Quasigroups and Related Systems **14** (2006), 43 – 59

Geometric means and reflection quasigroups

Jimmie Lawson and Yongdo Lim

Abstract

In this paper we show how the category of reflection quasigroups forms a natural and suitable context for the development of an abstract theory of the geometric mean, as it appears in matrix and operator theory. We provide a new characterization of those quasigroups that arise when reflection quasigroups are endowed with the mean operation. We also show how the notion of the geometric mean can be enlarged to that of weighted means, develop basic properties of the latter, and illustrate their usefulness in solving equations involving the mean operation.

1. Introduction

The notion of the geometric mean of two positive real numbers, $a\#b = \sqrt{ab}$, as the solution of the equation $x^2 = ab$ can be profitably extended to much more general contexts. A natural approach in the setting of a noncommutative group G is to "symmetrize" the equation and define the geometric mean $a\#b$ of a and b to be the unique solution of the equation $xa^{-1}x = b$, provided such a unique solution exists. In the matrix group setting the equation thus assumes the form of the simplest of the matrix Riccati equations.

The Riccati equation has a natural alternative form in the setting of nonassociative algebra. Recall that the *core* of a group G is defined to be the group equipped with the binary operation $a \bullet b := ab^{-1}a$. In the core setting we are seeking a unique solution of the equation $x \bullet a = b$. The condition that this equation always have a unique solution is just the condition that $(G, \bullet)$ be a *right quasigroup* (see Section 2). Since one quickly realizes that this is frequently not the case, it is natural to look for restricted settings in which it is. Thus we are led to look among $\bullet$-closed subsets containing the

2000 Mathematics Subject Classification: 20N05
Keywords: Quasigroup, dyadic symmetric set, geometric mean, weighted mean

identity, the *twisted subgroups* (terminology courtesy of M. Aschbacher [3]), for ones that are right quasigroups. The recognition that many familiar sets of “positive objects” such as the set of positive definite symmetric or Hermitian matrices or the set of positive elements in a C^*-algebra form twisted subgroups of the multiplicative group of invertible elements suggests that this approach is worth pursuing.

At this stage it becomes most helpful to identify in an axiomatic method the type of algebraic structures that are arising as twisted subgroups of the core of a group. For a twisted subgroup A of a group G, the following properties of $(A, \bullet)$ are readily verified: for all $a, b, c \in A$,

(1) (idempotency) $a \bullet a = a$.

(2) (left symmetry) $a \bullet (a \bullet b) = b$.

(3) (left distributivity) $a \bullet (b \bullet c) = (a \bullet b) \bullet (a \bullet c)$.

These turn out to be familiar axioms. Indeed they are the first three of the four axioms of Ottmar Loos in his axiomization of symmetric spaces [14, Chapter II], the fourth one being the topological requirement that the fixed points of L_a, solutions of $a \bullet x = x$, be isolated. In the symmetric space setting the left translation $L_a(b) := a \bullet b$ represents the symmetry or point reflection of the space through the point a, so the first two axioms at least have very natural geometric motivation.

After Loos, groupoids (or magmas as some would have it) that are idempotent, left symmetric, and left distributive received rather extensive study. N. Nobusawa [15, 16] and collaborators [6], who called them symmetric sets, were among the earliest, if not the earliest. They were also investigated by R. Pierce [17, 18] under the name of “symmetric groupoids.” Through the pioneering work of Joyce [5] they resurfaced in knot theory, where they are referred to as *involutory quandles*, although such structures, called “kei’s,” were studied as far back as 1945 by M. Takasaki [20]. (For *quandles* in general, Axiom (2) is weakened to require only that L_a be bijective; such structures are called *pseudo-symmetric sets* by Nobusawa.) The recent dissertation of D. Stanovský [19] also contains considerable information about such structures. Other references could be mentioned as well, but we return to our train of thought.

Pasting together the abstracted properties of twisted subgroups with the property of being a right quasigroup, we are led to groupoids $(X, \bullet)$ satisfying for all $a, b, c \in X$

(1) $a \bullet a = a \quad (S_a a = a)$;

(2) $a \bullet (a \bullet b) = b \quad (S_a S_a = \mathrm{id}_X)$;

(3) $a \bullet (b \bullet c) = (a \bullet b) \bullet (a \bullet c) \quad (S_a S_b = S_{S_a b} S_a)$;

(4) the equation $x \bullet a = b$ $(S_x a = b)$ has a unique solution $x \in X$, called the *geometric mean*, *mean* for short, or *midpoint* of a and b, and denoted $a\#b$.

