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Fuzzy ideals in ordered semigroups

Niovi Kehayopulu and Michael Tsingelis

Abstract

The right (left) ideals, quasi- and bi-ideals play an essential role in studying the structure of some ordered semigroups. In an attempt to show how similar is the theory of ordered semigroups based on ideals or ideal elements with the theory of ordered semigroups based on fuzzy ideals, keeping the usual definitions of fuzzy right ideal, fuzzy left ideal, fuzzy quasi-ideal and fuzzy bi-ideal, we show here that in ordered groupoids, the fuzzy right (resp. left) ideals are fuzzy quasi-ideals and in ordered semigroups, the fuzzy quasi-ideals are fuzzy bi-ideals. Moreover, we prove that in regular ordered semigroups, the fuzzy quasi-ideals and the fuzzy bi-ideals coincide. We finally show that in an ordered semigroup the fuzzy quasi-ideals are just intersections of fuzzy right and fuzzy left ideals.

1. Introduction and prerequisites

The notion of ideals created by Dedekind for the theory of algebraic numbers, was generalized by Emmy Noether for associative rings. The one- and two-sided ideals introduced by her, are still central concepts in ring theory. A further generalization of ideals, the concept of quasi-ideals, was introduced by Ottó Steinfeld. Steinfeld remarked first that the concept of quasi-ideals could be defined not only for rings, but for semigroups as well, and that a quasi-ideal of a semigroup was just the intersection of a right and a left ideal –generalizing a correspondence result given by L. Kovács for rings. Since then many papers on ideals for rings and semigroups appeared showing the importance of the concept [A.H. Clifford, L.M. Gluskin, M.–P. Schützenberger, S. Lajos, K. Iséki and many others]. Further generalization

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of ideals by lattice-theoretical methods was given by G. Birkhoff, O. Steinfeld, and N. Kehayopulu. The concepts of fuzzy one- and two-sided ideals in groupoids have been introduced by A. Rosenfeld in [11], the concepts of fuzzy bi-ideals and fuzzy quasi-ideals in semigroups have been introduced by N. Kuroki in [8] and [9], respectively. Fuzzy ideals in semigroups have been first studied by N. Kuroki, later by other authors as well (for a detailed exposition see the introduction in [7]). Fuzzy ideals in ordered groupoids-semigroups have been introduced by Kehayopulu and Tsingelis in [5]. For a recent work on fuzzy ideals see also [3, 4].

In semigroups the right (resp. left) ideals are quasi-ideals, and the quasi-ideals are bi-ideals. In regular (in the sense of J.v. Neumann) semigroups the bi-ideals and the quasi-ideals coincide [10]. Analogous results are true for ordered semigroups as well. In ordered semigroups the right (resp. left) ideals are quasi-ideals and the quasi-ideals are bi-ideals, and in regular ordered semigroups the bi-ideals and the quasi-ideals coincide. Moreover, in lattice ordered semigroups having a greatest element, the right (resp. left) ideal elements are quasi-ideal elements, the quasi-ideal elements are bi-ideal elements, and in regular lattice ordered semigroups which have a greatest element the bi-ideal elements and the quasi-ideal elements are the same. It might be noted that the concept of right and left ideal elements in an ordered groupoid has been introduced by G. Birkhoff (see, for example [1] p. 328). Ideals play an important role in studying the structure of some ordered semigroups. In an attempt to show how similar is the theory of fuzzy ordered semigroups based on ideals (right, quasi- etc.) with the theory of ordered semigroups based on ideals or the theory of lattice ordered semigroups based on ideal elements, keeping the usual definitions of fuzzy right ideal, fuzzy left ideal, fuzzy quasi-ideal and fuzzy bi-ideal, we show here that in ordered groupoids the fuzzy right (resp. fuzzy left) ideals are fuzzy quasi-ideals, in ordered semigroups the fuzzy quasi-ideals are fuzzy bi-ideals, and in regular ordered semigroups the fuzzy quasi-ideals and the fuzzy bi-ideals coincide. Moreover, we show that if S is an ordered semigroup, then a fuzzy subset f is a fuzzy quasi-ideal of S if and only if there exist a fuzzy right ideal g and a fuzzy left ideal h of S such that $f = g \cap h$.

