Since $e \in S$ then $e \in (St]$ and $e \in (tS]$. Then $e \leqslant xt$ and $e \leqslant ty$ for some $x, y \in S$. If $e \leqslant xt$ then $e^2 = ee \leqslant (xt)(xt) = (xtx)t$ and $(xtx, t) \in A_{e^2}$. If $e \leqslant ty$ then $e^2 = ee \leqslant (ty)(ty) = t(yty)$ and $(t, yty) \in A_{e^2}$. Since $A_{e^2} \neq \emptyset$, and f is an $\mathcal{N}$-fuzzy quasi-ideal of S, we have

$$\begin{aligned}
f(e^2) &\leqslant ((fN\alpha) \vee (\alpha Nf))(e^2) = \max[(fN\alpha)(e^2), (\alpha Nf)(e^2)] \\
&= \max\Big[\bigwedge_{(y_1,z_1)\in A_{e^2}} \max\{f(y_1), \alpha(z_1)\}, \bigwedge_{(y_2,z_2)\in A_{e^2}} \max\{\alpha(z_2), f(y_2)\}\Big] \\
&\leqslant \max[\max\{f(t), \alpha(yty)\}, \max\{f(t), \alpha(xtx)\}] \\
&= \max[\max\{f(t), -1\}, \max\{f(t), -1\}] = \max[f(t), f(t)] = f(t).
\end{aligned}$$

Since $e \in E_\Omega$, we have $e^2 \geqslant e$ and $f(e^2) \geqslant f(e)$. Thus $f(e) \leqslant f(t)$. On the other hand since S is left and right simple and $e \in S$, we have $S = (Se]$ and $S = (eS]$. Since $t \in S$ we have $t \in (Se]$ and $t \in (eS]$. Then $t \leqslant ze$ and $t \leqslant es$ for some $z, s \in S$. If $t \leqslant ze$ then $t^2 = tt \leqslant (ze)(ze) = (zez)e$ and $(zez, e) \in A_{t^2}$. If $t \leqslant es$ then $t^2 = tt \leqslant (es)(es) = e(ses)$ and $(e, ses) \in A_{t^2}$. Since $A_{t^2} \neq \emptyset$, we have

$$\begin{aligned}
f(t^2) &\leqslant ((fN\alpha) \vee (\alpha Nf))(t^2) = \max[(fN\alpha)(t^2), (\alpha Nf)(t^2)] \\
&= \max\Big[\bigwedge_{(p_1,q_1)\in A_{t^2}} \max\{f(p_1), \alpha(q_1)\}, \bigwedge_{(p_2,q_2)\in A_{t^2}} \max\{\alpha(p_2), f(q_2)\}\Big] \\
&\leqslant \max[\max\{f(e), \alpha(xex)\}, \max\{\alpha(yey), f(e)\}] \\
&= \max[\max\{f(e), -1\}, \max\{-1, f(e)\}] = \max[f(e), f(e)] = f(e).
\end{aligned}$$

Since $t \in E_\Omega$ then $t^2 \geqslant t$ and $f(t^2) \geqslant f(t)$. Thus $f(t) \leqslant f(e)$. Consequently, $f(t) = f(e)$.

(B). Now, we prove that f is a constant $\mathcal{N}$-function on S. That is, $f(t) = f(a)$ for every $a \in S$. Since S is regular and $a \in S$, there exists $x \in S$ such that $a \leqslant axa$. We consider the elements ax and xa of S. Then by $(OS3)$, we have

$$(ax)^2 = (axa)x \geqslant ax \text{ and } (xa) = x(axa) \geqslant xa,$$

then ax, $xa \in E_\Omega$ and by (A) we have $f(ax) = f(t)$ and $f(xa) = f(t)$. Since

$$(ax)(axa) \geqslant axa \geqslant a,$$

then $(ax, axa) \in A_a$ and $(axa)(xa) \geqslant axa \geqslant a$, then $(axa, xa) \in A_a$ and hence $A_a \neq \emptyset$. Since f is an $\mathcal{N}$-fuzzy quasi-ideal of S, we have

