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Redefined fuzzy Lie subalgebras

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Abstract

This paper introduces a new concept of a Lie subalgebra of a Lie algebra using the notion of an anti fuzzy point and its besideness to and non-quasi-coincidence with a fuzzy set, and presents some of its useful properties.

1. Introduction

The theory of Lie algebras is an area of mathematics in which we can see a harmonious between the methods of classical analysis and modern algebra. This theory, a direct outgrowth of a central problem in the calculus, has today become a synthesis of many separate disciplines, each of which has left its own mark. Theory of Lie groups were developed by the Norwegian mathematician Sophus Lie in the late nineteenth century in connection with his work on systems of differential equations. Lie algebras were also discovered by Sophus Lie when he first attempted to classify certain smooth subgroups of general linear groups. The groups he considered are called Lie groups. The importance of Lie algebras for applied mathematics and for applied physics has also become increasingly evident in recent years. In applied mathematics, Lie theory remains a powerful tool for studying differential equations, special functions and perturbation theory. Lie theory finds applications not only in elementary particle physics and nuclear physics, but also in such diverse fields as continuum mechanics, solid-state physics, cosmology and control theory. Lie algebra is also used by electrical engineers, mainly in the mobile robot control. For the basic information of Lie algebras, the readers are refereed to [7, 12, 17].

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In 1965, Zadeh [26] introduced the notion of a fuzzy subset of a set as a method for representing uncertainty. Since then it has become a vigorous area of research in different domains such as engineering, medical science, social science, physics, statistics, graph theory, artificial intelligence, signal processing, multiagent systems, pattern recognition, robotics, computer networks, expert systems, decision making, automata theory. Yehia introduced the notions of fuzzy ideals and fuzzy subalgebras of Lie algebras in [24] and studied some results. Since then, the concepts and results of Lie algebras have been broadened to the fuzzy setting frames (see, [1, 2, 3, 4, 6, 13, 14, 18, 20, 21, 24]).

This paper introduces a new concept of a subalgebra of a Lie algebra using the notion of an anti fuzzy point and its besideness to and non-quasi-coincidence with a fuzzy set, and presents some of its useful properties.

2. Preliminaries

A *Lie algebra* is a vector space L over a field F (equal to $\mathbf{R}$ or $\mathbf{C}$) on which is defined the multiplication $L \times L \to L$, denoted by $(x, y) \to [x, y]$, satisfying the following axioms:

(L_1) $[x, y]$ is bilinear,

(L_2) $[x, x] = 0$ for all $x \in L$,

(L_3) $[[x, y], z] + [[y, z], x] + [[z, x], y] = 0$ for all $x, y, z \in L$ (Jacobi identity).

In this paper by L will be denoted a Lie algebra. We note that the multiplication in a Lie algebra is not associative, but it is *anti commutative*, i.e., $[x, y] = -[y, x]$ for all $x, y \in L$. A subspace H of L closed under $[\cdot, \cdot]$ will be called a *Lie subalgebra*.

Definition 2.1. A *fuzzy set* ν on L, i.e., a real mapping $\nu : L \to R$ such that $0 \leqslant \nu(x) \leqslant 1$ for all $x \in L$, is called an *anti fuzzy Lie subalgebra* of L if

(I) $\nu(x + y) \leqslant \max\{\nu(x), \nu(y)\}$,

(II) $\nu(\alpha x) \leqslant \nu(x)$,

(III) $\nu([x, y]) \leqslant \min\{\nu(x), \nu(y)\}$

hold for all $x, y \in L$ and $\alpha \in F$.

As a consequence of the Transfer Principle proved in [22] we obtain

Theorem 2.2. *Let ν be a fuzzy set on L. Then ν is a fuzzy Lie subalgebra of L if and only if*

$$L(\nu; t) = \{x \in L : \nu(x) \leqslant t\}$$

is a Lie subalgebra of L for all $t \in (0, 1]$. □

The membership degree expresses the degree of belongingness of elements to a fuzzy set. The membership degree 1 denote that an element completely belongs to its corresponding fuzzy set, and the membership degree 0 denote that an element does not belong to the fuzzy set. The membership degrees on the interval $(0, 1)$ denote the partial membership to the fuzzy set. Sometimes, the membership degree means the satisfaction degree of elements to some property or constraint corresponding to a fuzzy set. The membership degrees on the interval $(0, 1]$ denote that elements somewhat satisfy the property.