Following the terminology given by L.A. Zadeh, if $(S, \cdot, \leqslant)$ is an ordered groupoid, we say that f is a fuzzy subset of S (or a fuzzy set in S) if S is a mapping of S into the real closed interval [0,1] (cf. [5]). For $a \in S$, we define $A_a = \{(y, z) \in S \times S \mid a \leqslant yz\}$. For two fuzzy subsets f and g of S, we define the multiplication of f and g as the fuzzy subset of S defined by:

$$(f \circ g)(a) = \begin{cases} \sup\limits_{(y,z)\in A_a} \{\min\{f(y), g(z)\}\} & \text{if} \quad A_a \neq \emptyset, \\ 0 & \text{if} \quad A_a = \emptyset \end{cases}$$

and in the set of all fuzzy subsets of S we define the order relation as follows: $f \subseteq g$ if and only if $f(x) \leqslant g(x)$ for all $x \in S$. Finally for two fuzzy subsets f and g of S we define the operations $f \cap g$ and $f \cup g$ as the fuzzy subsets of S defined by:

$$(f \cap g)(x) = \min\{f(x), g(x)\} \text{ and } (f \cup g)(x) = \max\{f(x), g(x)\}.$$

For an ordered groupoid S, the fuzzy subset 1 of S is defined by $1(x) = 1$ for all $x \in S$. If $F(S)$ is the set of fuzzy subsets of S, it is clear that the fuzzy subset 1 of S is the greatest element of the ordered set $(F(S), \subseteq)$. Moreover, as we have already seen in [6], if S is an ordered groupoid (resp. ordered semigroup), then the set $F(S)$ of all fuzzy subsets of S with the multiplication $\circ$ and the order $\subseteq$ on S defined above is an ordered groupoid (resp. ordered semigroup) as well.

2. Main results

Definition 1. (cf. [5]) Let $(S, \cdot, \leqslant)$ be an ordered groupoid. A fuzzy subset f of S is called a *fuzzy right ideal* (resp. *fuzzy left ideal*) of S if

(1) $f(xy) \geqslant f(x)$ (resp. $f(xy) \geqslant f(y)$) for every $x, y \in S$ and

(2) $x \leqslant y$ implies $f(x) \geqslant f(y)$.

Definition 2. (cf. [5]) Let $(S, \cdot, \leqslant)$ be an ordered groupoid. A fuzzy subset f of S is called a *fuzzy quasi-ideal* of S if

(1) $(f \circ 1) \cap (1 \circ f) \subseteq f$ and

(2) $x \leqslant y$ implies $f(x) \geqslant f(y)$.

Definition 3. Let $(S, \cdot, \leqslant)$ be an ordered semigroup. A fuzzy subset f of S is called a *fuzzy bi-ideal* of S if the following assertions are satisfied:

(1) $f(xyz) \geqslant \min\{f(x), f(z)\}$ for all $x, y, z \in S$ and

(2) $x \leqslant y$ implies $f(x) \geqslant f(y)$.

Proposition 1. *If $(S, \cdot, \leqslant)$ is an ordered groupoid, then the fuzzy right (resp. left) ideals of S are fuzzy quasi-ideals of S.*

Proof. Let f be a fuzzy right ideal of S and $x \in S$. First of all,

$$((f \circ 1) \cap (1 \circ f))(x) = \min\{(f \circ 1)(x), (1 \circ f)(x)\}.$$

If $A_x = \emptyset$ then we have $(f \circ 1)(x) = 0 = (1 \circ f)(x)$ and, since f is a fuzzy right ideal of S, we have $\min\{(f \circ 1)(x), (1 \circ f)(x)\} = 0 \leqslant f(x)$.

Let $A_x \neq \emptyset$. Then

$$(f \circ 1)(x) = \sup_{(u,v) \in A_x} \{\min\{f(u), 1(v)\}\}.$$

On the other hand,

$$f(x) \geqslant \min\{f(u), 1(v)\} \ \ \forall\, (u,v) \in A_x.$$

Indeed, if $(u,v) \in A_x$, then $x \leqslant uv$, $f(x) \geqslant f(uv) \geqslant f(u) = \min\{f(u), 1(v)\}$. Hence we have

$$\begin{aligned} f(x) &\geqslant \sup_{(u,v) \in A_x} \{\min\{f(u), 1(v)\}\} = (f \circ 1)(x) \\ &\geqslant \min\{(f \circ 1)(x), (1 \circ f)(x)\} \\ &= ((f \circ 1) \cap (1 \circ f))(x). \end{aligned}$$

