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Fuzzy (strong) congruence relations on hypergroupoids and hyper BCK-algebras

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Abstract

We define the concept of fuzzy (strong) congruence relations on hypergroupoids and hyper BCK-algebras and construct a quotient hyperstructure on a hypergroupoid. In particular, we prove that if H is a (semi) hypergroup and R is a fuzzy (strong) congruence relation on H, then H/R is a (semi) group. Finally, by considering the notion of a hyper BCK-algebra, we construct a quotient hyper BCK-algebra.

1. Introduction

The notion of a hyperstructure was introduced by F. Marty [13] in 1934 at the 8$^{\text{th}}$ congress of Scandinavian Mathematicians and the notion of a fuzzy set was introduced by Zadeh [16] in 1965. The study of BCK-algebras was initiated by Y. Imai and K. Iséki [7] in 1966 as a generalization of the concept of the set-theoretic difference and propositional calculi. In this paper, we use the notion of a fuzzy set and define the concept of a fuzzy (strong) congruence relation on hypergroupoids and hyper BCK-algebras and we obtain some results as mentioned in the abstract.

2. Fuzzy (strong) congruence relations

Definition 1. By a *hypergroupoid* we mean a nonempty set H endowed with a binary hyperoperation "$\circ$" (i.e., a function $\circ : H \times H \longrightarrow P(H)$,

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where $P(H)$ is the set of all nonempty subsets of H.)

Let Θ be a binary relation on a hypergroupoid H and $A, B \subseteq H$. Then:

(a) $A\Theta B$ means that there exist $a \in A$ and $b \in B$ such that $a\Theta b$,

(b) $A\overline{\Theta}B$ means that for $a \in A$ there exists $b \in B$ and for $b \in B$ there exists $a \in A$ such that $a\Theta b$,

(c) $A\overline{\overline{\Theta}}B$ means that $a\Theta b$ for each $a \in A$ and for $b \in B$,

(d) Θ is *left* (*resp. right*) *compatible* if $x\Theta y$ implies $a \circ x\overline{\Theta}a \circ y$ (resp. $x \circ a\overline{\Theta}y \circ a$) for all $x, y, a \in H$,

(e) Θ is *strong left* (resp. *right*) *compatible* if $x\Theta y$ implies $a \circ x\overline{\overline{\Theta}}a \circ y$ (resp. $x \circ a\overline{\overline{\Theta}}y \circ a$),

(f) Θ is (resp. *strong*) *compatible* if it is both (resp. strong) left and right compatible,

(g) Θ is a (resp. *strong*) *congruence* relation on H if it is a (resp. strong) compatible equivalence relation on H.

Definition 2. Let H be a nonempty set and R be a fuzzy relation on H. We say that R satisfies the *sup property* if for every subset T of H there exists $(u, v) \in T^2$ such that $\sup\limits_{(x,y)\in T^2} R(x, y) = R(u, v)$. R is said to be a *fuzzy equivalence relation* if

$R(x, x) = \bigvee\limits_{(y,z)\in H^2} R(y, z)$, (*fuzzy reflexive*)

$R(y, x) = R(x, y)$, (*fuzzy symmetric*)

$R(x, y) \geqslant \bigvee\limits_{z\in H} (R(x, z) \wedge R(z, y))$, (*fuzzy transitive*).

Definition 3. Let H be a nonempty set and R be a fuzzy relation on H. Then, for all $\alpha \in [0, 1]$, the *α-level subset* and *strong α-level subset* of R respectively, is defined as follows:

$$R^{\alpha} = \{(x, y) \in H^2 : R(x, y) \geqslant \alpha\}$$
$$R^{\alpha^{>}} = \{(x, y) \in H^2 : R(x, y) > \alpha\}$$

Lemma 1. *Let R be a fuzzy relation on a nonempty set H. Then:*

$$R^{\alpha} = \bigcap_{\beta\in[0,\alpha)} R^{\beta^{>}} \quad \text{and} \quad R^{\alpha^{>}} = \bigcup_{\beta\in(\alpha,1]} R^{\beta}$$

for all $\alpha \in [0, 1]$.

Proof. Let $\alpha \in [0,1]$ and $\beta < \alpha$. Then $R^\alpha \subseteq R^\beta$ and so $R^\alpha \subseteq \bigcap_{\beta\in[0,\alpha)} R^\beta$. Conversely, let $\varepsilon > 0$ be given and $(x,y) \in \bigcap_{\beta\in[0,\alpha)} R^\beta$. Then $R(x,y) \geqslant \alpha-\varepsilon$, which implies that $R(x,y) \geqslant \alpha$ and hence $(x,y) \in R^\alpha$. Similarly, the other part can be proved. □

Theorem 1. (cf. [3]) *Let R be a fuzzy relation on nonempty set H. Then the following properties are equivalent:*