A fuzzy set ν on L of the form

$$\nu(y) = \begin{cases} t \in [0, 1) & \text{if } y = x, \\ 1, & \text{if } y \neq x \end{cases}$$

is called an *anti fuzzy point* with support x and value t and is denoted by x_t. A fuzzy set ν in L is said to be *non-unit* if there exists $x \in L$ such that $\nu(x) < 1$. An anti fuzzy point x_t is said to "*besides to*" a fuzzy set ν, written as $x_t \prec \nu$ if $\nu(x) \leqslant t$. An anti fuzzy point x_t is said to be "*non-quasicoincident with*" a fuzzy set ν, denoted by $x_t \vdash \nu$ if $\nu(x) + t \leqslant 1$.

3. Redefined fuzzy Lie subalgebras

Let α and β denote one of the symbols $\prec$, $\vdash$, $\prec\vee\vdash$ or $\prec\wedge\vdash$ unless otherwise specified.

Definition 3.1. A fuzzy set ν in L is called an $(\alpha, \beta)^*$*-fuzzy Lie subalgebra* of L if it satisfies the following conditions:

(1) $x_s \alpha \nu, y_t \alpha \nu \Rightarrow (x + y)_{\max(s,t)} \beta \nu$,

(2) $x_s \alpha \nu \Rightarrow (mx)_s \beta \nu$,

(3) $x_s \alpha \nu, y_t \alpha \nu \Rightarrow ([x, y])_{\min(s,t)} \beta \nu$

for all x, $y \in L$, $m \in F$, s, $t \in [0, 1)$.

Notations: The following notations will be used:

- "$x_t \prec \nu$" and "$x_t \vdash \nu$" will be denoted by $x_t \prec\wedge\vdash \nu$.
- "$x_t \prec \nu$" or "$x_t \vdash \nu$" will be denoted by $x_t \prec\vee\vdash \nu$.
- The symbol $\overline{\prec\wedge\vdash}$ means neither $\prec$ nor $\vdash$ hold.

Remark. If ν is a fuzzy set in L such that $\nu(x) \geqslant 0.5$ for all $x \in L$. Then $\{x_t | x_t \prec\wedge\vdash \mu\} = \emptyset$.

The proof of the following proposition is trivial.

Proposition 3.2. *For any fuzzy set ν in L, Definition 2.1 is equivalent to the following conditions:*

(4) $x_s, y_t \prec \nu \Rightarrow (x+y)_{\max(s,t)} \prec \nu$,

(5) $x_s \prec \nu \Rightarrow (mx)_s \prec \nu$,

(6) $x_s, y_t \prec \nu \Rightarrow ([x,y])_{\min(s,t)} \prec \nu$,

for all $x, y \in L$, $m \in F$, $s, t \in [0,1)$. □

For a fuzzy set ν in a Lie algebra L, we denote $L^* = \{x \in L : \nu(x) < 1\}$.

Proposition 3.3. *If ν is a non-unit $(\prec,\prec)^*$-fuzzy Lie subalgebra of L, then L^* is a Lie subalgebra of L.*

Proof. Let $x, y \in L^*$. Then $\nu(x) < 1$ and $\nu(y) < 1$.

(1) Assume $\nu(x+y) = 1$. Then we can see that $x_{\nu(x)} \prec \nu$ and $y_{\nu(y)} \prec \nu$, but $(x+y)_{\max(\nu(x),\nu(y))} \overline{\prec} \nu$ since $\nu(x+y) = 1 > \max(\nu(x), \nu(y))$. This is clearly a contradiction, and hence $\nu(x+y) < 1$, which shows that $x+y \in L^*$.

(2) Assume $\nu(mx) = 1$. Then we can see that $x_{\nu(x)} \prec \nu$, but $(mx)_{\nu(x)} \overline{\prec} \nu$ since $\nu(mx) = 1 > \nu(x)$. This is clearly a contradiction, and hence $\nu(mx) < 1$, which shows that $mx \in L^*$.

(3) Assume $\nu([x,y]) = 1$. Then we can see that $x_{\nu(x)} \prec \nu$ and $y_{\nu(y)} \prec \nu$, but $([x,y])_{\min(\nu(x),\nu(y))} \overline{\prec} \nu$ since $\nu([x,y]) = 1 > \min(\nu(x), \nu(y))$. This is clearly a contradiction, and hence $\nu([x,y]) < 1$, which shows that $[x,y] \in L^*$. Hence L^* is a Lie subalgebra of L. □

Proposition 3.4. *If ν is a non-unit $(\prec,\vdash)^*$-fuzzy Lie subalgebra of L, then the set L^* is a Lie subalgebra of L.*

Proof. Let x, $y \in L^*$. Then $\nu(x) < 1$ and $\nu(y) < 1$.

(1) Suppose that $\nu(x+y) = 1$, then

$$\nu(x+y) + \max(\nu(x), \nu(y)) \geqslant 1.$$

Hence $(x+y)_{\max(\nu(x),\nu(y))}\overline{\vdash}\nu$, which is a contradiction since $x_{\nu(x)} \prec \nu$ and $y_{\nu(y)} \prec \nu$. Thus $\nu(x+y) < 1$, so $x+y \in L^*$.