$$\begin{aligned}
f(a) &\leqslant ((fN\alpha) \vee (\alpha N f))(a) = \max[(fN\alpha)(a), (\alpha N f)(a)] \\
&= \max\Big[\bigwedge_{(y_1,z_1)\in A_a} \max\{f(y_1), \alpha(z_1)\}, \bigwedge_{(y_2,z_2)\in A_a} \max\{\alpha(y_2), f(z_2)\} \Big] \\
&\leqslant \max[\max\{f(ax), \alpha(axa)\}, \max\{\alpha(axa), f(xa)\}] \\
&= \max[\max\{f(ax), -1\}, \max\{-1, f(xa)\}] \\
&= \max[f(ax), f(xa)] = \max[f(t), f(t)] = f(t).
\end{aligned}$$

Since S is left and right simple we have $(Sa] = S$, and $(aS] = S$. Since $t \in S$, we have $t \in (Sa]$ and $t \in (aS]$. Then $t \leqslant pa$ and $t \leqslant aq$ for some $p, q \in S$. Then $(p, a) \in A_t$ and $(a, q) \in A_t$. Since $A_t \neq \emptyset$ and f is an $\mathcal{N}$-fuzzy quasi-ideal of S, we have

$$\begin{aligned}
f(t) &\leqslant ((fN\alpha) \vee (\alpha N f))(t) = \max[(fN\alpha)(t), (\alpha N f)(t)] \\
&= \max\Big[\bigwedge_{(y_1,z_1)\in A_t} \max\{f(y_1), \alpha(z_1)\}, \bigwedge_{(y_2,z_2)\in A_t} \max\{\alpha(y_2), f(z_2)\}\Big] \\
&\leqslant \max[\max\{f(a), \alpha(p)\}, \max\{\alpha(q), f(a)\}] \\
&= \max[\max\{f(a), -1\}, \max\{-1, f(a)\}] = \max[f(a), f(a)] = f(a).
\end{aligned}$$

Thus $f(t) \leqslant f(a)$ and $f(t) = f(a)$.

$(ii) \Rightarrow (i)$. Let $a \in S$. Then the set $(aS]$ is a quasi-ideal of S. Indeed, (a) $(aS] \cap (Sa] \subseteq (aS]$ and (b) If $x \in (aS]$ and $S \ni y \leqslant x \in (aS]$, then $y \in ((aS]] = (aS]$. Since $(aS]$ is quasi-ideal of S, by Theorem 3.3, the characteristic $\mathcal{N}$-function $\kappa_{(aS]}$ of $(aS]$ is an $\mathcal{N}$-fuzzy quasi-ideal of S. By hypothesis, $\kappa_{(aS]}$ is a constant $\mathcal{N}$-function, so $\kappa_{(aS]}(x) = -1$ or $\kappa_{(aS]}(x) = -1$ for every $x \in S$.

Let $(aS] \subset S$ and a be an element of S such that $a \notin (aS]$, then $\kappa_{(aS]}(x) = 0$. On the other hand, since $a^2 \in (aS]$ then $\kappa_{(aS]}(a^2) = -1$. A contradiction to the fact that $\kappa_{(aS]}$ is a constant $\mathcal{N}$-function. Thus $(aS] = S$. By symmetry we can prove that $(Sa] = S$.