(i) *R is a fuzzy equivalence relation on H,*

(ii) *$R^\alpha \neq \emptyset$ is an equivalence relation on H for all $\alpha \in [0,1]$,*

(iii) *$R^{\alpha^>} \neq \emptyset$ is an equivalence relation on H for all $\alpha \in [0,1]$.* □

Definition 4. Fuzzy relation R on hypergroupoid H is said to be

(i) *fuzzy left compatible* iff

$$\Big(\bigwedge_{u\in c\circ a}\bigvee_{v\in c\circ b} R(u,v)\Big) \wedge \Big(\bigwedge_{v\in c\circ b}\bigvee_{u\in c\circ a} R(u,v)\Big) \geqslant R(a,b) \quad \forall a,b,c \in H,$$

and *fuzzy right compatible* iff

$$\Big(\bigwedge_{u\in a\circ c}\bigvee_{v\in b\circ c} R(u,v)\Big) \wedge \Big(\bigwedge_{v\in b\circ c}\bigvee_{u\in a\circ c} R(u,v)\Big) \geqslant R(a,b) \quad \forall a,b,c \in H,$$

(ii) *fuzzy strong left compatible* iff

$$\bigwedge_{u\in c\circ a, v\in c\circ b} R(u,v) \geqslant R(a,b) \quad \forall a,b,c \in H.$$

and *fuzzy strong right compatible* iff

$$\bigwedge_{u\in a\circ c, v\in b\circ c} R(u,v) \geqslant R(a,b), \quad \forall a,b,c \in H$$

Clearly, every fuzzy strong left (resp. right) compatible relation is a fuzzy left (resp. right) compatible relation, but the converse is not true.

Theorem 2. *Let R be a fuzzy relation on a hypergroupoid H that satisfies the sup property. Then the following statements are equivalent:*

(i) *R is fuzzy left (resp. right) compatible,*

(ii) $R^{\alpha} \neq \emptyset$ *is left (resp. right) compatible, for all* $\alpha \in [0,1]$,

(iii) $R^{\alpha^{>}} \neq \emptyset$ *is left (resp. right) compatible, for all* $\alpha \in [0,1]$.

Proof. We prove only for "left" compatible, the other cases can be proved in a similar way.

$(i) \Longrightarrow (ii)$ Let $R^{\alpha} \neq \emptyset$. For $\alpha \in [0,1]$ and $x, y, a \in H$ let $xR^{\alpha}y$ and $u \in x \circ a$. Since by (i), R is fuzzy left compatible, then

$$\Big(\bigwedge_{u\in a\circ x}\bigvee_{v\in a\circ y} R(u,v)\Big) \wedge \Big(\bigwedge_{v\in a\circ y}\bigvee_{u\in a\circ x} R(u,v)\Big) \geqslant R(x,y) \geqslant \alpha$$

and so

$$\bigwedge_{u\in a\circ x}\bigvee_{v\in a\circ y} R(u,v) \geqslant \alpha \quad \text{and} \quad \bigwedge_{v\in a\circ y}\bigvee_{u\in a\circ x} R(u,v) \geqslant \alpha.$$

Hence, for all $u \in a\circ x$, $\bigvee_{v\in a\circ y} R(u,v) \geqslant \alpha$ and for all $v \in a\circ y$, $\bigvee_{u\in a\circ x} R(u,v) \geqslant \alpha$. Since, R satisfies the sup property, then there exist $v_0 \in a\circ y$ and $u_0 \in a \circ x$ such that $R(u, v_0) = \bigvee_{v\in a\circ y} R(u,v) \geqslant \alpha$ for all $u \in a\circ x$ and $R(u_0, v) = \bigvee_{u\in a\circ x} R(u,v) \geqslant \alpha$ for all $v \in a \circ y$. Hence, $(u, v_0) \in R^{\alpha}$ and $(u_0, v) \in R^{\alpha}$, for all $u \in a\circ x$ and $v \in a\circ y$. This implies that R^{α} is left compatible.

$(ii) \Longrightarrow (iii)$ Let $R^{\alpha^{>}} \neq \emptyset$, for $\alpha \in [0,1]$ and $x, y, a \in H$ be such that $xR^{\alpha^{>}}y$ and $u \in a\circ x$. Thus by Lemma 1, there exists $\beta \in (\alpha, 1]$ such that $xR^{\beta}y$. Since R^{β} is left compatible, then $a\circ x\bar{R}^{\beta} a\circ y$, and so there exists $v \in a\circ y$ such that $uR^{\beta}v$. Thus, $R(u,v) \geqslant \beta > \alpha$. This shows that $uR^{\alpha^{>}}v$. Similarly, if $v \in a\circ y$, then there exists $u \in a \circ x$ such that $R(u,v) > \alpha$ and so $uR^{\alpha^{>}}v$. Therefore, $R^{\alpha^{>}}$ is left compatible.

