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On n-ary semigroups with adjoint neutral element

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Abstract

We prove that we can adjoint an n-ary neutral element to an n-ary semigroup iff this semigroup is derived from a binary semigroup.

According to the general convention used in the theory of n-ary groupoids the sequence of elements $x_i, x_{i+1}, \ldots, x_j$ will be denoted by x_i^j. For $j < i$ it is the empty symbol. If $x_{i+1} = x_{i+2} = \ldots = x_{i+t} = x$, then instead of x_{i+1}^{i+t} we will write $\overset{(t)}{x}$. In this convention the symbol $f(x_1, \ldots, x_n)$ will be written as $f(x_1^n)$. Similarly, the symbol $f(x_1^i, \overset{(t)}{x}, x_{i+t+1}^n)$ means $f(x_1, \ldots, x_i, \underbrace{x, \ldots, x}_{t}, x_{i+t+1}, \ldots, x_n)$.

An n-ary groupoid (G, f) is called (i,j)-*associative* if

$$f(x_1^{i-1}, f(x_i^{n+i-1}), x_{n+i}^{2n-1}) = f(x_1^{j-1}, f(x_j^{n+j-1}), x_{n+j}^{2n-1}) \tag{1}$$

holds for all $x_1, \ldots, x_{2n-1} \in G$. If this identity holds for all $1 \leqslant i < j \leqslant n$, then we say that the operation f is *associative* and (G, f) is called an n-*ary semigroup*. It is clear that an n-ary groupoid is associative if and only if it is $(1, j)$-associative for all $j = 2, \ldots, n$. In the binary case (i.e. for $n = 2$) it is a usual semigroup.

An n-ary semigroup (G, f) in which for all $x_0, x_1, \ldots, x_n \in G$ and all $i \in \{1, \ldots, n\}$ there exists an element $z_i \in G$ such that

$$f(x_1^{i-1}, z, x_{i+1}^n) = x_0, \tag{2}$$

is called an n-*ary group*. It is clear that for $n = 2$ we obtain a usual group.

Note by the way that in many papers n-ary semigroups (n-ary groups) are called n-semigroups (n-groups, respectively). Moreover, in many papers,

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where the arity of the basic operation does not play a crucial role, is used the term *polyadic semigroups* (*polyadic groups*) (cf. [8]).

In the paper [1] written by W. Dörnte (under inspiration of Emmy Noether), he observed that any n-ary groupoid (G,f) of the form

$$f(x_1^n) = x_1 \circ x_2 \circ \ldots \circ x_n \circ b,$$

where $(G,\circ)$ is a group and b belongs to the center of this group, is an n-ary group but for every $n > 2$ there are n-ary groups which are not of this form. In the first case we say that an n-ary groupoid (G,f) is *b-derived* (or *derived* if b is the identity of $(G,\circ)$) from the group $(G,\circ)$, in the second – *irreducible*. Obviously, an n-ary operation derived from a binary associative operation is also associative in the above sense. An n-ary operation b-derived from an associative operation can be associative also in the case when b is not in the center. For example, the ternary operation b-derived from the multiplication of a nilpotent associative algebra of index 7 (the product of any 7 elements is 0) is trivially associative for every b.

In some n-ary groupoids there exists an element e (called an *n-ary neutral element*) such that

$$f(\overset{(i-1)}{e}, x, \overset{(n-i)}{e}) = x \tag{3}$$

holds for all $x \in G$ and for all $i = 1, \ldots, n$. There are n-ary semigroups (groups) with two, three and more neutral elements [9]. Also there are n-ary semigroups (groups too) in which all elements are neutral. All n-ary groups with this property are derived from the commutative group of the exponent $k|(n-1)$ [2]. In n-ary group the set of neutral elements (if it is non-empty) forms an n-ary subgroup [5, 6]. In ternary groups each two neutral elements form a ternary subgroup. Other important properties of neutral elements one can find in [7] and [12].