Here we follow the notation of Loos and denote by S_a (instead of L_a) left translation by a to indicate its geometric interpretation as a symmetry or point reflection. Each axiom is then given an alternative formulation in terms of these symmetries. The terminology "geometric mean" is motivated by our entire previous discussion, while the terminology "midpoint" is suggested by that fact that it is reasonable to expect that reflection through the midpoint between a and b would carry a to b.

Note that it is an immediate consequence of Axiom (2) that the left translation S_a is a bijection (with inverse S_a); hence $(X, \bullet)$ is also a left quasigroup. Thus Axioms (2) and (4) together imply that $(X, \bullet)$ is a quasigroup.

Definition 1.1. If $(X, \bullet)$ satisfies Axioms (1)-(4), then it is called a *reflection quasigroup*. The reflection quasigroups form the objects of a category for which the morphisms are $\bullet$-preserving homomorphisms.

We note that we have previously investigated reflection quasigroups in some detail in [9, 10], where we called them "dyadic symmetric sets."

We can also consider the modified category of pointed reflection quasigroups with point-preserving $\bullet$-homomorphisms for morphisms. This viewpoint allows a weakening of Axiom (4) that is often useful for verification of the axiom in specific examples.

Lemma 1.2. *Let* $(X, \bullet, \varepsilon)$, $\varepsilon \in X$, *satisfy Axioms* $(1)-(3)$ *and for all* $b \in X$, *and*

(4_ε) *the equation* $x \bullet \varepsilon = b$ $(S_x \varepsilon = b)$ *has a unique solution* $x \in X$.

Then $(X, \bullet)$ *is a reflection quasigroup.*

Proof. For $a, b \in X$, pick y such that $y \bullet \varepsilon = S_y \varepsilon = a$; then $S_y a = S_y S_y \varepsilon = \varepsilon$. It follows that

$$S_y x \bullet a = b \Leftrightarrow x \bullet \varepsilon = S_y S_y x \bullet S_y a = S_y b.$$

The lemma follows from hypothesis and the fact S_y is bijective. □

The lemma leads to a characterization of those twisted subgroups admitting geometric means. Note first that if A is a twisted subgroup and e is the identity, then c is a solution of $x \bullet e = b$ if and only if $ce^{-1}c = c^2 = b$. Thus the equation $x \bullet e = b$ has a unique solution if and only if every element of A has a unique square root in A, i.e., A is *uniquely* 2*-divisible*. From the preceding lemma we conclude that

Proposition 1.3. *For a twisted subgroup A of a group G, $(A, \bullet)$ is a reflection quasigroup if and only if A is uniquely* 2*-divisible.*

Example 1.4. The twisted subgroups of positive definite symmetric real matrices, positive definite Hermitian matrices, and positive elements of a C^*-algebra are all uniquely 2-divisible, a standard and well-known fact. They thus yield examples of reflection quasigroups and admit geometric means.

The geometric mean in these contexts is a known quantity in the literature and has a variety of characterizations. In a uniquely 2-divisible twisted subgroup it can be defined directly by

$$a\#b = a^{1/2}(a^{-1/2}ba^{-1/2})^{1/2}a^{1/2};$$

one sees by direct computation that this satisfies the equation $x \bullet a = xa^{-1}x = b$. There are a variety of references where the geometric mean appears (see, for example, [1, 2, 4, 7, 8, 13]). In [8] the treatment begins with the matrix Riccati equation defining the geometric mean, as we have done here.

2. The quasigroup theory of means

After our initial foray into the study of the geometric mean [8], we began to look for a suitable categorical “home” for a general theory of the geometric mean. In the preceding section we have seen that reflection quasigroups provide a natural axiomatic theory for geometric means. In this section we demonstrate that much of the basic algebraic theory of the geometric mean can be worked out in an elementary and straightforward way in the quasigroup context. We thus set forth some elementary, mostly well-known facts about quasigroups that will be pertinent and useful for our study of reflection quasigroups.

A set M equipped with a binary operation $\bullet : M \times M \to M$ is a *quasigroup* if for all $a, b \in M$, the equations $x \bullet a = b$ and $a \bullet y = b$ have unique solutions, the *right and left quotients*, usually denoted by $x = b/a$ and $y = a \setminus b$.