(2) Suppose that $\nu(mx) = 1$, then

$$\nu(mx) + \nu(x) \geqslant 1.$$

Hence $mx_{\nu(x)}\overline{\vdash}\nu$, a contradiction since $x_{\nu(x)} \prec \nu$. Thus $\nu(mx) < 1$, so $mx \in L^*$.

(3) Suppose that $\nu([x,y]) = 1$, then

$$\nu([x,y]) + \min(\nu(x), \nu(y)) \geqslant 1.$$

Hence $[x,y]_{\min(\nu(x),\nu(y))}\overline{\vdash}\nu$, which is a contradiction since $x_{\nu(x)} \prec \nu$ and $y_{\nu(y)} \prec \nu$. Thus $\nu([x,y]) < 1$, so $[x,y] \in L^*$. Hence L^* is a Lie subalgebra of L. □

Proposition 3.5. *If ν is a non-unit $(\vdash, \prec)^*$-fuzzy Lie subalgebra of L, then L^* is a Lie subalgebra of L.*

Proof. Let $x, y \in L^*$. Then $\nu(x) < 1$ and $\nu(y) < 1$. Thus $x_0 \vdash \nu$ and $y_0 \vdash \nu$.

(1) If $\nu(x+y) = 1$, then $\nu(x+y) = 1 > 0 = \max(0,0)$. Therefore, $(x+y)_{\max(0,0)}\overline{\prec}\nu$, which is a contradiction. It follows that $\nu(x+y) < 1$ so that $x+y \in L^*$.

(2) If $\nu(mx) = 1$, then $\nu(mx) = 1 > 0$. Therefore, $mx_0\overline{\prec}\nu$, a contradiction. It follows that $\nu(mx) < 1$ so that $mx \in L^*$.

(3) If $\nu([x,y]) = 1$, then $\nu([x,y]) = 1 > 0 = \min(0,0)$. Therefore, $[x,y]_{\min(0,0)}\overline{\prec}\nu$, which is a contradiction. It follows that $\nu([x,y]) < 1$ so that $[x,y] \in L^*$. Hence L^* is a Lie subalgebra of L. □

Proposition 3.6. *If ν is a non-unit $(\vdash, \vdash)^*$-fuzzy Lie subalgebra of L, then L^* is a Lie subalgebra of L.*

Proof. Let $x, y \in L^*$. Then $\nu(x) < 1$ and $\nu(y) < 1$.

(1) If $\nu(x+y) = 1$, then $\nu(x+y)+\max(0,0) = 1$, and so $(x+y)_{\max(0,0)}\overline{\vdash}\nu$. This is impossible, and hence $\nu(x+y) < 1$, i.e., $x+y \in L^*$.

(2) If $\nu(mx) = 1$, then $\nu(mx) + 0 = 1$, and so $(mx)_0\overline{\vdash}\nu$. This is impossible, and hence $\nu(mx) < 1$, i.e., $mx \in L^*$.

(3) If $\nu([x,y]) = 1$, then $\nu([x,y])+\min(0,0) = 1$, and so $[x,y]_{\min(0,0)}\overline{\vdash}\nu$. This is impossible, and hence $\nu([x,y]) < 1$, i.e., $[x,y] \in L^*$. Hence L^* is a Lie subalgebra of L. □

Proposition 3.7. *If ν is a non-unit $(\prec, \prec\vee\vdash)^*$-fuzzy Lie subalgebra of L, then L^* is a Lie subalgebra of L.*

Proof. Let x, $y \in L^*$. Then $\nu(x) < 1$ and $\nu(y) < 1$. Thus $\nu(x) = s_1$ and $\nu(y) = s_1$ for some s_1, $s_2 \in [0, 1)$. It follow that $x_{s_1} \prec \nu$ and $y_{s_2} \prec \nu$ so that $(x+y)_{\max(s_1,s_2)} \prec\vee\vdash \nu$, i.e., $(x+y)_{\max(s_1,s_2)} \prec \nu$ or $(x+y)_{\max(s_1,s_2)} \vdash \nu$. If $(x+y)_{\max(s_1,s_2)} \prec \nu$, then $\nu(x+y) \leqslant \max(s_1, s_2) < 1$ and hence $x+y \in L^*$. On the other hand, If $(x + y)_{\max(s_1,s_2)} \vdash \nu$, then $\nu(x + y) \leqslant \nu(x + y) + \max(s_1, s_2) < 1$, and hence $x + y \in L^*$. Verification of conditions (2) and (3) in Definition 3.1 is similar, we omit the details. $\square$

By using similar argumentations we can also prove the following two propositions.