As is it well known, to any semigroup $(G,\circ)$ we can adjoint the identity $e \notin G$ in this way that $(G \cup \{e\}, \diamond)$ is a semigroup containing $(G,\circ)$ as its semigroup. For this it is sufficient to define the operation $\diamond$ as the extension of $\circ$ putting $x \diamond y = x \circ y$ for $x, y \in G$, $e \diamond e = e$ and $x \diamond e = e \diamond x = x$ for $x \in G$.

Natural question is: *Is it possible to find the analogous construction for n-ary semigroups?* We prove below that the answer is positive.

First we characterize n-ary semigroups containing at least one n-ary neutral element.

Lemma 1. *An n-ary semigroup containing the neutral element is derived from a binary semigroup.*

Proof. Let e be the neutral element of an n-ary semigroup (G, f). It is clear that $(G, \circ)$, where $x \circ y = f(x, \overset{(n-2)}{e}, y)$, is a semigroup and e is its neutral element. Direct computations shows that (G, f) is derived from $(G, \circ)$. $\square$

From the above proposition we can deduce the following result firstly proved by W. Dörnte.

Corollary 1. *An n-ary group is derived from a binary group if and only if it has the neutral element.*

Note that any (i, j)-associative n-ary groupoid (G, f) with the neutral element in the center is an n-ary semigroup [3, 10, 11]. Such groupoid is associative also in the case when in the center of (G, f) lies at least one *neutral polyad* (*sequence*), i.e., the sequence of elements $a_2^n \in G$ such that $f(x, a_2^n) = f(a_2^n, x) = x$ holds for all $x \in G$ [3, 11]. Neutral sequences are in all n-ary groups ([8]), but not in all n-ary semigroups.

Lemma 2. *An n-ary semigroup derived from a binary semigroup possess a neutral sequence if and only if it contains the neutral element.*

Proof. Let (G, f) be derived from a semigroup $(G, \circ)$. If a_2^n is a neutral sequence of (G, f), then $e = a_2 \circ a_3 \circ \ldots \circ a_n$ belongs to G and $x \circ e = x \circ a_2 \circ a_3 \circ \ldots \circ a_n = f(x, a_2^n) = x$ for all $x \in G$. Similarly $e \circ x = x$. This means that e is the identity of $(G, \circ)$. Hence it is the neutral element of an n-ary semigroup derived from $(G, \circ)$.

The converse statement is obvious. $\square$

Corollary 2. *If an n-ary semigroup without neutral elements is derived from a binary semigroup then it does not possess any neutral sequence.*

Proposition 1. *A neutral element can be adjoint to any n-ary semigroup derived from a binary semigroup.*

Proof. Let n-ary semigroup (G, f) be derived from a binary semigroup $(G, \circ)$. Then to $(G, \circ)$ we can add the identity $e \notin G$ in this way that $(G \cup \{e\}, \circ)$ becomes a semigroup with $(G, \circ)$ as its subsemigroup. In an n-ary semigroup $(G \cup \{e\}, g)$ derived from $(G \cup \{e\}, \circ)$ the element e is neutral and $f(x_1^n) = g(x_1^n)$ for $x_1^n \in G$. So, to (G, f) we can adjoint the neutral element $e \notin G$. $\square$

Proposition 2. *If an n-ary semigroup (G, f) do not contains any neutral elements, then to (G, f) we can adjoint the neutral element if and only if (G, f) is derived from a binary semigroup.*

Proof. If to an n-ary semigroup (G, f) we can adjoint the neutral element $e \notin G$, then on $G \cup \{e\}$ we can define the n-ary operation g such that $g(x_1^n) = f(x_1^n)$ for all $x_1^n \in G$. By Lemma 1, an n-ary semigroup $(G \cup \{e\}, g)$ is derived from the semigroup $(G \cup \{e\}, *)$, where $x * y = g(x, \overset{(n-2)}{e}, y)$. Obviously $(G, *)$ is a subsemigroup of $(G \cup \{e\}, *)$. If not, then there are $a, b \in G$ such that $e = a * b$ which contradicts to the assumption on e. This means that (G, f) is an n-ary subsemigroup of $(G \cup \{e\}, g)$ and it is derived from $(G, *)$.