Remark 2.1. Alternatively one can define a quasigroup by requiring that the left and right translation maps $L_a, R_a : X \to X$ defined by $L_a(x) = a \bullet x$ and $R_a(x) = x \bullet a$ are bijections. It follows that quasigroups are cancellative.

Quasigroups form the objects of a category with corresponding morphisms those functions that are $\bullet$-homomorphisms.

We develop the quasigroup theory that pertains to reflection quasigroups in the context of general quasigroups. We assume as we go along that $(X, \bullet)$ denotes a reflection quasigroup, and point out applications of our developments to reflection quasigroups. There will be some variation in notation in the general quasigroup setting and the reflection quasigroup setting. For example, left translation is denoted L_a in the general setting and S_a in the reflection quasigroup setting. The right quotient of b by a is denoted b/a, but this same element is called the mean and denoted by $a\#b$ in the reflection quasigroup setting.

Let $(M, \bullet)$ be a quasigroup. We consider the right quotient operation $x \bullet_r y := x/y$. That is,

$$x \bullet_r y = w \iff x/y = w \iff x = w \bullet y. \tag{2.1}$$

Lemma 2.2. *Let $(M, \bullet)$ be a quasigroup.*

(i) *$(M, \bullet_r)$ is a quasigroup.*

(ii) *$(\bullet_r)_r = \bullet$.*

(iii) *A function is a $\bullet$-homomorphism if and only if it is a $\bullet_r$-homomorphism.*

(iv) *L_a distributes over $\bullet$ if and only if it distributes over $\bullet_r$.*

Proof. (i) The quasigroup property of $(M, \bullet_r)$ follows from (2.1):

$$x \bullet_r a = b \iff x = b \bullet a,$$

$$a \bullet_r y = b \iff a = b \bullet y \iff y = b \setminus a.$$

(ii) From

$$a = b(\bullet_r)_r c \iff a \bullet_r c = b \iff a = b \bullet c,$$

we have $(\bullet_r)_r = \bullet$.

(*iii*) Suppose $f : (M, \bullet) \to (Y, \bullet)$ is a homomorphism. From (2.1)

$$\begin{aligned} x \bullet_r y = w \Longleftrightarrow x = w \bullet y \Longrightarrow f(x) = f(w \bullet y) = f(w) \bullet f(y) \\ \Longleftrightarrow f(x) \bullet_r f(y) = f(w). \end{aligned}$$

The converse follows from (*ii*).

(*iv*) Immediate from (*iii*). □

Lemma 2.3. *Let* $(M, \bullet)$ *be a quasigroup.*

(*i*) $(M, \bullet)$ *is idempotent if and only if* $(M, \bullet_r)$ *is.*

(*ii*) $(M, \bullet)$ *is left symmetric* ($L_x \circ L_x = id_M$ *for all* x) *if and only if* $(M, \bullet_r)$ *is commutative.*

Proof. (*i*) From (2.1) $x = x \bullet x$ if and only if $x \bullet_r x = x$.

(*ii*) Let $a, b \in M$. There exists a unique x such that $L_x(a) = x \bullet a = b$. Assuming that L_x is involutive and applying it to the previous equation, we conclude that $a = L_x(b) = x \bullet b$. From these two equations we conclude that

$$b \bullet_r a = b/a = x = a/b = a \bullet_r b.$$

Conversely assume that $(M, \bullet_r)$ is commutative and let $x \in M$. Let $a \in M$ and set $b := x \bullet a = L_x(a)$. Then

$$x = b/a = b \bullet_r a = a \bullet_r b = a/b,$$

so $L_x(L_x(a)) = L_x(b) = x \bullet b = (a/b) \bullet b = a$. □

Remark 2.4. Consider a reflection quasigroup $(X, \bullet)$. By definition $a\#b = b/a = b \bullet_r a = a \bullet_r b$, where the last equality follows from part (*ii*) of the preceding lemma. Thus the operation # is commutative, equal to $\bullet_r$, and idempotent. It follows from Lemma 2.2(*i*) that $(X, \#)$ is also a quasigroup and from part (*ii*) that $(X, \#_r) = (X, \bullet)$.