Proposition 3.8. *If ν is a non-unit $(\vdash, \prec \vee \vdash)^*$-fuzzy Lie subalgebra of L, then L^* is a Lie subalgebra of L.* $\square$

Proposition 3.9. *If ν is a non-unit $(\prec, \prec\wedge\vdash)^*$-,$(\prec\vee\vdash, \vdash)^*$-, $(\prec\vee\vdash, \prec)^*$-, $(\prec\vee\vdash, \prec\wedge\vdash)^*$-, $(\vdash, \prec\wedge\vdash)^*$-, or $(\prec\vee\vdash, \prec \vee \vdash)^*$-fuzzy Lie subalgebra of L, then L^* is a Lie subalgebra of L.* $\square$

Definition 3.10. A fuzzy set ν in L is called an $(\prec, \prec \vee \vdash)^*-$*fuzzy Lie subalgebra* of L if the following conditions are satisfied:

(a) $x_s, y_t \prec \nu \Rightarrow (x+y)_{\max(s,t)} \prec\vee\vdash \nu$,

(b) $x_s \prec \nu \Rightarrow (mx)_s \prec\vee\vdash \nu$,

(c) $x_s, y_t \prec \nu \Rightarrow ([x, y])_{\min(s,t)} \prec\vee\vdash \nu$

for all $x, y \in L$, $m \in F$, $s, t \in [0, 1)$.

Example 3.11. Let V be a vector space over a field F such that $dim(V) = 5$. Let $V = \{e_1, e_2, \ldots, e_5\}$ be a basis of a vector space over a field F with Lie brackets as follows:

$$[e_1, e_2] = e_3, \quad [e_1, e_3] = e_5, \quad [e_1, e_4] = e_5, \quad [e_1, e_5] = 0,$$
$$[e_2, e_3] = e_5, \quad [e_2, e_4] = 0, \quad [e_2, e_5] = 0, \quad [e_3, e_4] = 0,$$
$$[e_3, e_5] = 0, \quad [e_4, e_5] = 0, \quad [e_i, e_j] = -[e_j, e_i]$$

and $[e_i, e_j] = 0$ for all $i = j$. Then V is a Lie algebra over F. We define a fuzzy set $\nu : V \to [0, 1]$ by

$$\nu(x) := \begin{cases} 0.25 & \text{if } x = 0, \\ 0.46 & \text{if } x \in \{e_3, e_5\}, \\ 0 & \text{if } x \in \{e_1, e_2, e_4\}. \end{cases}$$

By routine computations, it is easy to see that ν is an $(\prec, \prec\vee\vdash)^*$-fuzzy Lie subalgebra of L.

Theorem 3.12. *Let ν be a fuzzy set in a Lie algebra L. Then ν is an $(\prec, \prec\vee\vdash)^*-$fuzzy Lie subalgebra of L if and only if*

(d) $\nu(x+y) \leqslant \max(\nu(x), \nu(y), 0.5)$,

(e) $\nu(mx) \leqslant \max(\nu(x), 0.5)$,

(f) $\nu([x,y]) \leqslant \min(\nu(x), \nu(y), 0.5)$

hold for all $x, y \in L$, $m \in F$.

Proof. $(a) \Rightarrow (d)$: Let $x, y \in L$. We consider the following two cases:

(1) $\max(\nu(x), \nu(y)) > 0.5$,

(2) $\max(\nu(x, \nu(y)) \leqslant 0.5$.

Case (1): Assume that $\nu(x+y) > \max(\nu(x), \nu(y), 0.5)$ Then $\nu(x+y) > \max(\nu(x), \nu(y))$. Take s such that $\nu(x+y) > s > \max(\nu(x), \nu(y))$. Then $x_s \prec \nu$, $y_s \prec \nu$, but $(x+y)_s \overline{\prec\vee\vdash} \nu$, which is contradiction with (a).

Case (2): Assume that $\nu(x+y) > 0.5$. Then $x_{0.5}$, $y_{0.5} \prec \nu$ but $(x+y)_{0.5} \overline{\prec\vee\vdash} \nu$, a contradiction. Hence (d) holds.

$(d) \Rightarrow (a)$: Let x_s, $y_t \prec \nu$, then $\nu(x) \leqslant s$, $\nu(y) \leqslant t$. Now, we have

$$\nu(x+y) \leqslant \max(\nu(x), \nu(y), 0.5) \leqslant \max(s, t, 0.5).$$

If $\max(s,t) < 0.5$, then $\nu(x+y) \leqslant 0.5 \Rightarrow \nu(x+y) + \max(s,t) < 1$. On the other hand, if $\max(s,t) \geqslant 0.5$, then $\nu(x+y) \leqslant \max(s,t)$. Hence $(x+y)_{\max(s,t)} \prec\vee\vdash \nu$.