Since $a \in S$ and $S = (aS] = (Sa]$, we have $a \in (aS] = (a(Sa]] \subseteq (aSa]$, hence S is regular. $\square$

Theorem 4.2. *An ordered semigroup $(S, \cdot, \leqslant)$ is completely regular if and only if for every $\mathcal{N}$-fuzzy quasi-ideal f of S we have $f(a) = f(a^2)$ for every $a \in S$.*

Proof. Let S be a completely regular ordered semigroup and f an $\mathcal{N}$-fuzzy quasi-ideal of S. Since S is left and right regular we have $a \in (Sa^2]$ and $a^2 \in (a^2S]$ for every $a \in S$. Then there exists $x, y \in S$ such that $a \leqslant xa^2$ and $a \leqslant a^2y$. Then (x, a^2), $(a^2, y) \in A_a$. Since $A_a \neq \emptyset$, we have

$$\begin{aligned} f(a) &\leqslant ((fN\alpha) \vee (\alpha N f))(a) = \max[(fN\alpha)(a), (\alpha N f)(a)] \\ &= \max\Big[\bigwedge_{(y,z)\in A_a} \max\{f(y), \alpha(z)\}, \bigwedge_{(y,z)\in A_a} \max\{\alpha(y), f(z)\}\Big] \\ &\leqslant \max[\max\{f(a^2), \alpha(y)\}, \max\{\alpha(x), f(a^2)\}] \\ &= \max[\max\{f(a^2), -1\}, \max\{-1, f(a^2)\}] \\ &= \max[f(a^2), f(a^2)] = f(a^2) = f(aa) \leqslant \max\{f(a), f(a)\} = f(a). \end{aligned}$$

Thus $f(a) = f(a^2)$.

Conversely, let $a \in S$. We consider the quasi-ideal $Q(a^2)$ generated by a^2 $(a \in S)$. That is, the set $Q(a^2) = (a^2 \cup (a^2S \cap Sa^2)]$. By Theorem 3.3, the characteristic $\mathcal{N}$ -function $\kappa_{Q(a^2)}$ is an $\mathcal{N}$-fuzzy quasi-ideal of S. By hypothesis

$$\kappa_{Q(a^2)}(a) = \kappa_{Q(a^2)}(a^2).$$

Since $a^2 \in Q(a^2)$, we have $\kappa_{Q(a^2)}(a^2) = -1$ then $\kappa_{Q(a^2)}(a) = -1$ and $a \in Q(a^2) = (a^2 \cup (a^2S \cap Sa^2)]$. Then $a \leqslant a^2$ or $a \leqslant a^2x$ and $a \leqslant ya^2$ for some $x, y \in S$. If $a \leqslant a^2$ then $a \leqslant a^2 = aa \leqslant a^2a^2 = aaa^2 \leqslant a^2aa^2 \in a^2Sa^2$ and $a \in (a^2Sa^2]$. If $a \leqslant a^2x$ and $a \leqslant ya^2$ then $a \leqslant (a^2x)(ya^2) = a^2(xy)a^2 \in a^2Sa^2$ and $a \in (a^2Sa^2]$. □

A subset T of an ordered semigroup S is called *semiprime* if for every $a \in S$ from $a^2 \in T$ it follows $a \in T$.

Definition 4.3. An $\mathcal{N}$-fuzzy subset f of an ordered semigroup $(S, \cdot, \leqslant)$ is called *semiprime* if $f(a) \leqslant f(a^2)$ for all $a \in S$.

Theorem 4.4. *An ordered semigroup $(S, \cdot, \leqslant)$ is completely regular if and only if every $\mathcal{N}$-fuzzy quasi-ideal f of S is semiprime.*

Proof. Let S be a completely regular ordered semigroup and f an $\mathcal{N}$-fuzzy quasi-ideal of S. Let $a \in S$. Then $f(a) \leqslant f(a^2)$. Indeed, since S is left and right regular, there exist x, $y \in S$ such that $a \leqslant xa^2$ and $a \leqslant a^2y$ then $(x, a^2) \in A_a$ and $(a^2, y) \in A_a$. Since $A_a \neq \emptyset$, then we have

$$\begin{aligned} f(a) &\leqslant ((fN\alpha) \vee (\alpha N f))(a) = \max[(fN\alpha)(a), (\alpha N f)(a)] \\ &= \max\Big[\bigwedge_{(y,z)\in A_a} \max\{f(y), \alpha(z)\}, \bigwedge_{(y,z)\in A_a} \max\{\alpha(y), f(z)\}\Big] \end{aligned}$$

$$\leqslant \max\left[\max\{f(a^2), \alpha(y)\}, \max\{\alpha(x), f(a^2)\}\right]$$
$$= \max\left[\max\{f(a^2), -1\}, \max\{-1, f(a^2)\}\right] = \max\left[f(a^2), f(a^2)\right] = f(a^2).$$

