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Duality for central piques

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Abstract

A duality for locally compact central piques is established, based on Pontryagin duality for locally compact abelian groups. The duality restricts to yield Suvorov duality for quasigroup modes.

1. Introduction

Piques are quasigroups with a pointed idempotent element. As such, they form an intermediate class between loops and general quasigroups. A quasigroup Q is *central* if the diagonal $\widehat{Q}$ is a normal subquasigroup of the square Q^2, or an equivalence class of a quasigroup congruence relation on Q^2 [1, Ch. III]. Each central quasigroup Q is very tightly related to a central pique, namely the quotient $Q^2/\widehat{Q}$ with $\widehat{Q}$ as the pointed idempotent. (The exact relationship is central isotopy [1, Ch. III]. In particular, $Q \times Q$ and $Q \times Q^2/\widehat{Q}$ are isomorphic.) Moreover, the class of central piques includes many of the quasigroups encountered in practice, such as abelian groups under subtraction or the standard constructions of orthogonal quasigroups based on finite fields. It also includes the class of *quasigroup modes*, idempotent and entropic quasigroups [3].

As part of a general determination of the character tables of finite (discrete) central piques, the paper [4] sketched a duality for them, and raised the problem of

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> extension of the duality theory from finite (discrete) central piques to general locally compact central piques.

The purpose of the current paper is to present such an extension (Theorem 5.5). Earlier, Suvorov exhibited a duality for locally compact topological quasigroup modes on the basis of Pontryagin duality [7]. Suvorov duality may now be seen as a special case of the duality for central piques (Section 6).

Readers are referred to [6] for concepts and notational conventions not otherwise defined explicitly in this paper.

2. Quasigroups and piques

A *quasigroup* $(Q, \cdot)$ is considered combinatorially as a set Q equipped with a binary *multiplication* operation denoted by $\cdot$ or simple juxtaposition of the two arguments, in which specification of any two of x, y, z in the equation $x \cdot y = z$ determines the third uniquely. Equationally, a quasigroup $(Q, \cdot, /, \backslash)$ is a set Q equipped with three binary operations of multiplication, *right division* $/$ and *left division* $\backslash$, satisfying the identities:

$$\begin{array}{ll} \text{(IL)} & y \backslash (y \cdot x) = x; \\ \text{(IR)} & x = (x \cdot y)/y; \\ \text{(SL)} & y \cdot (y \backslash x) = x; \\ \text{(SR)} & x = (x/y) \cdot y. \end{array}$$

(Suppressing explicit mention of the division operations of a quasigroup, one may denote it merely as $(Q, \cdot)$ instead.) The equational definition of quasigroups means that they form a variety in the sense of universal algebra, and are thus susceptible to study by the concepts and methods of universal algebra [6]. In particular, a quasigroup $(Q, \cdot, /, \backslash)$ is *topological* if the underlying set Q is a topological space, and if the operations $Q^2 \to Q; (x, y) \mapsto x \cdot y$, $Q^2 \to Q; (x, y) \mapsto x/y$ and $Q^2 \to Q; (x, y) \mapsto x \backslash y$ are continuous maps from the product space Q^2.

An element e of a quasigroup Q is said to be *idempotent* if $\{e\}$ forms a singleton subquasigroup of Q. A *pique* or *pointed idempotent quasigroup* [1, §III.5] is a quasigroup P, containing an idempotent element 0, that has its quasigroup structure of multiplication and the divisions enriched by a nullary operation selecting the idempotent element 0. Note that piques also form a variety.

For each element q of a quasigroup $(Q, *)$, the *right multiplication* $R_*(q)$ or

$$R(q) : Q \to Q; x \mapsto x * q$$

and *left multiplication* $L_*(q)$ or

$$L(q) : Q \to Q; x \mapsto q * x$$

are elements of the group $Q!$ of bijections from the set Q to itself. The subgroup of $Q!$ generated by all the right and left multiplications is called the *multiplication group* $\operatorname{Mlt} Q$ of Q. For a pique P with pointed idempotent 0, it is conventional to set $R = R(0)$ and $L = L(0)$. The stabilizer of 0 in the permutation group $\operatorname{Mlt} P$ is called the *inner multiplication group* $\operatorname{Inn} P$ of P. For example, if P is a group, then the inner multiplication group of the pique P is just the inner automorphism group of the group P.

3. Central piques

A *loop* L is a pique in which the pointed idempotent element 1 acts as an identity. For a general pique $(P, \cdot, 0)$, the *cloop* or *corresponding loop* is the loop $B(P)$ or $(P, +, 0)$ in which the "multiplication" operation $+$ is defined by

$$x + y = xR^{-1} \cdot yL^{-1}. \tag{3.1}$$

Inverting (3.1), the multiplication of a pique is recovered from the cloop by

$$x \cdot y = xR + yL. \tag{3.2}$$

Definition 3.1. A pique $(P, \cdot, 0)$ is said to be *central*, or to lie in the class $\mathfrak{Z}$, if $B(P)$ is an abelian group, and $\operatorname{Inn} P$ is a group of automorphisms of $B(P)$.