Therefore f is a fuzzy quasi-ideal of S. □

Proposition 2. *If $(S, \cdot, \leqslant)$ is an ordered semigroup, then the fuzzy quasi-ideals are fuzzy bi-ideals of S.*

Proof. Let f be a fuzzy quasi-ideal of S and $x, y, z \in S$. Then we have

$$f(xyz) \geqslant ((f \circ 1) \cap (1 \circ f))(xyz) = \min\{(f \circ 1)(xyz), (1 \circ f)(xyz)\}.$$

Since $(x, yz) \in A_{xyz}$, we have

$$(f \circ 1)(xyz) = \sup_{(u,v) \in A_{xyz}} \{\min\{f(u), 1(v)\}\} \geqslant \min\{f(x), 1(yz)\} = f(x).$$

Since $(xy, z) \in A_{xyz}$, we have

$$(1 \circ f)(xyz) = \sup_{(u,v) \in A_{xyz}} \{\min\{1(u), f(v)\}\} \geqslant \min\{1(xy), f(z)\} = f(z).$$

Thus we have

$$f(xyz) \geqslant \min\{(f \circ 1)(xyz), (1 \circ f)(xyz)\} \geqslant \min\{f(x), f(z)\}.$$

So f is a fuzzy bi-ideal of S. $\square$

Proposition 3. *In a regular ordered semigroup S, the fuzzy quasi-ideals and the fuzzy bi-ideals coincide.*

Proof. Let f be a fuzzy bi-ideal of S and $x \in S$. We will prove that

$$((f \circ 1) \cap (1 \circ f))(x) \leqslant f(x). \tag{1}$$

First of all, we have

$$((f \circ 1) \cap (1 \circ f))(x) = \min\{(f \circ 1)(x), (1 \circ f)(x)\}.$$

If $A_x = \emptyset$ then, as we have already seen in Proposition 1, condition (1) is satisfied.

Let $A_x \neq \emptyset$. Then

$$(f \circ 1)(x) = \sup_{(z,w) \in A_x} \{\min\{f(z), 1(w)\}\} \tag{2}$$

$$(1 \circ f)(x) = \sup_{(u,v) \in A_x} \{\min\{1(u), f(v)\}\} \tag{3}$$

Let $(f \circ 1)(x) \leqslant f(x)$. Then we have

$$\begin{aligned} f(x) &\geqslant (f \circ 1)(x) \geqslant \min\{(f \circ 1)(x), (1 \circ f)(x)\} \\ &= ((f \circ 1) \cap (1 \circ f))(x), \end{aligned}$$

and condition (1) is satisfied.

Let $(f \circ 1)(x) > f(x)$. Then, by (2), there exists $(z, w) \in A_x$ such that

$$\min\{f(z), 1(w)\} > f(x) \tag{4}$$

(otherwise $f(x) \leqslant (f \circ 1)(x)$, which is impossible).

Since $(z, w) \in A_x$, we have $z, w \in S$ and $x \leqslant zw$. Similarly, from $\min\{f(z), 1(w)\} = f(z)$, by (4), we obtain

$$f(z) > f(x). \tag{5}$$

We will prove that $(1 \circ f)(x) \leqslant f(x)$, Then

$$\min\{(f \circ 1)(x), (1 \circ f)(x)\} \leqslant (1 \circ f)(x) \leqslant f(x),$$

so $((f \circ 1) \cap (1 \circ f))(x) \leqslant f(x)$, and condition (1) is satisfied.

By (3), it is enough to prove that

$$\min\{1(u), f(v)\} \leqslant f(x) \quad \forall \ (u,v) \in A_x.$$

Let $(u,v) \in A_x$. Then $x \leqslant uv$ for some $u,v \in S$. Since S is regular, there exists $s \in S$ such that $x \leqslant xsx$. Then $x \leqslant zwsuv$. Then, since f is a fuzzy bi-ideal of S, we have

$$f(x) \geqslant f(zwsuv) \geqslant \min\{f(z), f(v)\}.$$

If $\min\{f(z), f(v)\} = f(z)$, then $f(z) \leqslant f(x)$ which is impossible by (5). Thus we have $\min\{f(z), f(v)\} = f(v)$, then $f(x) \geqslant f(v) = \min\{1(u), f(v)\}$. $\square$

In the following, using the usual definitions of ideals mentioned above, we show that the fuzzy quasi-ideals of an ordered semigroup are just intersections of fuzzy right and fuzzy left ideals.