The verification of $(b) \Leftrightarrow (e)$ and $(c) \Leftrightarrow (f)$ is analogous and we omit the details. This completes the proof. $\square$

Theorem 3.13. *Let ν be an $(\prec, \prec\vee\vdash)^*$-fuzzy Lie subalgebra of L.*

(i) *If there exists $x \in L$ such that $\nu(x) \leqslant 0.5$, then $\nu(0) \leqslant 0.5$.*

(ii) *If $\nu(0) > 0.5$, then ν is an $(\prec, \prec)^*$-fuzzy Lie subalgebra of L.*

Proof. (i) Let $x \in L$ such that $\nu(x) \leqslant 0.5$. Then $\nu(-x) = \max(\nu(x), 0.5) = 0.5$. Hence $\nu(0) = \nu(x-x) \leqslant \max(\nu(x), \nu(-x), 0.5) = 0.5$.

(ii) If $\nu(0) > 0.5$ then $\nu(x) > 0.5$ for all $x \in L$. Thus we conclude that $\nu(x+y) \leqslant \max(\nu(x), \nu(y))$, $\nu(mx) \leqslant \nu(x)$, $\nu([x,y]) \leqslant \min(\nu(x), \nu(y))$ for all $x, y \in L$, $m \in F$. Hence ν is an $(\prec, \prec)^*$-fuzzy Lie subalgebra of L. $\square$

Theorem 3.14. *Let ν be a fuzzy set of fuzzy Lie subalgebra of L. Then ν is an $(\prec, \prec \vee \vdash)^*$-fuzzy Li subalgebra of L if and only if each nonempty $L(\nu;t)$, $t \in [0.5, 1)$ is a Lie subalgebra of L.*

Proof. Assume that ν is an $(\prec, \prec \vee \vdash)^*$ fuzzy Lie subalgebra of L and let $t \in [0.5, 1)$. If $x, y \in L(\nu;t)$ and $m \in F$, then $\nu(x) \leq t$ and $\nu(y) \leq t$. Thus,

$$\nu(x+y) \leqslant \max(\nu(x), \nu(y), 0.5) \leqslant \max(t, 0.5) = t,$$

$$\nu(mx) \leqslant \max(\nu(x), 0.5) \leqslant \max(t, 0.5) = t,$$

$$\nu([x,y]) \leqslant \max(\nu(x), \nu(y), 0.5) \leqslant \max(t, 0.5) = t,$$

and so $x+y, mx, [x,y] \in L(\nu;t)$. This shows that $L(\nu;t)$ is a Lie subalgebra of L.

Conversely, let ν be a fuzzy set such that $L(\nu;t)$ is a Lie subalgebra of L, for all $t \in [0.5, 1)$. If there exist $x, y \in L$ such that such that $\nu(x+y) > \max(\nu(x), \nu(y), 0.5)$, then we can take $t \in (0, 1)$ such that

$$\nu(u+v) > t > \max(\nu(x), \nu(y), 0.5).$$

Thus x, $y \in L(\nu;t)$ and $t > 0.5$, and so $x + y \notin L(\nu;t)$, which contradicts to the assumption that all $L(\nu;t)$ are Lie ideals. Therefore,

$$\nu(x+y) \leqslant \max(\nu(x), \nu(y), 0.5).$$

The verification is analogous for other conditions and we omit the details. Hence ν is an $(\prec, \prec \vee \vdash)^*$ fuzzy Lie subalgebra of L. $\square$

Theorem 3.15. *Let ν be a fuzzy set in a Lie algebra L. Then $L(\nu;t)$ is a Lie subalgebra of L if and only if*

(g) $\min(\nu(x+y), 0.5) \leqslant \max(\nu(x), \nu(y))$,

(h) $\min(\nu(mx), 0.5) \leqslant \nu(x)$,

(i) $\min(\nu([x,y]), 0.5)) \leqslant \max(\nu(x), \nu(y))$

for all $x, y \in L$, $m \in F$.

Proof. Suppose that $L(\nu;t)$ is a Lie subalgebra of L. Let $\min(\nu(x+y), 0.5) > \max(\nu(x), \nu(y)) = t$ for some $x, y \in L$, then $t \in [0.5, 1)$, $\nu(x+y) > t$, $x \prec L(\nu;t)$ and $y \prec L(\nu;t)$. Since $x, y \prec L(\nu;t)$ and $L(\nu;t)$ is a Lie subalgebra of L, so $x + y \prec L(\nu;t)$ or $\nu(x+y) \leqslant t$, which is contradiction with $\nu(x+y) > t$. Hence (d) holds. For (e), (f) the verification is analogous.