Remark 3.2. The syntactical Definition 3.1 of pique centrality is chosen for its concreteness, and because it is well suited to the purposes of the current paper. The equivalence of this definition with the structural characterization given in the introduction is discussed in [1, §III.5], [5, §3.5].

In a central pique P, the right division is given by

$$x/y = (x - yL)R^{-1} \tag{3.3}$$

and the left division by

$$x \backslash y = (y - xR)L^{-1} \tag{3.4}$$

for x, y in P, using the subtraction of the cloop $B(P)$. Topological central piques are easily characterized.

Proposition 3.3. *A central pique P is topological if and only if the cloop $B(P)$ is a topological abelian group, and the maps $R : P \to P$ and $L : P \to P$ are homeomorphisms.*

Proof. If P is topological, then the maps $R : P \to P; x \mapsto x \cdot 0$ and $L : P \to P; x \mapsto 0 \cdot x$ are certainly continuous, as are their respective inverses $R^{-1} : P \to P; x \mapsto x/0$ and $L^{-1} : P \to P; x \mapsto 0\backslash x$. The cloop addition (3.1) is then continuous, as is the negation $-y = 0/(yRL^{-1})$ according to (3.3).

Conversely, if the cloop is a topological group and the maps R, L are homeomorphisms, then it is apparent from (3.1), (3.3) and (3.4) that the pique is topological. □

Let P be a central pique. Let $\langle R, L \rangle$ denote the free group on the 2-element set $\{R, L\}$. Then the group homomorphism

$$\langle R, L \rangle \to \operatorname{Inn} P; \quad R \mapsto R(0), \quad L \mapsto L(0)$$

makes P a right $\langle R, L \rangle$-module. Conversely, a right $\langle R, L \rangle$-module P yields a pique with multiplication given by (3.2).

Proposition 3.4. *Let $\langle R, L \rangle$ denote the free group on the 2-element set $\{R, L\}$.*

(a) *The category of central piques is equivalent to the category of right $\langle R, L \rangle$-modules.*

(b) *The category of locally compact central piques is equivalent to the category of locally compact right $\langle R, L \rangle$-modules.*

Proof. Statement (a) follows immediately from the preceding considerations, while (b) holds by Proposition 3.3. □

4. Schizophrenic objects and diagrammatic duality

Let $\mathbf{D}$ and $\mathbf{E}$ be concrete categories with products. Objects of $\mathbf{D}$ and $\mathbf{E}$ are often just denoted by their underlying sets. A *schizophrenic object* is a set S which is the underlying set of an object S or $S_{\mathbf{D}}$ of $\mathbf{D}$ and an object S or $S_{\mathbf{E}}$ of $\mathbf{E}$. Now for an object D of $\mathbf{D}$, the morphism set $D^* = \mathbf{D}(D, S)$ is a subset of the product S^D. If this subset inherits the structure of $S_{\mathbf{E}}^D$ as a product object of $\mathbf{E}$, then the assignment $D \mapsto D^*$ may become the object part of a contravariant functor $F : \mathbf{D} \to \mathbf{E}$. Similarly, for an object E of $\mathbf{E}$, define $E^* = \mathbf{E}(E, S)$. The assignment $E \mapsto E^*$ may then become the object part of a contravariant functor $G : \mathbf{E} \to \mathbf{D}$. If the composites FG and GF are naturally isomorphic to the identity functors on their domains, then the categories $\mathbf{D}$ and $\mathbf{E}$ are said to be *dual* or *dually equivalent* via the schizophrenic object S.

Let T be the *one-dimensional torus*, the group S^1 of complex numbers of unit modulus under multiplication, or the quotient abelian group $\mathbb{R}/\mathbb{Z}$. The torus may be equipped with the subset topology induced from the Euclidean topology on $\mathbb{C}$, or the quotient topology induced from the Euclidean topology on $\mathbb{R}$. Then *Pontryagin duality* is the dual equivalence of the category $\mathbf{A}$ of locally compact abelian topological groups with itself that is given by T as schizophrenic object. For a locally compact abelian group A, the dual object A^* is called the *character group* of A. In particular, the compact group T is the character group of the discrete group $\mathbb{Z}$ of integers, while the locally compact group $\mathbb{R}$ of reals is its own character group.