Lemma 2.5. *In a quasigroup* $(M, \bullet)$, *the following are equivalent.*

(*i*) $R^{-1}_{x \bullet y} = R^{-1}_x \bullet R^{-1}_y$, *where* $(R^{-1}_x \bullet R^{-1}_y)(z) = R^{-1}_x(z) \bullet R^{-1}_y(z)$.

(*ii*) *If* $a \bullet c = b \bullet d := m$, *then* $(a \bullet b) \bullet (c \bullet d) = m$.

Proof. (i) implies (ii): Suppose that $a \bullet c = b \bullet d := m$. Then

$$a = R_c^{-1}(a \bullet c) = R_c^{-1}(m), \quad b = R_d^{-1}(b \bullet d) = R_d^{-1}(m).$$

This implies that

$$a \bullet b = R_c^{-1}(m) \bullet R_d^{-1}(m) = (R_c^{-1} \bullet R_d^{-1})(m) = R_{c \bullet d}^{-1}(m)$$

and thus $m = R_{c \bullet d}(a \bullet b) = (a \bullet b) \bullet (c \bullet d)$.

(ii) implies (i): Let x, y and m be given. Let $a = R_x^{-1}(m), b = R_y^{-1}(m)$. Then $m = a \bullet x = b \bullet y$ and then by (ii)

$$(a \bullet b) \bullet (x \bullet y) = m.$$

This implies that

$$R_{x \bullet y}^{-1}(m) = a \bullet b = R_x^{-1}(m) \bullet R_y^{-1}(m) = (R_x^{-1} \bullet R_y^{-1})(m).$$

Since m was arbitrary, we have

$$R_{x \bullet y}^{-1} = R_x^{-1} \bullet R_y^{-1}.$$

□

Remark 2.6. A binary operation $\bullet$ is called *medial* if for all a, b, c, d,

$$(a \bullet b) \bullet (c \bullet d) = (a \bullet c) \bullet (b \bullet d).$$

Observe that if the operation $\bullet$ is idempotent and $a \bullet c = m = b \bullet d$, then the right hand side of the preceding equation is also m, so the left must be also. Hence condition (ii) of Lemma 2.5, in the setting of idempotent operations, can be viewed as weakened mediality condition, called *limited mediality.* If one interprets geometric means as midpoints of some sort, then the notion of limited mediality has an intuitive geometric interpretation.

Proposition 2.7. *In a quasigroup $(M, \bullet)$, the following are equivalent.*

- (i) *$(M, \bullet)$ is left distributive, i.e., $x \bullet (y \bullet z) = (x \bullet y) \bullet (x \bullet z)$ for all x, y, z.*
- (ii) *In $(M, \bullet_r)$, for all x, y, $R_{x \bullet_r y}^{-1} = R_x^{-1} \bullet_r R_y^{-1}$.*
- (iii) *For all $a, b, c, d \in M$, if $a \bullet_r c = b \bullet_r d := m$, then $(a \bullet_r b) \bullet_r (c \bullet_r d) = m$.*

If, in addition, M is left symmetric, then $(i) - (iii)$ are equivalent to (*iv*) *For all* $x, y, z \in M$, $(x \bullet y) \bullet z = x \bullet (y \bullet (x \bullet z))$.

Proof. The equivalence of (*ii*) and (*iii*) follows from the preceding lemma.

Suppose that (*i*) holds. We have $w = R_x^{-1}(z)$ in $(M, \bullet_r)$ if and only if $w \bullet_r x = z$ if and only if $w = z \bullet x$. Thus $R_x^{-1}(z) = z \bullet x$. From Lemma 2.2(*iv*), we have

$$R_x^{-1}(z) \bullet_r R_y^{-1}(z) = (z \bullet x) \bullet_r (z \bullet y) = z \bullet (x \bullet_r y) = R_{x \bullet_r y}^{-1}(z).$$

Thus (*ii*) holds.

Conversely assume that (*ii*) holds. Then from the previous paragraph and the hypothesis

$$z \bullet (x \bullet_r y) = R_{x \bullet_r y}^{-1}(z) = R_x^{-1}(z) \bullet_r R_y^{-1}(z) = (z \bullet x) \bullet_r (z \bullet y).$$

By Lemma 2.2(*iv*) $(M, \bullet)$ is left distributive.