Lemma 1. *Let $(S, \cdot, \leqslant)$ be an ordered semigroup and f a fuzzy subset of S. Then we have the following:*

(i) $(1 \circ f)(xy) \geqslant f(y)$ *for all* $x, y \in S$,

(ii) $(1 \circ f)(xy) \geqslant (1 \circ f)(y)$ *for all* $x, y \in S$.

Proof. (i) Let $x, y \in S$. Since $(x,y) \in A_{xy}$, we have

$$(1 \circ f)(xy) = \sup_{(w,z) \in A_{xy}} \{\min\{1(w), f(z)\}\} \geqslant \min\{1(x), f(y)\} = f(y).$$

(ii) Let $x, y \in S$. If $A_y = \emptyset$, then $(1 \circ f)(y) = 0$. Since $1 \circ f$ is a fuzzy subset of S, we have $(1 \circ f)(xy) \geqslant 0 = (1 \circ f)(y)$.

Let now $A_y \neq 0$. Then

$$(1 \circ f)(y) = \sup_{(w,z) \in A_y} \{\min\{1(w), f(z)\}\}.$$

On the other hand,

$$(1 \circ f)(xy) \geqslant \min\{1(w), f(z)\} \quad \forall \ (w,z) \in A_y. \tag{6}$$

Indeed, let $(w,z) \in A_y$. Since $(x,y) \in A_{xy}$, we have

$$(1 \circ f)(xy) = \sup_{(s,t) \in A_{xy}} \{\min\{1(s), f(t)\}\}.$$

Since $(w,z) \in A_y$, we have $y \leqslant wz$, then $xy \leqslant xwz$, and $(xw,z) \in A_{xy}$. Hence we have

$$(1 \circ f)(xy) \geqslant \min\{1(xw), f(z)\} = f(z) = \min\{1(w), f(z)\}.$$

By (6), we have

$$(1 \circ f)(xy) \geqslant \sup_{(w,z)\in A_y} \{\min\{1(w), f(z)\}\} = (1 \circ f)(y). \qquad \square$$

In a similar way we prove the following:

Lemma 2. *Let $(S, \cdot, \leqslant)$ be an ordered semigroup and f a fuzzy subset of S. Then we have the following:*

(i) $(f \circ 1)(xy) \geqslant f(x)$ *for all* $x, y \in S$,

(ii) $(f \circ 1)(xy) \geqslant (f \circ 1)(x)$ *for all* $x, y \in S$.

Lemma 3. *Let $(S, \cdot, \leqslant)$ be an ordered semigroup, f a fuzzy subset of S and $x \leqslant y$. Then we have*

$$(1 \circ f)(x) \geqslant (1 \circ f)(y).$$

Proof. If $A_y = \emptyset$, then $(1 \circ f)(y) = 0$. Since $1 \circ f$ is a fuzzy subset of S, we have $(1 \circ f)(x) \geqslant 0$, then $(1 \circ f)(x) \geqslant (1 \circ f)(y)$.

Let $A_y \neq \emptyset$. Then

$$(1 \circ f)(y) = \sup_{(w,z)\in A_y} \{\min\{1(w), f(z)\}\} = \sup_{(w,z)\in A_y} \{f(z)\}.$$

On the other hand,

$$(1 \circ f)(x) \geqslant f(z) \quad \forall\ (w,z) \in A_y. \tag{7}$$

Indeed, let $(w,z) \in A_y$. Since $x \leqslant y \leqslant wz$, we have $(w,z) \in A_x$. Then

$$(1 \circ f)(xy) = \sup_{(s,t)\in A_{xy}} \{\min\{1(s), f(t)\}\} \geqslant \min\{1(w), f(z)\} = f(z).$$

Thus, by (7), we have

$$(1 \circ f)(x) \geqslant \sup_{(w,z)\in A_y} \{f(z)\} = (1 \circ f)(y). \qquad \square$$