The converse statement follows from Proposition 1. $\square$

As a consequence of the above two propositions we obtain the following

Theorem 1. *To an n-ary semigroup (G, f) we can adjoint the neutral element if and only if (G, f) is derived from a binary semigroup.*

From the above proofs it follows that in an n-ary semigroup (G, f) derived from a semigroup $(G, \circ)$ the adjoint n-ary neutral element is the adjoint identity of $(G, \circ)$.

Corollary 3. *An n-ary semigroup (G, f) can be embedded into an n-ary semigroup with neutral element if and only if it is derived from a binary semigroup.*

Corollary 4. *An n-ary group (G, f) can be embedded into an n-ary group derived from a binary group if and only if (G, f) has at least one neutral element.*

This means that to n-ary groups without neutral element we do not adjoint any neutral element.

Theorem 2. *For every $n > 2$ there exists at least one n-ary semigroup* (*group*) *to which any n-ary neutral element cannot be adjoint.*

Proof. It is sufficient to prove that for every $n > 2$ there exists at least one n-ary group without neutral elements.

At first consider the multiplicative group $G = T(3, \mathbb{K})$ of triangular matrices of the form $\begin{bmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{bmatrix}$, where $\mathbb{K}$ is a field of non-zero characteristic p. Then the map

$$\theta \begin{bmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & \alpha x & y \\ 0 & 1 & \beta z \\ 0 & 0 & 1 \end{bmatrix},$$

where α is a primitive root of unity of degree $n-1$ and $\alpha\beta = 1$, is an automorphism of this group. It is not difficult to verify that the set G with the operation

$$f(A_1, A_2, \ldots, A_n) = A_1 \cdot \theta(A_2) \cdot \theta^2(A_3) \cdot \ldots \cdot \theta^{n-1}(A_n) \cdot B, \tag{4}$$

where $B = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, is an n-ary group.

This group do not contains any n-ary neutral element. Indeed, if A is its n-ary neutral element, then we have $f(X, A, A, \ldots, A) = f(A, X, A, \ldots, A)$ for all $X \in G$. Whence, according to (4), we conclude $X \cdot \theta A = A \cdot \theta X$. Taking the identity matrix as X, we get $\theta A = A$. This proves that the matrix A belongs to the center of the group $(G, \cdot)$. Thus $X \cdot A = A \cdot \theta X = \theta X \cdot A$, which implies $\theta X = X$ for all $X \in G$. This is not true. So, (G, f) is an n-ary group without neutral elements.

Now we give the another example of n-ary group without n-ary neutral elements.

Let $\mathbb{C}$ be the set of complex numbers and let ω be the primitive $(n-1)$-th root of unity. Then $G = \mathbb{C}^3$ with the operation

$$\mathbf{x} \bullet \mathbf{y} = (x_1, x_2, x_3) \bullet (y_1, y_2, y_3) = (x_1 + y_1, x_2 + y_2 + x_1 y_3, x_3 + y_3)$$

is a group and $\theta(x_1, x_2, x_3) = (\omega x_1, \omega^2 x_2, \omega x_3)$ is its automorphism.

It is not difficult to verify that (G, g), where

$$g(\mathbf{x_1}, \mathbf{x_2}, \ldots, \mathbf{x_n}) = \mathbf{x_1} \bullet \theta(\mathbf{x_2}) \bullet \theta^2(\mathbf{x_3}) \bullet \ldots \bullet \theta^{n-1}(\mathbf{x_n}),$$

is an n-ary group. It is isomorphic to an n-ary group of triangular matrices from the proof of Theorem 3 in [4].

Similarly as in the previous case we can prove that (G, g) is not derived from any binary group. $\square$

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