Suppose that categories $\mathbf{D}$ and $\mathbf{E}$ are dually equivalent by a pair of contravariant functors F and G. Let J be a small category. Then objects of the functor category $\mathbf{D}^J$ are considered as *diagrams* in $\mathbf{D}$, images of the small category J. Similarly, objects of the functor category $\mathbf{E}^{J^{\mathrm{op}}}$ (consisting of covariant functors to $\mathbf{E}$ from the opposite J^{op} of J, or contravariant functors from J to $\mathbf{E}$) are images of the opposite category J^{op}. For an object $\theta : J \to \mathbf{D}$ of $\mathbf{D}^J$, the duality between $\mathbf{D}$ and $\mathbf{E}$ gives a *dual diagram* $\theta F : J \to \mathbf{E}$ in $\mathbf{E}^{J^{\mathrm{op}}}$. Similarly, a diagram $\varphi : J^{\mathrm{op}} \to \mathbf{E}$ in $\mathbf{E}$ yields a dual diagram $\varphi G : J \to \mathbf{D}$ in D. The double duals $\theta FG : J \to \mathbf{D}$ and $\varphi GF : J \to \mathbf{E}$ are naturally isomorphic to their respective primals θ and φ. Thus one obtains a *diagrammatic duality* between the functor categories $\mathbf{D}^J$ and $\mathbf{E}^{J^{\mathrm{op}}}$.

5. Duality for central piques

Let P be a locally compact central pique, with inner multiplication group H and pointed idempotent 0. The cloop $B(P)$ of P is a locally compact abelian group, with dual abelian group $B(P)^*$. Let H act from the left on $B(P)^*$ by $(b)^h\beta = (b^h)\beta$ for $h \in H$, $\beta \in B(P)^*$, and $b \in P$. A quasigroup operation is defined on $B(P)^*$ by

$$\xi \cdot \eta = {}^R\xi + {}^L\eta. \tag{5.5}$$

Definition 5.1. The pique P^* *dual* to P consists of the space $B(P)^*$ equipped with the quasigroup operation (5.5), and pointed by the trivial character 0 of $B(P)$.

Example 5.2. Consider the pique $(\mathbb{R}, -, 0)$ of real numbers under subtraction. It is a topological pique under the usual, locally compact (Euclidean) topology on $\mathbb{R}$. Since $xR(0) = x - 0 = x$ and $xL(0) = 0 - x = -x$, the $\langle R, L\rangle$-module structure on $\mathbb{R}$ is given by $xR = x$ and $xL = -x$ for each real number x. The character group of the cloop $B(\mathbb{R}, -, 0) = \mathbb{R}$ is again the usual abelian group $\mathbb{R}$. For a real number ξ considered as the character $\mathbb{R} \to S^1; x \mapsto \exp(2\pi i x\xi)$, the corresponding left actions are given by ${}^R\xi = \xi$ and ${}^L\xi = -\xi$. By (5.5), the dual pique $(\mathbb{R}, -, 0)^*$ has the subtraction $(\xi, \eta) \mapsto \xi - \eta$ as its quasigroup operation. Similar considerations show that the dual of the pique $(\mathbb{Z}, -, 0)$ of integers under subtraction is the unit circle $(S^1, \circ, 1)$ under $z \circ w = z\overline{w}$ for complex numbers z, w of unit modulus. Here z is considered as the character $\mathbb{Z} \to S^1; n \mapsto z^n$.

Let J be the free group on two generators R and L. Consider J as a small category with a single object $\circ$, and with morphisms corresponding to the elements of the group [2, Ch.1, §2]. The following proposition identifies locally compact central piques as J-diagrams in the category $\mathbf{A}$ of locally compact abelian groups.

Proposition 5.3. *The category of locally compact central piques is equivalent to the functor category $\mathbf{A}^J$. Under this equivalence, a pique P with cloop $B(P)$ determines the* covariant *functor $J \to \mathbf{A}$ specified uniquely by*

$$\theta : \begin{cases} R \mapsto \big(R : B(P) \to B(P)\big), \\ L \mapsto \big(L : B(P) \to B(P)\big). \end{cases} \tag{5.6}$$

Conversely, a functor $\varphi : J \to \mathbf{A}$ with $P = \circ\varphi$ determines a locally compact central pique on P with

$$x \cdot y = xR^{\varphi} + yL^{\varphi} \tag{5.7}$$

for x, y in P.

Proof. The functor (5.6) maps to $\mathbf{A}$, since the cloop of a locally compact central pique is a locally compact abelian group, and since the right and left multiplications by the pointed idempotent are homeomorphic homomorphisms. Conversely, (5.7) defines a locally compact central pique, since the automorphisms R^{φ}, L^{φ}, and their inverses are continuous. $\square$

Proposition 5.3 has a dual counterpart.