Conversely, suppose that (d), (e) and (f) hold. Assume that $t \in [0.5, 1)$, $x, y \prec L(\nu; t)$. Then

$$0.5 > t \geqslant \max(\nu(x), \nu(y)) \geqslant \min(\nu(x+y), 0.5) \Rightarrow \nu(x+y) \leqslant t,$$

$$0.5 > t \geqslant \nu(x) \geqslant \min(\nu(mx), 0.5) \Rightarrow \nu(mx) \leqslant t,$$

$$0.5 > t \geqslant \max(\nu(x), \nu(y)) \geqslant \min(\nu([x, y], 0.5) \Rightarrow \nu([x, y]) \leqslant t,$$

and so $x+y \prec L(\nu; t)$, $mx \prec L(\nu; t)$,$[x, y] \prec L(\nu; t)$. This shows that $L(\nu; t)$ is a Lie subalgebra of L. □

Definition 3.16. An $(\prec, \prec \vee \vdash)^*$-fuzzy Lie subalgebra of L is called *proper* if Im ν has at least two elements. Two $(\prec, \prec \vee \vdash)^*$- fuzzy Lie subalgebras ν_1 and ν_2 are said to be *equivalent* if they have the same family of level Lie subalgebras.

Theorem 3.17. *Any proper $(\prec, \prec \vee \vdash)^*$-fuzzy Lie subalgebra of L for which the cardinality of $\{\nu(x) : \nu(x) > 0.5\} \leqslant 2$ can be expressed as the union of two proper non-equivalent $(\prec, \prec \vee \vdash)^*$-fuzzy Lie subalgebras of L.*

Proof. Let ν be a proper $(\prec, \prec \vee \vdash)^*$-fuzzy Lie subalgebra of L such that $\{\nu(x) : \nu(x) > 0.5\} = \{t_1, t_2, \ldots, t_n\}$ where $t_1 < t_2 < \ldots < t_n$ and $n \geqslant 2$. Then

$$\nu_{0.5} \subseteq \nu_{t_1} \subseteq \ldots \subseteq \nu_{t_n} = L$$

is the chain of $(\prec, \prec \vee \vdash)^*$-Lie subalgebras of ν. Define μ_1 and μ_2 by

$$\mu_1(x) = \begin{cases} t_1, & \text{if } x \in \nu_{t_1}, \\ t_2, & \text{if } x \in \nu_{t_2} \setminus \nu_{t_1}, \\ \vdots & \\ t_n, & \text{if } x \in \nu_{t_n} \setminus \nu_{t_{n-1}}, \end{cases}$$

$$\mu_2(x) = \begin{cases} \nu(x), & \text{if } x \in \nu_{0.5}, \\ n, & \text{if } x \in \nu_{t_2} \setminus \nu_{0.5}, \\ t_3, & \text{if } x \in \nu_{t_3} \setminus \nu_{t_2}, \\ \vdots & \\ t_n, & \text{if } x \in \nu_{t_n} \setminus \nu_{t_{n-1}}, \end{cases}$$

respectively, where $t_3 > n > t_2$. Then μ_1 and μ_2 are $(\prec, \prec \vee \vdash)^*$-fuzzy Lie subalgebras of L with

$$\nu_{t_1} \subseteq \nu_{t_2} \subseteq \ldots \subseteq \nu_{t_n}$$

and

$$\nu_{t_{0.5}} \subseteq \nu_{t_2} \subseteq \ldots \subseteq \nu_{t_n}$$

being respectively chains of $(\prec, \prec \vee \vdash)^*$-fuzzy Lie subalgebras of μ_1 and μ_2.

Hence ν can be expressed as the union of two proper non-equivalent $(\prec, \prec \vee \vdash)^*$-fuzzy subalgebras of L. $\square$

Theorem 3.18. *Let $\{\nu_i : i \in \Lambda\}$ be a family of $(\prec, \prec)^*$-fuzzy Lie subalgebras of L. Then $\nu = \bigcup_{i\in\Lambda} \nu_i$ is an $(\prec, \prec)^*$-fuzzy Lie subalgebra of L.*

Proof. Let $x_s \prec \nu$ and $y_t \prec \nu$, where $s, t \in [0, 1)$. Then $\nu(x) \leqslant s$ and $\nu(y) \leqslant t$. Thus we have $\nu_i(x) \leqslant s$ and $\nu_i(y) \leqslant t$ for all $i \in \Lambda$. Hence $\nu_i(x + y) \leqslant \max(s, t)$. Therefore, $\nu(x + y) \leqslant \max(s, t)$, which implies that $(x + y)_{\max\{s,t\}} \prec \nu$. For other conditions the verification is analogous. $\square$