$(iii) \Longrightarrow (i)$ Suppose that $x, y, a \in H$ are such that $R(x,y) = \alpha$. Then by Lemma 1, for all $\beta \in [0, \alpha)$ we have $xR^{\beta^{>}}y$. So, by (iii) we have $a\circ x\bar{R}^{\beta^{>}} a\circ y$, and so for all $u \in a\circ x$ there exists $v \in a\circ y$ such that $uR^{\beta^{>}}v$ i.e., $R(u,v) > \beta$. This implies that $\bigwedge_{u\in a\circ x}\bigvee_{v\in a\circ y} R(u,v) > \beta$, for all $\beta \in [0,\alpha)$. Similarly, for all $v \in a\circ y$ there exists $u \in a\circ x$ such that $uR^{\beta^{>}}v$ and so $\bigwedge_{v\in a\circ y}\bigvee_{u\in a\circ x} R(u,v) > \beta$, for all $\beta \in [0,\alpha)$. Hence, $\bigwedge_{u\in a\circ x}\bigvee_{v\in a\circ y} R(u,v) \geqslant \alpha =$

$R(x,y)$ and $\bigwedge\limits_{v\in a\circ y}\bigvee\limits_{u\in a\circ x} R(u,v) \geqslant \alpha = R(x,y)$, which implies

$$\Big(\bigwedge_{u\in a\circ x}\bigvee_{v\in a\circ y} R(u,v)\Big) \wedge \Big(\bigwedge_{v\in a\circ y}\bigvee_{u\in a\circ x} R(u,v)\Big) \geqslant R(x,y).$$

Thus, R is fuzzy left compatible. $\square$

Theorem 3. *For a fuzzy relation R on a hypergroupoid H satisfying the sup property the following properties are equivalent:*

(i) *R is fuzzy strong left (resp. right) compatible,*

(ii) *$R^{\alpha} \neq \emptyset$ is strong left (resp. right) compatible, for all $\alpha \in [0,1]$,*

(iii) *$R^{\alpha^{>}} \neq \emptyset$ is strong left (resp. right) compatible, for all $\alpha \in [0,1]$.*

Proof. $(i) \Longrightarrow (ii)$ Let R be a fuzzy strong left compatible relation on H, $a \in H$ and $x, y \in H$ be such that $xR^{\alpha}y$, for some $\alpha \in [0,1]$. Then for all $u \in a\circ x$ and $v \in a \circ y$,

$$R(u,v) \geqslant \bigwedge_{w\in a\circ x,\, w'\in a\circ y} R(w,w') \geqslant R(x,y) \geqslant \alpha$$

that is $uR^{\alpha}v$. This shows that R^{α} is a strong left compatible relation on H.

$(ii) \Longrightarrow (iii)$ Let $R^{\alpha} \neq \emptyset$ be a strong left compatible relation on H, for $\alpha \in L$, $x, y \in H$ be such that $xR^{\alpha^{>}}y$ and $a \in H$. Then, there exists $\beta \in (\alpha, 1]$ such that $xR^{\beta}y$ and so by (ii), $a\circ x\,\overline{\overline{R}}^{\beta}a\circ y$. This implies that for all $u \in a\circ x$ and for all $v \in a\circ y$, $R(u,v) \geqslant \beta > \alpha$ and so $uR^{\alpha^{>}}v$. Hence, $a\circ x\,\overline{\overline{R}}^{\alpha^{>}}a\circ y$, which implies that $R^{\alpha^{>}}$ is a strong left compatible relation on H.

$(iii) \Longrightarrow (i)$ Let $a \in H$ and $x, y \in H$ be such that $R(x,y) = \alpha$, for $\alpha \in [0,1]$. Then, by Lemma 1, for all $\beta \in [0,\alpha)$ we have $xR^{\beta^{>}}y$ and so by (iii), $a\circ x\,\overline{\overline{R}}^{\beta^{>}}a\circ y$; i.e., for all $u \in a\circ x$ and for all $v \in a\circ y$, $uR^{\beta^{>}}v$ i.e., $R(u,v) > \beta$, for all $\beta \in [0,\alpha)$. Thus $R(u,v) \geqslant \alpha$, and hence

$$\bigwedge_{u\in a\circ x, v\in a\circ y} R(u,v) \geqslant \alpha = R(x,y).$$

Therefore, R is a fuzzy strong left compatible relation on H. $\square$

Definition 5. Let R be a fuzzy relation on a hypergroupoid H. Then, R is said to be

(i) *fuzzy compatible* if

$$\Big(\bigwedge_{u\in a\circ c}\bigvee_{v\in b\circ d} R(u,v)\Big)\wedge\Big(\bigwedge_{v\in b\circ d}\bigvee_{u\in a\circ c} R(u,v)\Big) \geqslant R(a,b)\wedge R(c,d),\ \ \forall a,b,c,d \in H,$$

(ii) *fuzzy strong compatible* if

$$\bigwedge_{u\in a\circ c, v\in b\circ d} R(u,v) \geqslant R(a,b)\wedge R(c,d), \quad \forall a,b,c,d\in H.$$

Definition 6. By a *fuzzy* (resp. *strong*) *congruence relation* we mean a fuzzy (resp. strong) compatible equivalence relation.