In a similar way we prove the following:

Lemma 4. *Let $(S, \cdot, \leqslant)$ be an ordered semigroup, f a fuzzy subset of S and $x \leqslant y$. Then*

$$(f \circ 1)(x) \geqslant (f \circ 1)(y).$$

Lemma 5. *Let $(S, \cdot, \leqslant)$ be an ordered semigroup and f a fuzzy subset of S. Then*

$$(f \cup (1 \circ f))(xy) \geqslant (f \cup (1 \circ f))(y) \quad \forall\ x, y \in S.$$

Proof. Let $x, y \in S$. Since $1 \circ f \subseteq f \cup (1 \circ f)$, we have

$$(f \cup (1 \circ f))(xy) \geqslant (1 \circ f)(xy).$$

By Lemma 1, $(1 \circ f)(xy) \geqslant f(y)$ and $(1 \circ f)(xy) \geqslant (1 \circ f)(y)$, so we have

$$(1 \circ f)(xy) \geqslant \max\{f(y), (1 \circ f)(y)\} = (f \cup (1 \circ f))(y).$$

Therefore $(f \cup (1 \circ f))(xy) \geqslant (f \cup (1 \circ f))(y)$. □

In a similar way we prove the following:

Lemma 6. *Let $(S, \cdot, \leqslant)$ be an ordered semigroup and f a fuzzy subset of S. Then*

$$(f \cup (f \circ 1))(xy) \geqslant (f \cup (f \circ 1))(x) \quad \forall\ x, y \in S.$$

Lemma 7. *Let $(S, \cdot, \leqslant)$ be an ordered semigroup and f a fuzzy subset of S such that for all $x, y \in S$ such that $x \leqslant y$, we have $f(x) \geqslant f(y)$. Then the fuzzy subset $f \cup (1 \circ f)$ is a fuzzy left ideal of S.*

Proof. Let $x, y \in S$. By Lemma 5, we have $(f \cup (1 \circ f))(xy) \geqslant (f \cup (1 \circ f))(y)$. Let now $x \leqslant y$. Then $(f \cup (1 \circ f))(x) \geqslant (f \cup (1 \circ f))(y)$. Indeed: Since f is a fuzzy subset of S and $x \leqslant y$, by Lemma 3, we get $(1 \circ f)(x) \geqslant (1 \circ f)(y)$ and, by hypothesis, $f(x) \geqslant f(y)$. Then

$$\begin{aligned}(f \cup (1 \circ f))(x) &= \max\{f(x), (1 \circ f)(x)\} \geqslant \max\{f(y), (1 \circ f)(y)\} \\ &= (f \cup (1 \circ f))(y).\end{aligned}$$

□

In a similar way we prove the following:

Lemma 8. *Let $(S, \cdot, \leqslant)$ be an ordered semigroup and f a fuzzy subset of S such that for all $x, y \in S$ such that $x \leqslant y$, we have $f(x) \geqslant f(y)$. Then the fuzzy subset $f \cup (f \circ 1)$ is a fuzzy right ideal of S.*

Lemma 9. *If a, b, c are real numbers, then*

(*i*) $\min\{a, \max\{b, c\}\} = \max\{\min\{a, b\}, \min\{a, c\}\}$ *and*

(*ii*) $\max\{a, \min\{b, c\}\} = \min\{\max\{a, b\}, \max\{a, c\}\}$.

Proof. (*i*) Let $a \leqslant \max\{b, c\}$. Then $\min\{a, \max\{b, c\}\} = a$. If $b \leqslant c$, then $\max\{b, c\} = c$, $a \leqslant c$, $\min\{a, b\} \leqslant a = \min\{a, c\}$, and

$$\max\{\min\{a, b\}, \min\{a, c\}\} = a,$$

so condition (*i*) is satisfied. If $c < b$ then, in a similar way we prove that condition (*i*) is satisfied.

Let now $a > \max\{b, c\}$. Then $\min\{a, \max\{b, c\}\} = \max\{b, c\} \geqslant b$, $a > b$, and $\min\{a, b\} = b$. Similarly $\min\{a, c\} = c$. Then

$$\max\{\min\{a, b\}, \min\{a, c\}\} = \max\{b, c\},$$

and condition (*i*) is satisfied.