Theorem 3.19. *Let $\{\nu_i : i \in \Lambda\}$ be a family of $(\prec, \prec \vee \vdash)^*$-fuzzy Lie subalgebra of L. Then $\nu := \bigcap_{i\in\Lambda} \nu_i$ is an $(\prec, \prec\vee\vdash)^*$-fuzzy Lie subalgebra of L.*

Proof. By Theorem 3.12, we have $\nu(x+y) \leqslant \max(\nu(x), \nu(y), 0.5)$, and hence

$$\begin{aligned}
\nu(x + y) &= \inf_{i\in\Lambda} \nu_i(x + y)\\
&\leqslant \inf_{i\in\Lambda} \max(\nu_i(x), \nu_i(y), 0.5)\\
&= \max(\inf_{i\in\Lambda} \nu_i(x), \inf_{i\in\Lambda} \nu_i(y), 0.5)\\
&= \max(\textstyle\bigcap_{i\in\Lambda} \nu_i(x), \bigcap_{i\in\Lambda} \nu_i(y), 0.5)\\
&= \max(\nu(x), \nu(y), 0.5).
\end{aligned}$$

For other conditions the verification is analogous. By Theorem 3.12, it follows that ν is an $(\prec, \prec\vee\vdash)^*$-fuzzy Lie subalgebra of L. $\square$

Remark. Let $\{\nu_i : i \in \Lambda\}$ be a family of $(\prec, \prec\vee\vdash)^*$-fuzzy Lie subalgebras of L. Is $\nu = \bigcup_{i\in\Lambda} \nu_i$ an $(\prec, \prec\vee\vdash)^*$-fuzzy Lie subalgebra of L? When? The following example shows that it is not an $(\prec, \prec\vee\vdash)^*$-fuzzy Lie subalgebra in general.

Example 3.20. Let V be a vector space over a field F such that $dim(V) = 5$. Let $V = \{e_1, e_2, e_3, e_4, e_5\}$ be its basis and let Lie brackets will be defined as in Example 3.11. If we define fuzzy sets $\mu_1, \mu_2 : V \to [0, 1]$ by putting

$$\mu_1(x) := \begin{cases} 0.6 & \text{if } x = 0,\\ 1 & \text{if } x \in \{e_3, e_5\},\\ 0 & \text{if } x \in \{e_1, e_2, e_4\},\end{cases}$$

$$\mu_2(x) := \begin{cases} 0.3 & \text{if } x = 0, \\ 1 & \text{if } x \in \{e_3, e_5\}, \\ 0 & \text{if } x \in \{e_1, e_2, e_4\}, \end{cases}$$

then both μ_1 and μ_2 will be $(\prec, \prec \vee \vdash)^*$-fuzzy Lie subalgebras of L, but $\mu_1 \cup \mu_2$ is not an $(\prec, \prec\vee\vdash)^*$-fuzzy Lie subalgebra of L since

$$\begin{aligned} 1 = \max(\mu_1(e_3), \mu_2(e_3)) &= (\mu_1 \cup \mu_2)(e_3) = (\mu_1 \cup \mu_2)([e_1, e_2]) \\ &\leqslant \min((\mu_1 \cup \mu_2)(e_1), (\mu_1 \cup \mu_2)(e_2), 0.5) = \min(0, 0, 0.5) = 0. \end{aligned}$$

Theorem 3.21. *Let $\{\nu_i : i \in \Lambda\}$ be a family of $(\prec, \prec \vee \vdash)^*$-fuzzy Lie subalgebras of L such that $\nu_i \subseteq \nu_j$ or $\nu_j \subseteq \nu_i$ for all $i, j \in \Lambda$. Then the fuzzy set $\nu := \bigcup_{i\in\Lambda} \nu_i$ is an $(\prec, \prec\vee\vdash)^*$-fuzzy Lie subalgebra of L.*

Proof. By Theorem 3.12, we have $\nu(x+y) \leqslant \max(\nu(x), \nu(y), 0.5)$, and hence

$$\begin{aligned} \nu(x + y) &= \sup\nolimits_{i\in\Lambda} \nu_i(x + y) \\ &\leqslant \sup\nolimits_{i\in\Lambda} \max(\nu_i(x), \nu_i(y), 0.5) \\ &= \max(\sup\nolimits_{i\in\Lambda} \nu_i(x), \sup\nolimits_{i\in\Lambda} \nu_i(y), 0.5) \\ &= \max(\textstyle\bigcup_{i\in\Lambda} \nu_i(x), \bigcup_{i\in\Lambda} \nu_i(y), 0.5) \\ &= \max(\nu(x), \nu(y), 0.5). \end{aligned}$$

It is easy to see that

$$\sup_{i\in\Lambda} \max(\nu_i(x), \nu_i(y), 0.5) \geqslant \bigcup_{i\in\Lambda} \max(\nu_i(x), \nu_i(y), 0.5).$$