Theorem 4. *A fuzzy relation R is a (resp. strong) fuzzy congruence relation if and only if it is both a (resp. strong) left and right fuzzy compatible equivalence relation.*

Proof. Let R be a fuzzy congruence relation on H and $a,x,y\in H$. Then

$$\Big(\bigwedge_{u\in a\circ x}\bigvee_{v\in a\circ y} R(u,v)\Big)\wedge\Big(\bigwedge_{v\in a\circ y}\bigvee_{u\in a\circ x} R(u,v)\Big) \geqslant R(x,y)\wedge R(a,a) = R(x,y)$$

which shows that R is a fuzzy left compatible relation on H. Similarly, it can be shown that R is a fuzzy right compatible relation on H.

Conversely, suppose that R is both a fuzzy left and right compatible equivalence relation on H and $a,b,c,d\in H$. Now, for every $u\in a\circ c$ and every $v\in b\circ d$, by transitivity of R, we have

$$R(u,v)\geqslant \bigvee_{y\in H}(R(u,y)\wedge R(y,v)) \geqslant R(u,w)\wedge R(w,v), \quad \forall w\in b\circ c$$

and so

$$R(u,v)\geqslant\Big(\bigvee_{w\in b\circ c} R(u,w)\Big)\wedge\Big(\bigvee_{w\in b\circ c} R(w,v)\Big).$$

Thus

$$\bigvee_{v\in b\circ d} R(u,v)\geqslant\bigwedge_{v\in b\circ d} R(u,v)\geqslant\Big(\bigvee_{w\in b\circ c} R(u,w)\Big)\wedge\Big(\bigwedge_{v\in b\circ d}\bigvee_{w\in b\circ c} R(w,v)\Big)$$

and hence

$$\begin{aligned}\bigwedge_{u\in a\circ c}\bigvee_{v\in b\circ d} R(u,v) &\geqslant \Big(\bigwedge_{u\in a\circ c}\bigvee_{w\in b\circ c} R(u,w)\Big)\wedge\Big(\bigwedge_{v\in b\circ d}\bigvee_{w\in b\circ c} R(w,v)\Big)\\ &\geqslant R(a,b)\wedge R(c,d).\end{aligned}$$

Therefore, R is a fuzzy congruence relation on H.

Now, let R be a fuzzy strong congruence relation on H and $x,y,a\in H$. Then,

$$\bigwedge_{u\in a\circ x,\ v\in a\circ y} R(u,v)\geqslant R(a,a)\wedge R(x,y)=R(x,y).$$

Hence, R is fuzzy strong left compatible. The proof for "fuzzy strong right" is similar.

Conversely, let R be a fuzzy strong left and right compatible, $a,b,c,d\in H$. Then,

$$R(a,b)\leqslant \bigwedge_{u\in a\circ c,\ v\in b\circ c} R(u,v) \quad\text{and}\quad R(c,d)\leqslant \bigwedge_{u\in b\circ c,\ v\in b\circ d} R(u,v)$$

and so

$$R(a,b)\wedge R(c,d)\leqslant\Big(\bigwedge_{u\in a\circ c,\ v\in b\circ c} R(u,v)\Big)\wedge\Big(\bigwedge_{u\in b\circ c,\ v\in b\circ d} R(u,v)\Big).$$

For every $u\in a\circ c$ and $v\in b\circ d$, by transitivity of R, we have

$$\begin{aligned}R(u,v)&\geqslant\bigvee_{y\in H}(R(u,y)\wedge R(y,v))\geqslant R(u,w)\wedge R(w,v),\quad\forall w\in b\circ c\\ &\geqslant(\bigwedge_{u\in a\circ c,\, v\in b\circ c} R(u,v))\wedge(\bigwedge_{w\in b\circ c,\, z\in b\circ d} R(u,v))\geqslant R(a,b)\wedge R(c,d).\end{aligned}$$

Thus R is a fuzzy strong congruence relation on H. □

By Theorems 1, 2, 3 and 4 we have the following corollary.

Corollary 1. *Let R be a fuzzy relation on a hypergroupoid H that satisfies the sup property. Then,*

- (i) *R is a fuzzy congruence relation on H if and only if every nonempty α-level set R^α of R is both left and right compatible equivalence relation,*
- (ii) *R is a fuzzy strong congruence relation on H if and only if every nonempty α-level set R^α of R is both strong left and right compatible equivalence relation on H.* □

Let R be a fuzzy relation on H. For all $x \in H$, define a fuzzy subset μ on H by $\mu_x(y) = R(y, x)$, for all $y \in H$.