The proof of (*ii*) is similar. $\square$

Lemma 10. *Let S be an ordered semigroup and f, g, h fuzzy subsets of S. Then*

$$f \cap (g \cup h) = (f \cap g) \cup (f \cap h).$$

Proof. Indeed,

$$\begin{aligned}
(f \cap (g \cup h))(x) &= \min\{f(x), (g \cup h)(x)\} \\
&= \min\{f(x), \max\{g(x), h(x)\}\} \\
&= \max\{\min\{f(x), g(x)\}, \min\{f(x), h(x)\}\} \text{ (by Lemma 9)} \\
&= \max\{f \cap g)(x),\ (f \cap h)(x)\} \\
&= ((f \cap g) \cup (f \cap h))(x). \qquad \square
\end{aligned}$$

By Lemma 10, we have the following:

Corollary 1. *If S is an ordered semigroup, then the set of all fuzzy subsets of S is a distributive lattice.*

Proposition 4. *Let $(S, \cdot, \leqslant)$ be and ordered semigroup. A fuzzy subset f of S is a fuzzy quasi-ideal of S if and only if there exist a fuzzy right ideal g and a fuzzy left ideal h of S such that $f = g \cap h$.*

Proof. $\Longrightarrow$. By Lemmas 7 and 8, $f \cup (1 \circ f)$ is a fuzzy left ideal and $f \cup (f \circ 1)$ is a fuzzy right ideal of S. Moreover, we have

$$f = (f \cup (1 \circ f)) \cap (f \cup (f \circ 1)).$$

In fact, by Corollary 1, we have

$$\begin{aligned}(f \cup (1 \circ f)) \cap (f \cup (f \circ 1)) &= ((f \cup (1 \circ f)) \cap f) \cup ((f \cup (1 \circ f)) \cap (f \circ 1)) \\ &= (f \cap f) \cup ((1 \circ f) \cap f) \cup (f \cap (f \circ 1)) \cup ((1 \circ f) \cap (f \circ 1)) \\ &= f \cup ((1 \circ f) \cap f) \cup (f \cap (f \circ 1)) \cup ((1 \circ f) \cap (f \circ 1)).\end{aligned}$$

Since f is a fuzzy quasi-ideal of S, we have $(f \circ 1) \cap (1 \circ f) \subseteq f$. Besides, $(1 \circ f) \cap f \subseteq f$ and $f \cap (f \circ 1) \subseteq f$. Hence

$$(f \cup (1 \circ f)) \cap (f \cup (f \circ 1)) = f.$$

$\Longleftarrow$. Let $x \in S$. Then

$$((f \circ 1) \cap (1 \circ f))(x) \leqslant f(x) \tag{8}$$

In fact, $((f \circ 1) \cap (1 \circ f))(x) = \min\{(f \circ 1)(x), (1 \circ f)(x)\}$. If $A_x = \emptyset$, then $(f \circ 1)(x) = 0 = (1 \circ f)(x)$. So, in this case condition (8) is satisfied.

If $A_x \neq \emptyset$, then

$$(f \circ 1)(x) = \sup_{(y,z) \in A_x} \{\min\{f(y), 1(z)\}\} = \sup_{(y,z) \in A_x} \{f(y)\}. \tag{9}$$

We have

$$f(y) \leqslant h(x) \quad \forall \ (y,z) \in A_x. \tag{10}$$

Indeed, for $(y,z) \in A_x$ we have $x \leqslant yz$ and $h(x) \geqslant h(yz) \geqslant h(y)$ because h is a fuzzy left ideal of S.

Thus, applying (10) to (9), we obtain

$$(f \circ 1)(x) = \sup_{(y,z) \in A_x} \{f(y)\} \leqslant h(x).$$

In a similar way, we get $(1 \circ f)(x) \leqslant g(x)$. Hence

$$\begin{aligned}((f \circ 1) \cap (1 \circ f))(x) &= \min\{(f \circ 1)(x), (1 \circ f)(x)\} \\ &\leq \min\{h(x), g(x)\} \\ &= (h \cap g)(x) \\ &= f(x),\end{aligned}$$

which completes the proof of (8). □

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University of Athens
Department of Mathematics
157 84 Panepistimiopolis
Athens, Greece
E-mail: nkehayop@math.uoa.gr

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