Quasigroups and Related Systems **16** (2008), 13 – 16

Counting loops with the inverse property

Asif Ali and John Slaney

Abstract

The numbers of isomorphism classes of IP loops of order up to 13 have been obtained by exhaustive enumeration, and are presented here along with some basic observations concerning IP loops.

1. Introduction

An *IP loop* is a set L and a binary operation $*$, where L contains an *identity* e such that $a * e = a = e * a$ for all $a \in L$, and where each $x \in L$ has a *two-sided inverse* x^{-1} such that for all $y \in L$

$$x^{-1} * (x * y) = y = (y * x) * x^{-1}.$$

For an account of the properties of IP loops, see Bruck's survey [3]). Clearly every group is an IP loop, but the converse is not the case. *Steiner loops* are also IP loops, satisfying the extra condition $x^{-1} = x$. IP loops form a very important class, not only in that they represent a strong generalization of both groups and Steiner loops, but also in that the Moufang nucleus (the set of $a \in L$ such that $a[(xy)a] = (ax)(ya)$ for all $x, y \in L$) of such loops behaves as a nilpotency function for this class. Moreover IP loops are exactly those groupoids whose power sets are the semiassociative relation algebras [7].

The present paper reports the numbers of non-isomorphic IP loops having order up to 13. Since these were obtained by exhaustive enumeration, they are available for inspection.

2000 Mathematics Subject Classification: 20N05
Keywords: loops, inverse property
This research was supported by National ICT Australia (NICTA). NICTA is funded through the Australian Government's *Backing Australia's Ability* initiative, in part through the Australian Research council.

2. History of counting loops

The number of non-isomorphic loops up to order 6 was found by Schönhardt [12] in 1930, but this was not noticed by Albert [1] or Sade [11] who obtained weaker results much later. Dénes and Keedwell [5] present counts of "quasigroups" up to order 6, but in fact count loops owing to their assumption that each "quasigroup" is isomorphic to a reduced square, which is obviously untrue of quasigroups in general. The loops of order 7 were counted in 1985 by Brant and Mullen [2]. In 2001, "QSCGZ" announced the number of loops of order 8 in an electronic forum [10], and the same value was found independently by Gujerin. For more on the history of counting loops, see McKay *et al* [9].

3. IP loops of small order

The smallest IP loop which is not a group is of order 7:

$*$	1	2	3	4	5	6	7
$e=1$	1	2	3	4	5	6	7
2	2	3	1	6	7	5	4
3	3	1	2	7	6	4	5
4	4	7	6	5	1	2	3
5	5	6	7	1	4	3	2
6	6	4	5	3	2	7	1
7	7	5	4	2	3	1	6

x	x^{-1}
1	1
2	3
3	2
4	5
5	4
6	7
7	6

This structure has proper subalgebras $\{1,2,3\}$, $\{1,4,5\}$ and $\{1,6,7\}$. Note that the order of these subloops does not divide the order of the loop, marking a significant difference between IP loops and groups.

Note also that the only element which is its own inverse is the identity e. This is a general feature of IP loops of odd order, as may be shown by a simple counting argument:

Observation 1. *IP loops of odd order have no subloops of even order.*

Proof. Let $(L,*)$ be an IP loop and let $(S,\underline{*})$ be a subloop of $(L,*)$ of even order. Clearly, S consists of e and some subset of elements of L along with their inverses. For this subset to be of even cardinality, some element in it other than e must be self-inverse and thus of order 2. Let $a \in L$ be such an element of order 2. Let $\dagger x$ be defined as $a * x$. Then the operation $\dagger$ is of period 2, because $\dagger\dagger x = a*(a*x) = a^{-1}*(a*x) = x$. Moreover, $\dagger$ has no fixed points, because if $\dagger x = x$ then $a * x = x$, so $a = e$, contradicting the assumption that a is of order 2. Hence $\dagger$ partitions L into pairs, so the cardinality of L must be even. □

The IP loops of small orders were counted by using a finite domain constraint solver to generate representatives of all isomorphism classes. The solver FINDER [13] has previously been used to generate results concerning the spectra of quasigroup identities [6]. It works by expressing each equation or other defining condition as the set of its ground instances on the domain of N elements, compiling these into constraints and then conducting a backtracking search for solutions to the constraint satisfaction problem using standard techniques such as forward checking and nogood learning [4].

Some symmetries were broken by enforcing conditions such as that e is always the first element. The remaining isomorphic copies were eliminated in a postprocessing phase. The results to order 11 were independently corroborated using the first order theorem prover PROVER9 and its associated propositional satisfiability solver MACE-4 [8]. In the cases of order 12 and order 13, the required searches are too hard for MACE and PROVER9, so we have only the results by FINDER in those cases.

size	groups	non − groups	total
1	1	0	1
2	1	0	1
3	1	0	1
4	2	0	2
5	1	0	1
6	2	0	2
7	1	1	2
8	5	3	8
9	2	5	7
10	2	45	47
11	1	48	49
12	5	2679	2684
13	1	10341	10342

Table 1. Numbers of IP loops of given order

The full list of these small IP loops, in a simple matrix format as for the order 7 example above, is available online.[1]

[1] `http://users.rsise.anu.edu.au/~jks/IPloops/`

References

[1] **A. A. Albert:** *Quasigroups. II*, Trans. Amer. Math. l Soc., **55** (1944), 401 – 409.

[2] **Brant and G. L. Mullen:** *A note on isomorphism classes of reduced latin squares of order* 7, Utilitas Math., **27** (1985), 261 – 263.

[3] **R. H. Bruck:** *A survey of binary systems*, Springer-Verlag, 1971.

[4] **R. Dechter:** *Constraint Processing*, Morgan Kaufmann, 2003.

[5] **J. Dénes and A. D. Keedwell:** *Latin squares and their applications*, Academic Press, 1974.

[6] **M. Fujita, J. Slaney, and F. Bennett:** *Automatic generation of some results in finite algebra*, Proc. 13th Internat. Joint Conference on Artificial Intelligence, 1993, 52 – 57.

[7] **R. Maddux:** *Some varieties containing relation algebras*, Trans. Amer. Math. Soc., **272** (1982), 501 – 526.

[8] **W. W. McCune:** *Prover9 Manual and Examples*, University of New Mexico, 2006. http://www.cs.unm.edu/ mccune/prover9/manual-examples.html.

[9] **B. D. McKay, A. Meynert, and W. Myrvold:** *Small latin squares, quasigroups and loops*, J. Combinatorial Designs, **15** (2007), 98 – 119.

[10] **QSCGZ** (pseudonym): *Anonymous electronic posting to Loopforum*, October 2001. http://groups.yahoo.com/group/loopforum/.

[11] **A. Sade:** *Morphismes de quasigroupes: Tables*, Revista da Faculdade de Ciências de Lisboa, 2: A – Ciências Matemáticas, **13** (1970/71), 149 – 172.

[12] **E. Schönhardt:** *Über lateinische quadrate und unionen*, J. für die reine und angewandte Mathematik, **163** (1930), 183 – 230.

[13] **J. Slaney:** *FINDER: Finite domain enumerator, system description*, Proc. 12th Confer. on Automated Deduction, 1994, 798 – 801.

Received October 12, 2007

Computer Sciences Laboratory
Research School of Infomation Sciences and Engineering
The Australian National University
Canberra, ACT 0200
Australia
e-mail: John.Slaney@anu.edu.au, dr_asif_ali@hotmail.com