To prove the converse, let f be an $\mathcal{N}$-fuzzy quasi-ideal of S such that $f(a) \leqslant f(a^2)$ for all $a \in S$. We consider the quasi-ideal $Q(a^2)$ generated by $a^2 (a \in S)$. That is, the set $Q(a^2) = (a^2 \cup (a^2 S \cap Sa^2)]$. Then by Theorem 3.3, $\kappa_{Q(a^2)}$ is an $\mathcal{N}$-fuzzy quasi-ideal of S. By hypothesis $\kappa_{Q(a^2)}(a) \leqslant \kappa_{Q(a^2)}(a^2)$. Since $a^2 \in Q(a^2)$, we have $\kappa_{Q(a^2)}(a^2) = -1$ and $\kappa_{Q(a^2)}(a) = -1$ we get $a \in Q(a^2)$. Then $a \leqslant a^2$ or $a \leqslant a^2 p$ and $a \leqslant qa^2$ for some $p, q \in S$. If $a \leqslant a^2$ then

$$a \leqslant a^2 = aa \leqslant a^2 a^2 = aaa^2 \leqslant a^2 aa^2 \in a^2 Sa^2 \text{ and } a \in (a^2 Sa^2].$$

If $a \leqslant a^2 p$ and $a \leqslant qa^2$. Then $a \leqslant (a^2 p)(qa^2) = a^2(pq)a^2 \in a^2 Sa^2$ and $a \in (a^2 Sa^2]$. □

5. Some semilattices of simple ordered semigroups

A subsemigroup F of an ordered semigroup S is called a *filter* of S if:

(i) $a, b \in S$ and $ab \in F$ implies $a \in F$ and $b \in F$.

(ii) If $a \in F$ and $c \in S$ such that $c \geqslant a$ then $c \in F$ (see [7]).

For $x \in S$, we denote by $N(x)$ the least filter of S generated x ($x \in S$) and by $\mathcal{N}$ the *equivalence relation*

$$\mathcal{N} := \{(x, y) \in S \times S \mid N(x) = N(y)\}.$$

Let S be an ordered semigroup. An equivalence relation σ on S is called *congruence* if $(a, b) \in \sigma$ implies $(ac, bc) \in \sigma$ and $(ca, cb) \in \sigma$ for every $c \in S$. A congruence σ on S is called *semilattice congruence* if $(a^2, a) \in \sigma$ and $(ab, ba) \in \sigma$ for each $a, b \in S$ (see [7]). If σ is a semilattice congruence on S then the σ-class $(x)_\sigma$ of S containing x is a subsemigroup of S for every $x \in S$ (see [7]). An ordered semigroup S is called a semilattice of *left and right simple semigroups* if there exists a semilattice congruence σ on S such that the *σ-class* $(x)_\sigma$ of S containing x is a left and right simple subsemigroup of S for every $x \in S$.

Equivalent definition:

There exists a semilattice Y and a family $\{S_\alpha\}_{\alpha \in Y}$ of left and right simple subsemigroups of S such that

(i) $S_\alpha \cap S_\beta = \emptyset \ \forall \alpha, \beta \in Y, \alpha \neq \beta$,

(ii) $S = \bigcup_{\alpha \in Y} S_\alpha$,

(iii) $S_\alpha S_\beta \subseteq S_{\alpha\beta} \ \forall \alpha, \beta \in Y$.