Proposition 5.4. *The category of locally compact central piques is equivalent to the functor category $\mathbf{A}^{J^{\mathrm{op}}}$. Under this equivalence, a pique P' with cloop $B(P')$ determines the* contravariant *functor $J \to \mathbf{A}$ specified uniquely by*

$$\theta : \begin{cases} R \mapsto \big(R : B(P') \to B(P')\big), \\ L \mapsto \big(L : B(P') \to B(P')\big). \end{cases} \tag{5.8}$$

Conversely, a contravariant functor $\varphi : J \to \mathbf{A}$ with $P' = \circ\varphi$ determines a locally compact central pique on P' with

$$x \cdot y = R^{\varphi}x + L^{\varphi}y \tag{5.9}$$

for x, y in P'. In particular, if a central pique P corresponds to the functor (5.6)*, then the dual pique P^* corresponds to the contravariant functor*

$$\theta^* : \begin{cases} R \mapsto \big(R^* : B(P)^* \to B(P)^*\big), \\ L \mapsto \big(L^* : B(P)^* \to B(P)^*\big). \end{cases} \tag{5.10}$$

Proof. If the primal pique P is given by the functor (5.6) according to Proposition 5.3, one obtains $B(P)$ as a right $\langle R, L\rangle$ module according to Proposition 3.4. Diagrammatic duality then gives the dual group $B(P)^*$ as the left $\langle R, L\rangle$-module corresponding to the dual pique P^* of Definition 5.1, equivalent to the functor (5.10). $\square$

Theorem 5.5. *The correspondence between the covariant functors θ of* (5.6) *and the contravariant functors θ^* of* (5.10) *yields a duality for the category of locally compact central piques.*

Proof. If P corresponds under the equivalence of Proposition 5.3 to the functor (5.6), then P^* corresponds to (5.10). In turn, P^{**} corresponds to the dual of this latter functor, namely the covariant functor θ^{**} with

$$\theta^{**} : \begin{cases} R \mapsto \big(R^{**} : B(P)^{**} \to B(P)^{**}\big)\,, \\ L \mapsto \big(L^{**} : B(P)^{**} \to B(P)^{**}\big)\,. \end{cases}$$

Since θ^{**} is naturally isomorphic to θ, the double dual pique P^{**} is naturally isomorphic to the primal P. □

6. Suvorov duality

An algebra is said to be *idempotent* if each singleton forms a subalgebra. It is said to be *entropic* if each operation is a homomorphism. Finally, it is said to be a *mode* if it is both idempotent and entropic [3]. For quasigroups, idempotence reduces to satisfaction of the identity

$$x \cdot x = x \tag{6.11}$$

[7, (3)], while entropicity reduces to satisfaction of the identity

$$xy \cdot zt = xz \cdot yt \tag{6.12}$$

[7, (2)]. Comparing with (3.2), it is apparent that a central pique P is a mode if and only if

$$R + L = 1 \tag{6.13}$$

and

$$RL = LR \tag{6.14}$$

in the endomorphism ring of the cloop $B(P)$. In fact P is a mode if and only if (6.13) alone is satisfied, since then $RL = R(1-R) = (1-R)R = LR$. Conversely, any quasigroup mode P is central, since the diagonal $\widehat{P}$ is the preimage of the singleton subquasigroup $\{\widehat{P}\}$ under the quasigroup homomorphism

$$P^2 \to P^2/\widehat{P}; \quad (x,y) \mapsto \widehat{P}(x,y)$$

to the set $\{\widehat{P}(x,y) \mid x,y \in P\}$ of cosets of $\widehat{P}$ under the multiplication

$$\widehat{P}(x,y) \cdot \widehat{P}(x',y') = \widehat{P}\widehat{P} \cdot \big((x,y)(x',y')\big) = \widehat{P}(xx',yy')$$

well-defined by the entropic law (6.12).

Suvarov duality is now recovered as follows.

Corollary 6.1. *The duality of Theorem 5.5 restricts to a self-duality for the category of locally compact quasigroup modes.*

Proof. If the image of the functor (5.6) satisfies (6.13), then so does the image of the dual functor (5.10). Since the images of the morphism parts of the functors are commutative groups, there is no distinction between covariant and contravariant functors. □

References

[1] **O. Chein et al**: *Quasigroups and Loops: Theory and Applications*, Heldermann, Berlin, 1990.

[2] **S. Mac Lane**: *Categories for the Working Mathematician*, Springer-Verlag, Berlin, 1971.

[3] **A. B. Romanowska and J. D. H. Smith**: *Modes*, World Scientific, Singapore, 2002.

[4] **J. D. H. Smith**: *Characters of central piques*, J. Algebra **279** (2004), 437 – 450.

[5] **J. D. H. Smith**: *An Introduction to Quasigroups and their Representations*, CRC Press, Boca Raton, FL, to appear.

[6] **J. D. H. Smith and A. B. Romanowska**: *Post-Modern Algebra*, Wiley, New York, NY, 1999.

[7] **N. M. Suvarov**: *A duality theorem for distributive transitive topological quasigroups* (Russian), Matem. Issled. **4** (1969), 162 – 166.

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