Lemma 2. *Let R be a fuzzy equivalence relation on a hypergroupoid H. Then, $\mu_x = \mu_y$ if and only if $R(x, y) = \bigvee_{u,v\in H} R(u, v)$.*

Proof. (i) Let $\mu_x = \mu_y$, for $x, y \in H$. Since, R is fuzzy reflexive, then

$$R(x, y) = \mu_y(x) = \mu_x(x) = R(x, x) = \bigvee_{u,v\in H} R(u, v).$$

Conversely, suppose that $R(x, y) = \bigvee_{u,v\in H} R(u, v)$, for $x, y \in H$ and $w \in H$. Since R is fuzzy symmetric and fuzzy transitive, we obtain

$$\begin{aligned}\mu_x(w) = R(w, x) = R(x, w) &\geqslant R(x, y) \wedge R(y, w)\\ &= \Big(\bigvee\nolimits_{u,v\in H} R(u, v)\Big) \wedge R(y, w) = R(y, w) = \mu_y(w).\end{aligned}$$

Similarly, we can show that $\mu_y(w) \geqslant \mu_x(w)$. Thus, $\mu_x(w) = \mu_y(w)$ and so $\mu_x = \mu_y$. □

Theorem 5. *Let R be a fuzzy congruence relation on H with the sup property and $H/R = \{\mu_x : x \in H\}$. Then $(H/R, \diamond)$ is a hypergroupoid, where binary hyperoperation "$\diamond$" is defined by*

$$\mu_x \diamond \mu_y = \{\mu_z : z \in x \circ y\} = \mu_{x\circ y}.$$

Proof. First, we show that "$\diamond$" is well-defined. Let $\mu_x = \mu_{x'}$ and $\mu_y = \mu_{y'}$, for $\mu_x, \mu_{x'}, \mu_y, \mu_{y'} \in H/R$. Then, by Lemma 2, $R(x, x') = \bigvee_{u,v\in H} R(u, v) = R(y, y')$. Let $\alpha = \bigvee_{u,v\in H} R(u, v)$. Then $xR^\alpha x'$ and $yR^\alpha y'$ and by Corollary 1, R^α is a congruence relation on H, then $x \circ y\bar{R}^\alpha x' \circ y'$. Now, let $\mu_z \in \mu_x \diamond \mu_y = \mu_{x\circ y}$. Then there exists $z' \in x \circ y$ such that $\mu_z = \mu_{z'}$. On the other hand, since $x \circ y\bar{R}^\alpha x' \circ y'$, then there exists $u \in x' \circ y'$ such that $z'R^\alpha u$ and so $R(z', u) \geqslant \alpha = \bigvee_{u,v\in H} R(u, v) \geqslant R(z', u)$. Hence, $R(z', u) = \alpha$. Now, for $w \in H$ we have

$$\begin{aligned}\mu_z(w) = \mu_{z'}(w) = R(w, z') = R(z', w) &\geqslant R(z', u) \wedge R(u, w) = \alpha \wedge R(u, w)\\ = R(u, w) = R(w, u) = \mu_u(w)\end{aligned}$$

and so $\mu_z \geqslant \mu_u$. Similarly $\mu_u \geqslant \mu_z$. Hence, $\mu_z = \mu_u$ and so $\mu_z = \mu_u \in \mu_{x' \circ y'} = \mu_{x'} \diamond \mu_{y'}$, since $u \in x' \circ y'$. Thus $\mu_x \diamond \mu_y \subseteq \mu_{x'} \diamond \mu_{y'}$. Analogously, $\mu_{x'} \diamond \mu_{y'} \subseteq \mu_x \diamond \mu_y$. Thus $\mu_x \diamond \mu_y = \mu_{x'} \diamond \mu_{y'}$. This completes the proof. $\square$

In the following, we briefly give some preliminaries about hypergroups.

Definition 7. (cf. [5]) Let $(H, \circ)$ be a hypergroupoid. Then H is called a *semihypergroup* if "$\circ$" is associative i.e., $(x \circ y) \circ z = x \circ (y \circ z)$, for all $x, y, z \in H$. Moreover, if H is a semihypegroup that satisfies the *reproduction axioms* that is, $x \circ H = H \circ x = H$, for all $x \in H$, then we say that H is a *hypergroup*. Now, let H be a hypergroup. An, element $e \in H$ is called an *identity* if for all $x \in H$, $x \in (x \circ e) \cap (e \circ x)$, an element $a \in H$ is said to be a *scalar identity* if for all $x \in H$, $|a \circ x| = |x \circ a| = 1$. Let H has an identity e, an element $a' \in H$ is said to be an *inverse* of $a \in H$ if $e \in (a \circ a') \cap (a' \circ a)$. H is called *regular* if it has at least one identity and each element has at least one inverse. H is said to be *reversible* if for all $x, y, z \in H$, $y \in a \circ x$ implies that there exists an inverse a' of a such that $x \in a' \circ y$ and $y \in x \circ a$ implies that there exists an inverse a" of a such that $x \in y \circ a$", a hypergroup $(H, \circ)$ is called *canonical* if it is commutative, with a scalar identity, such that every element has an unique inverse and it is reversible.