Theorem 5.1. *An ordered semigroup $(S, \cdot, \leqslant)$ is a semilattice of left and right simple semigroups if and only if for every $\mathcal{N}$-fuzzy quasi-ideal f of S, we have $f(a) = f(a^2)$ and $f(ab) = f(ba)$ for all $a, b \in S$.*

Proof. Suppose that S is a semilattice of left and right simple semigroups. Then by hypothesis, there exists a semilattice Y and a family $\{S_\alpha\}_{\alpha\in Y}$ of left and right simple subsemigroups of S satisfying (i), (ii) and (iii).

(A) Let f be an $\mathcal{N}$-fuzzy quasi-ideal of S and $a \in S$. Since $a \in S = \bigcup_{\alpha\in Y} S_\alpha$, then there exists $\alpha \in Y$ such that $a \in S_\alpha$. Since S_α is left simple we have $S_\alpha = (S_\alpha a]$. Since $a \in S_\alpha$, then $a \in (S_\alpha a]$ and so $a \leqslant xa$ for some $x \in S_\alpha$. Since $x \in S_\alpha$, we have $x \in (S_\alpha a]$, then $x \leqslant ya$ for some $y \in S_\alpha$. Thus $a \leqslant xa \leqslant (ya)a = ya^2$ and we have $(y, a^2) \in A_a$. Also S is right simple, we have $S_\alpha = (aS_\alpha]$, since $a \in S_\alpha$ then $a \in (aS_\alpha]$ and we have $a \leqslant az$ for some $z \in S_\alpha$. Since $z \in S_\alpha$ we have $z \in (aS_\alpha]$ then $z \leqslant at$ for some $t \in S$. Thus $a \leqslant az \leqslant a(at) = a^2t$, and we have $(a^2, t) \in A_a$. Since f is an $\mathcal{N}$-fuzzy quasi-ideal of S and $A_a \neq \emptyset$, we have

$$\begin{aligned}
f(a) &\leqslant ((fN\alpha) \vee (\alpha Nf))(a) = \max[(fN\alpha)(a), (\alpha Nf)(a)]\\
&= \max\Big[\bigwedge_{(y_1,z_1)\in A_a} \max\{f(y_1), \alpha(z_1)\}, \bigwedge_{(y_2,z_2)\in A_a} \max\{\alpha(y_2), f(z_2)\}\Big]\\
&\leqslant \max\left[\max\{f(a^2), \alpha(t)\}, \max\{\alpha(y), f(a^2)\}\right]\\
&= \max\left[\max\{f(a^2), -1\}, \max\{-1, f(a^2)\}\right] = \max\left[f(a^2), f(a^2)\right] = f(a^2).
\end{aligned}$$

On the other hand, since every $\mathcal{N}$-fuzzy quasi-ideal is an $\mathcal{N}$-fuzzy subsemigroup of S, we have $f(a^2) = f(aa) \leqslant \max\{f(a), f(a)\} = f(a)$. Thus $f(a) = f(a^2)$.

(B) Let $a, b \in S$. By (A), we have $f(ab) = f((ab)^2) = f((ab)^4)$. Since $(AB] = (BA] = A \cap B$, by Lemmas 2.5 an 2.8, we have

$$\begin{aligned}
(ab)^4 &= (aba)(babab) \in Q(aba)Q(babab) \subseteq (Q(aba)Q(babab)]\\
&= (Q(babab)Q(aba)] = ((babab \cup (bababS \cap Sbabab](aba \cup (abaS \cap Saba]]\\
&\subseteq ((babab \cup (bababS](aba \cup (Saba]] \subseteq ((baS \cup (baS](Sba \cup (Sba]]\\
&= ((baS](Sba]] = ((baSba]] = (baSba] = baS \cap Sba
\end{aligned}$$