Suppose that

$$\sup_{i\in\Lambda} \max(\nu_i(x), \nu_i(y), 0.5) \neq \bigcup_{i\in\Lambda} \max(\nu_i(x), \nu_i(y), 0.5),$$

then there exists s such that

$$\sup_{i\in\Lambda} \max(\nu_i(x), \nu_i(y), 0.5) > s > \bigcup_{i\in\Lambda} \max(\nu_i(x), \nu_i(y), 0.5).$$

Since $\nu_i \subseteq \nu_j$ or $\nu_j \subseteq \nu_i$ for all $i, j \in \Lambda$, there exists $k \in \Lambda$ such that $s > \max(\nu_k(x), \nu_k(y), 0.5)$. On the other hand, $\max(\nu_i(x), \nu_i(y), 0.5) > s$ for all $i \in \Lambda$, a contradiction. Hence

$$\begin{aligned} \sup\nolimits_{i\in\Lambda} \max\{\nu_i(x), \nu_i(y), 0.5\} &= \max(\textstyle\bigcup_{i\in\Lambda} \nu_i(x), \bigcup_{i\in\Lambda} \nu_i(y), 0.5) \\ &= \max\{\nu(x), \nu(y), 0.5\}. \end{aligned}$$

The verification of other conditions is analogous. By Theorem 3.12, it follows that ν is an $(\prec, \prec\vee\vdash)^*$-fuzzy Lie subalgebra of L. $\square$

Finally we study anti fuzzy Lie subalgebras with thresholds.

Definition 3.22. Let $m_1, m_2 \in [0,1]$ and $m_1 < m_2$. If ν is a fuzzy set of a Lie algebra L, then ν is called an *anti fuzzy Lie subalgebra with thresholds* (m_1, m_2) if

(1) $\min(\nu(x+y), m_1) \leqslant \max(\nu(x), \nu(y), m_2)$,

(2) $\min(\nu(mx), m_1) \leqslant \max(\nu(x), m_2)$,

(3) $\min(\nu([x,y]), m_1) \leqslant \max(\nu(x), \nu(y), m_2)$

for all $x, y \in L$, $m \in F$.

Theorem 3.23. *A fuzzy set ν of Lie algebra L is an anti fuzzy Lie subalgebra with thresholds (m_1, m_2) of L if and only if $L(\nu;t)(\neq \emptyset)$, for any $t \in (m_1, m_2]$, is a Lie subalgebra of L.*

Proof. Assume that ν is an anti fuzzy Lie subalgebra with thresholds (m_1, m_2) of L. Let $x, y \in L(\nu;t)$. Then $\nu(x) \leqslant t$ and $\nu(y) \leqslant t$, $t \in (m_1, m_2]$. Then it follows that

$$\min(\nu(x+y), m_1) \leqslant \max(\nu(x), \nu(y), m_2) = t \Longrightarrow \nu(x+y) \leqslant t,$$
$$\min(\nu(mx), m_1) \leqslant \max(\nu(x), m_2) = t \Longrightarrow \nu(mx) \leqslant t,$$
$$\min(\nu([x,y]), m_1) \leqslant \min(\nu(x), \nu(y), m_2) = t \Longrightarrow \nu([x,y]) \leqslant t,$$

and hence $x+y, mx, [x,y] \in L(\nu;t)$. This shows that $L(\nu;t)$ is a Lie subalgebra of L.

Conversely, assume that ν is a fuzzy set such that $L(\nu;t) \neq \emptyset$ is a Lie subalgebra of L for $m_1, m_2 \in [0,1]$ and $m_1 < m_2$. Suppose that $\min(\nu(x+y), m_1) > \max(\nu(x), \nu(y), m_2) = t$, then $\nu(x+y) > t$, $x \in L(\nu;t)$, $y \in L(\nu;t)$, $t \in (m_1, m_2]$. Since $x, y \in L(\nu;t)$ and $L(\nu;t)$ are Lie subalgebras, $x + y \in L(\nu;t)$, i.e., $\nu(x+y) \leqslant t$. This is a contradiction. Therefore condition (1) holds. The verification of (2) and (3) is analogous. $\square$

Remark. By Definition 3.22, we have the following result: If ν is an anti fuzzy subalgebra with thresholds (m_1, m_2), then we can conclude that: ν is an anti fuzzy subalgebra when $m_1 = 0$ and $m_2 = 1$; ν is an $(\prec, \prec\vee\vdash)^*$-fuzzy Lie subalgebra when $m_1 = 0.5$ and $m_2 = 1$.

By Definition 3.22, one can define other anti fuzzy subalgebra of L, such as $[0.2, 0.6)$-fuzzy subalgebra of L, $[0.3, 0.8)$- fuzzy subalgebra of L.

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