Theorem 6. *If $(H, \circ)$ is a semihypergroup and R is a fuzzy congruence relation on H, then H/R is a semihypergroup. In particular, if $(H, \circ)$ is a hypergroup then H/R is a hypergroup.*

Proof. Let $\mu_x, \mu_y, \mu_z \in H/R$ and $\mu_u \in (\mu_x \diamond \mu_y) \diamond \mu_z$. Then there exists $\mu_w \in \mu_x \diamond \mu_y$ such that $\mu_u \in \mu_w \diamond \mu_z = \mu_{w \circ z}$ and so there exists $v \in w \circ z$ such that $\mu_u = \mu_v$. But, $v \in w \circ z \subseteq (x \circ y) \circ z = x \circ (y \circ z)$ and so there exists $u' \in y \circ z$ such that $v \in x \circ u'$. Hence, $\mu_u = \mu_v \in \mu_{x \circ u'} = \mu_x \diamond \mu_{u'} \subseteq \mu_x \diamond (\mu_y \diamond \mu_z)$, which shows that $(\mu_x \diamond \mu_y) \diamond \mu_z \subseteq \mu_x \diamond (\mu_y \diamond \mu_z)$. By a similar way, we can show that $\mu_x \diamond (\mu_y \diamond \mu_z) \subseteq (\mu_x \diamond \mu_y) \diamond \mu_z$. Hence, $(\mu_x \diamond \mu_y) \diamond \mu_z = \mu_x \diamond (\mu_y \diamond \mu_z)$, which shows that "$\diamond$" is associative. Therefore, H/R is a semihypergroup.

Now, suppose that $(H, \circ)$ is a hypergroup and $\mu_x \in H/R$. Obviously $\mu_x \diamond H/R \subseteq H/R$. Now, let $\mu_u \in H/R$. Since, $u \in H = x \circ H$, then there exists $y \in H$ such that $u \in x \circ y$ and so $\mu_u \in \mu_{x \circ y} = \mu_x \diamond \mu_y \subseteq \mu_x \diamond H/R$. Hence, $H/R \subseteq \mu_x \diamond H/R$ and so $\mu_x \diamond H/R = H/R$. Similarly, $H/R \diamond \mu_x = H/R$ and hence H/R satisfies the reproduction axioms. Therefore, H/R is a hypergroup. $\square$

Theorem 7. *Let $(H, \circ)$ be a semihypergroup and R be a fuzzy strong congruence relation on H. Then:*

(*i*) *H/R is a semigroup,*

(*ii*) *if H is a hypergroup, then H/R is a group.*

Proof. (*i*) By Theorem 6, H/R is a semihypergroup. It is enough to show that $|\mu_x \diamond \mu_y| = 1$, for all $\mu_x, \mu_y \in H/R$. Let $\mu_x, \mu_y \in H/R$. Since, R is a fuzzy strong congruence relation, then

$$\bigwedge_{a\in x\circ y,\ b\in x\circ y} R(a,b) \geqslant R(x,x) \wedge R(y,y) = \bigvee_{u,v\in H} R(u,v).$$

Thus for all $a, b \in x \circ y$, $R(a,b) \geqslant \bigvee_{u,v\in H} R(u,v)$ and so $R(a,b) = \bigvee_{u,v\in H} R(u,v)$. Hence, by Lemma 1, $\mu_a = \mu_b$, for all $a, b \in x \circ y$, which implies that $|\mu_x \diamond \mu_y| = 1$.

(*ii*) Similar to the proof of (i), it is enough to show that for all $\mu_x, \mu_y \in H/R$, $|\mu_x \diamond \mu_y| = 1$. But, this immediately follows from (*i*). □

Theorem 8. *If $(H, \circ)$ is a canonical hypergroup, then H/R is a canonical hypergroup.*

Proof. Let H be a canonical hypergroup and $\mu_x, \mu_y \in H/R$. Then,

$$\mu_x \diamond \mu_y = \{\mu_z : z \in x \circ y\} = \{\mu_z : z \in y \circ x\} = \mu_y \diamond \mu_x$$

which shows that H/R is commutative. Since, H has a scalar identity, then there exists $e \in H$, such that $e \circ x = x \circ e = \{x\}$. Hence, for all $\mu_x \in H/R$,

$$\mu_x \diamond \mu_e = \mu_{x\circ e} = \mu_x = \mu_{e\circ x} = \mu_e \diamond \mu_x.$$