Then $(ab)^4 \leqslant (ba)x$ and $(ab)^4 \leqslant y(ba)$ for some $x, y \in S$. Then $(ba, x) \in A_{(ab)^4}$ and $(y, ba) \in A_{(ab)^4}$. Since f is an $\mathcal{N}$-fuzzy quasi-ideal and $A_{(ab)^4} \neq \emptyset$, we have

$$\begin{aligned}
f((ab)^4) &\leqslant ((fN\alpha) \vee (\alpha Nf))((ab)^4) = \max[(fN\alpha)(ab)^4, (\alpha Nf)(ab)^4]\\
&= \max\Big[\bigwedge_{(y_1,z_1)\in A_{(ab)^4}} \max\{f(y_1), \alpha(z_1)\}, \bigwedge_{(y_2,z_2)\in A_{(ab)^4}} \max\{\alpha(y_2), f(z_2)\}\Big]\\
&\leqslant \max\left[\max\{f(ba), \alpha(x)\}, \max\{\alpha(y), f(ba)\}\right]\\
&= \max\left[\max\{f(ba), -1\}, \max\{-1, f(ba)\}\right] = \max\left[f(ba), f(ba)\right] = f(ba).
\end{aligned}$$

By symmetry we can prove that $f(ba) \leqslant f((ab)^4) = f(ab)$.

Conversely, assume that conditions (1) and (2) are true. Then by (1) and Lemma 2.7, S is completely regular. Let A be a quasi-ideal of S and let $a \in A$.

Since S is completely and $a \in S$, there exists $x \in S$ such that $a \leqslant a^2xa^2$. Then

$$\begin{aligned} a &\leqslant a^2xa^2 \in (a^2Sa^2) = a(a(Sa)a) \subseteq a(aSa) \subseteq a(aSa] \\ &= a(aS \cap Sa) \subseteq A(AS \cap SA) \subseteq A(AS) \subseteq AA, \end{aligned}$$

and so $A \subseteq AA \subseteq (A^2]$. On the other hand, by Lemma 2.4, A is a subsemigroup of S, we have $A^2 \subseteq A \Longrightarrow (A^2] \subseteq (A] = A$.

Let A and B be any quasi-ideals of S and let $x \in (AB]$, then $x \leqslant ab$ for some $a \in A$ and $b \in B$. We consider the quasi-ideal $Q(ab) = (ab \cup (abS \cap Sab)]$ generated by ab. Then by Theorem 3.3, the characteristic $\mathcal{N}$-function $\kappa_{Q(ab)}$ of $Q(ab)$ is an $\mathcal{N}$-fuzzy quasi-ideal of S. By hypothesis $\kappa_{Q(ab)}(ba) = \kappa_{Q(ab)}(ab)$.

Since $ab \in Q(ab)$, we have $\kappa_{Q(ab)}(ab) = -1$ and $\kappa_{Q(ab)}(ba) = -1$. Therefore $ba \in Q(ab) = (ab \cup (abS \cap Sab)]$ and, by Lemma 2.9,

$$\begin{aligned} ba &\in (ab \cup (abS \cap Sab)] = (ab \cup (abSab]] \subseteq (AB \cup (ABSAB]] \\ &\subseteq (AB \cup ((AB]S(AB]] \subseteq (AB \cup (AB]] = ((AB]] = (AB]. \end{aligned}$$

Hence $(BA] \subseteq (AB]$. By symmetry we can prove that $(AB] \subseteq (BA]$. $\square$

Lemma 5.2. *Let $(S, \cdot, \leqslant)$ be an ordered semigroup such that $a \leqslant a^2$ for all $a \in S$. Then for every $\mathcal{N}$-fuzzy quasi-ideal f of S we have $f(a) = f(a^2)$ for every $a \in S$.*

Proof. Let $a \in S$ such that $a \leqslant a^2$. Let f be an $\mathcal{N}$-fuzzy quasi-ideal of S. Since f is an $\mathcal{N}$-fuzzy subsemigroup of S. Then $f(a) \leqslant f(a^2) \leqslant \max\{f(a), f(a)\} = f(a)$. $\square$