This shows that μ_e is a scalar identity. Let $\mu_x \in H/R$ and x' be the unique inverse of x. Since, $e \in (x\circ x')\cap(x'\circ x)$, then $\mu_e \in (\mu_x \diamond \mu_{x'}) \cap (\mu_{x'} \diamond \mu_x)$, which shows that $\mu_{x'}$ is an inverse of μ_x. Now, let μ_y be another inverse of μ_x. Then $\mu_e \in (\mu_x \diamond \mu_y) \cap (\mu_y \diamond \mu_x)$ and so there exists $b \in y \circ x$ such that $\mu_e = \mu_b$. Hence, by Lemma 1, $R(e,b) = \bigvee_{u,v\in H} R(u,v)$. Let $\alpha = \bigvee_{u,v\in H} R(u,v)$. Then, $eR^\alpha b$ i.e., $\{e\}R^\alpha y \circ x$. Since, R^α is compatible, then $e \circ x' \bar{R}^\alpha (y \circ x) \circ x'$ and so $x' \bar{R}^\alpha y \circ (x \circ x')$. Since, $y \in y \circ e \subseteq y \circ (x \circ x')$, then $x' R^\alpha y$ and so $R(x', y) \geqslant \alpha = \bigvee_{u,v\in H} R(u,v)$. Hence, $R(x', y) = \bigvee_{u,v\in H} R(u,v)$ and so by Lemma 1, $\mu_y = \mu_{x'}$, says that the inverse of μ_x is unique. Now, we show that H/R is reversible. For this, let $\mu_x, \mu_y, \mu_a \in H/R$ and $\mu_y \in \mu_a \diamond \mu_x = \mu_{a\circ x}$.

Then, there exists $u \in a \circ x$ such that $\mu_y = \mu_u$. Since, $u \in a \circ x$, then there exists an inverse a' of a such that $x \in a' \circ y$ and so $\mu_x \in \mu_{a'} \diamond \mu_y$, and $\mu_{a'}$ is an inverse of μ_a. Similarly, if $\mu_y \in \mu_x \diamond \mu_a$, then there exists an inverse a" of a such that $\mu_x \in \mu_y \diamond \mu_{a"}$. Hence, H/R is reversible. Therefore, H/R is a canonical hypergroup. $\square$

3. Fuzzy congruence relations on hyper BCK-algebras

Definition 8. (cf. [10, 11]) By a *hyper BCK-algebra* we mean a hypergroupoid $(H, \circ)$ equipped a constant element "0" that satisfies the following axioms:

(HK1) $(x \circ z) \circ (y \circ z) \ll x \circ y$,
(HK2) $(x \circ y) \circ z = (x \circ z) \circ y$,
(HK3) $x \circ H \ll \{x\}$,
(HK4) $x \ll y$ and $y \ll x$ imply $x = y$,

for all $x, y, z \in H$, where by $x \ll y$ we mean $0 \in x \circ y$ and for every $A, B \subseteq H$, $A \ll B$ is defined by $\forall a \in A$, $\exists b \in B$ such that $a \ll b$.

Definition 9. Let R be a fuzzy relation on a hyper BCK-algebra H. Then, R is said to be *fuzzy regular* if

$$R(x, y) \geqslant \Big(\bigvee_{a \in x \circ y} R(a, 0) \Big) \wedge \Big(\bigvee_{b \in y \circ x} R(b, 0) \Big).$$

Lemma 3. *Let R be a fuzzy relation on a hyper BCK-algebra H with the sup property. Then, R is fuzzy regular if and only if for all $\alpha \in [0, 1]$, each nonempty α-level subset R^α is regular.*

Proof. Let R be a fuzzy regular relation on H. Then $x \circ y R^\alpha \{0\}$ and $y \circ x R^\alpha \{0\}$, for $x, y \in H$ and $\alpha \in [0, 1]$. Then, there exist $a \in x \circ y$ and $b \in y \circ x$ such that $aR^\alpha 0$ and $bR^\alpha 0$. This implies that $R(a, 0), R(b, 0) > \alpha$ and so $\bigvee_{a \in x \circ y} R(a, 0) > \alpha$ and $\bigvee_{b \in y \circ x} R(b, 0) > \alpha$. Thus,

$$R(x, y) \geqslant \Big(\bigvee_{a \in x \circ y} R(a, 0) \Big) \wedge \Big(\bigvee_{b \in y \circ x} R(b, 0) \Big) > \alpha$$

and so $xR^\alpha y$, which shows that R^α is regular.

Conversely, suppose that

$$\Big(\bigvee_{a \in x \circ y} R(a, 0) \Big) \wedge \Big(\bigvee_{b \in y \circ x} R(b, 0) \Big) = \alpha$$

for $x, y \in H$. Then $\bigvee_{a\in x\circ y} R(a,0) \geqslant \alpha$ and $\bigvee_{b\in y\circ x} R(b,0) \geqslant \alpha$ and since R has the sup property, then there exist $a_0 \in x\circ y$ and $b_0 \in y \circ x$ such that $R(a_0,0) = \bigvee_{a\in x\circ y} R(a,0) \geqslant \alpha$ and similarly $R(b_0,0) = \bigvee_{b\in y\circ x} R(b,0) \geqslant \alpha$. Hence, $a_0R^\alpha 0$ and $b_0R^\alpha 0$ and so $x\circ yR^\alpha\{0\}$ and $y\circ xR^\alpha\{0\}$. Since R^α is regular, then $xR^\alpha y$ and so

$$R(x,y) \geqslant \alpha = \Big(\bigvee_{a\in x\circ y} R(a,0)\Big) \wedge \Big(\bigvee_{b\in y\circ x} R(b,0)\Big)$$

Therefore, R is a fuzzy regular relation. □

Theorem 9. *Let $(H,\circ)$ be a hyper BCK-algebra and R be a fuzzy regular congruence relation on H. Then, H/R is a hyper BCK-algeba.*

Proof. It is enough to establish the axioms of a hyper BCK-algebra.

(HK1) Let $\mu_x, \mu_y, \mu_z, \mu_v \in H/R$ be such that $\mu_v \in (\mu_x \diamond \mu_z)\diamond(\mu_y\diamond\mu_z)$. Then there exist $\mu_u \in \mu_x\diamond\mu_z$ and $\mu_w \in \mu_y\diamond\mu_z$ such that $\mu_v \in \mu_u\diamond\mu_w$ and so there exists $a \in u\circ w$ such that $\mu_v = \mu_a$. Since $a \in u\circ w \subseteq (x\circ z)\circ(y\circ z) \ll x\circ y$, then there exists $b\in x\circ y$ such that $a \ll b$ and so $0 \in a\circ b$. This implies that $\mu_0 \in \mu_a\diamond\mu_b = \mu_v\diamond\mu_b \subseteq (\mu_u\diamond\mu_w)\diamond(\mu_x\diamond\mu_y) \subseteq ((\mu_x\circ\mu_z)\diamond(\mu_y\diamond\mu_z))\diamond(\mu_x\diamond\mu_y)$. Thus $(\mu_x\diamond\mu_z)\diamond(\mu_y\diamond\mu_z) \ll \mu_x\diamond\mu_y$.

(HK2) Let $\mu_u \in (\mu_x\diamond\mu_y)\diamond\mu_z$. Then there exists $v \in (x\circ y)\circ z$ such that $\mu_u = \mu_v$. Since by (HK2) of H, $(x\circ y)\circ z = (x\circ z)\circ y$, then $v \in (x\circ z)\circ y$ and so $\mu_u = \mu_v \in (\mu_x\diamond\mu_z)\diamond\mu_y$. This implies that $(\mu_x\diamond\mu_y)\diamond\mu_z \subseteq (\mu_x\diamond\mu_z)\diamond\mu_y$. Similarly, we can show that $(\mu_x\diamond\mu_z)\diamond\mu_y \subseteq (\mu_x\diamond\mu_y)\diamond\mu_z$. Thus $(\mu_x\circ\mu_y)\diamond\mu_z = (\mu_x\diamond\mu_z)\diamond\mu_y$.

(HK3) Let $\mu_z \in \mu_x\diamond H/R$, for $\mu_x\in H/R$. Then there exists $\mu_y \in H/R$ such that $\mu_z\in\mu_x\diamond\mu_y$ and so there exists $w\in x\circ y$ such that $\mu_z = \mu_w$. Since by (HK3) of H, $x\circ y \ll x$, then $w \ll x$ and so $0\in w\circ x$. Thus $\mu_0 \in \mu_w\diamond\mu_x = \mu_z\diamond\mu_x$. This implies that $\mu_z \ll \mu_x$ and so $\mu_x\diamond H/R \ll \mu_x$.

(HK4) Let $\mu_x \ll \mu_y$ and $\mu_y \ll \mu_x$, for $\mu_x,\mu_y\in H/R$. Then $\mu_0\in\mu_x\diamond\mu_y$ and $\mu \in \mu_x\diamond\mu_y$. Hence there exist $z\in x\circ y$ and $w \in y\circ x$ such that $\mu_z = \mu_0 = \mu_w$. Since, $\mu_z = \mu$, then by Lemma 1, $R(z,w) = \bigvee_{u,v\in H} R(u,v)$. Since $\mu_z = \mu$ (and also $\mu_w = \mu$), then $R(z,0) = \bigvee_{u,v\in H} R(u,v) = R(w,0)$. Let $\alpha = \bigvee_{u,v\in H} R(u,v)$. Then $zR^\alpha 0$ and $wR^\alpha 0$, means that $x\circ yR^\alpha\{0\}$ and

$y \circ xR^{\alpha}\{0\}$ and since R^{α} is regular, then $xR^{\alpha}y$. Hence, $R(x,y) \geqslant \alpha = \bigvee_{u,v \in H} R(u,v)$ and so $R(x,y) = \bigvee_{u,v \in H} R(u,v)$, which implies that $\mu_x = \mu_y$, by Lemma 1. Therefore, H/R is a hyper BCK-algebra. $\square$

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