Theorem 5.3. *Let S be an ordered semigroup and $a \in S$ such that $a \leqslant a^2$ for all $a \in S$. Then the following are equivalent:*

(*i*) *$ab \in (baS] \cap (Sba]$ for each $a, b \in S$.*

(*ii*) *For every $\mathcal{N}$-fuzzy quasi-ideal f of S, we have $f(ab) = f(ba)$ for every $a, b \in S$.*

Proof. $(i) \Rightarrow (ii)$. Let f be an $\mathcal{N}$-fuzzy quasi-ideal of S. Since $ab \in (baS] \cap (Sba]$, then $ab \in (baS]$ and we have $ab \leqslant (ba)x$ for some $x \in S$. By (i), we have $(ba)x \in (xbaS] \cap (Sxba]$. Then $(ba)x \in (Sxba]$ and we have $(ba)x \leqslant (yx)(ba)$ and so, $ab \leqslant (yx)(ba) \Rightarrow (yx, ba) \in A_{ab}$. Again, since $ab \in (Sba]$, then $ab \leqslant z(ba)$ for some $z \in S$ and by (i) we have $z(ba) \in (bazS]$, then $z(ba) \leqslant (ba)(zt)$ for some $t \in S$. So we have $ab \leqslant (ba)(zt) \Rightarrow (ba, zt) \in A_{ab}$. Since f is an $\mathcal{N}$-fuzzy quasi-ideal of S and $A_{ab} \neq \emptyset$, then

$$\begin{aligned} f(ab) &\leqslant ((fN\alpha) \vee (\alpha Nf))(ab) = \max[(fN\alpha)(ab), (\alpha Nf)(ab)] \\ &= \max\Big[\bigwedge_{(y_1,z_1)\in A_{ab}} \max\{f(y_1), \alpha(z_1)\}, \bigwedge_{(y_2,z_2)\in A_{ab}} \max\{\alpha(y_2), f(z_2)\}\Big] \\ &\leqslant \max\left[\max\{f(ba), \alpha(zt)\}, \max\{\alpha(yx), f(ba)\}\right] \\ &= \max\left[\max\{f(ba), -1\}, \max\{-1, f(ba)\}\right] = \max[f(ba), f(ba)] = f(ba). \end{aligned}$$

By symmetry we can prove that $f(ba) \leqslant f(ab)$.

$(ii) \Rightarrow (i)$. Let f be an $\mathcal{N}$-fuzzy quasi-ideal of S. Since $a \leqslant a^2$ for all $a \in S$, by Lemma 5.2, we have $f(a) = f(a^2)$. By (ii), we obtain $f(ba) = f(ab)$ for each $a, b \in S$. By Theorem 5.1, it follows that S, is a semilattice of left and right simple semigroups. Thus by hypothesis, there exists a semilattice Y and a family $\{S_\alpha\}_{\alpha \in Y}$ of left and right simple subsemigroups satisfying (i), (ii) and (iii) from the equivalent definition of a semilattice of simple semigroups.

Let $a, b \in S$, we have to show that $a \in (baS] \cap (Sba]$. Let $\alpha, \beta \in Y$ be such that $a \in S_\alpha$ and $b \in S_\beta$. Then $ab \in S_\alpha S_\beta \subseteq S_{\alpha\beta}$ and $ba \in S_\beta S_\alpha \subseteq S_{\beta\alpha} = S_{\alpha\beta}$. Since $S_{\alpha\beta}$ is left and right simple we have $S_{\alpha\beta} = (S_{\alpha\beta}c]$ and $S_{\alpha\beta} = (cS_{\alpha\beta}]$ for each $c \in S_{\alpha\beta}$. Since $ab, ba \in S_{\alpha\beta}$, we have $ab \in (baS_{\alpha\beta}] \cap (S_{\alpha\beta}ba] \subseteq (baS] \cap (Sba]$. This complete the proof. □

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