The equivalence of (*i*) and (*iv*) follows by replacing z by $x \bullet z$ in each and reducing it to the other by means of left symmetry. □

We gather together the preceding results into a characterization of the mean operation # on a reflection quasigroup. The theorem follows directly from our preceding results.

Theorem 2.8. *Let* $(X, \bullet)$ *be a reflection quasigroup. Then* $(X, \#)$ *equals* $(X, \bullet_r)$ *and is a quasigroup satisfying for all* $a, b, c, d \in X$*:*

(1) (*idempotency*) $a \# a = a$;

(2) (*commutivity*) $a \# b = b \# a$;

(3) (*limited mediality*) *If* $a \# c = m = b \# d$, *then* $(a \# b) \# (c \# d) = m$.

Furthermore, $(X, \#_r) = (X, \bullet)$.

Conversely, if $(X, \#)$ *is an idempotent, commutative, limited medial quasigroup, then* $(X, \bullet) := (X, \#_r)$ *is a reflection quasigroup and* $(X, \bullet_r) = (X, \#)$.

The preceding theorem is quite satisfactory. It provides a characterization of the geometric mean operations arising from reflection quasigroups.

Remark 2.9. It was shown in [2] that the geometric mean operation of positive definite matrices satisfies the limited medial property (2.6). We have extended this result to any reflection quasigroup.

3. Dyadic powers, weighted means, and dyadic symmetric sets

Let $(X, \bullet, \varepsilon)$ be a pointed reflection quasigroup. If X is, in particular, a uniquely 2-divisible twisted subgroup, $\bullet$ is the core operation, and ε is the identity, then by the defining equation of $\varepsilon \# x$,

$$x = (\varepsilon \# x) \bullet \varepsilon = (\varepsilon \# x)\varepsilon^{-1}(\varepsilon \# x) = (\varepsilon \# x)^2$$

and thus $\varepsilon \# x = x^{1/2}$. We thus have for all $x, y \in X$,

$$\varepsilon \# x = x^{1/2} \qquad x \bullet \varepsilon = x^2 \qquad \varepsilon \bullet x = x^{-1},$$

where the last two equalities follow directly from the definition $x \bullet y = xy^{-1}x$. We take these equalities to be defining equalities for the powers $x^{1/2}$, x^2 and x^{-1} in the case of general pointed reflection quasigroups.

Note that the squaring map is right translation R_ε, the inversion map is $L_\varepsilon = S_\varepsilon$ and the square root map is left translation $L^{\#}_\varepsilon$ by ε in $(X, \#)$. Since

$$\varepsilon \bullet (x \bullet \varepsilon) = (\varepsilon \bullet x) \bullet (\varepsilon \bullet \varepsilon) = (\varepsilon \bullet x) \bullet \varepsilon,$$

it follows that R_ε and S_ε commute. Since $(\varepsilon \# x) \bullet \varepsilon = x$ (by the defining equation), it follows that $L^{\#}_\varepsilon$ given by $L^{\#}_\varepsilon(x) = \varepsilon \# x = x^{1/2}$ is the inverse for the bijection R_ε. Thus the square root map $L^{\#}_\varepsilon$ commutes with R_ε, and also commutes with S_ε since R_ε does. The mutual commutativity of these three operators allows us unambiguously to define the dyadic powers $x^{m/2^n}$ by an appropriate composition of these three maps. We note that in the uniquely 2-divisible setting, these powers agree with those computed with respect to the group operation.

Example 3.1. The additive group $\mathbb{D}$ of dyadic rationals (rational numbers with denominator a power of 2) is a uniquely 2-divisible subgroup of itself, and hence provides an important special example of a (pointed) reflection quasigroup. The reflection operation is given by $r \bullet s = 2r - s$ and the corresponding geometric mean operation is the standard midpoint operation $r \# s = (r + s)/2$. The reflection quasigroup $\mathbb{D}$ is called the *dyadic line* and a $\bullet$-homomorphism from $\mathbb{D}$ into a reflection quasigroup X is called a *dyadic geodesic.*

Theorem 3.2 (Corollary 5.8, [9]). *Let $(X, \bullet)$ be a reflection quasigroup and let $x, y \in X$. Then there exists a unique $\bullet$-homomorphism (and hence* *also #-homomorphism) γ from the dyadic line $(\mathbb{D}, \bullet)$ to X such that $\gamma(0) = x$ and $\gamma(1) = y$. For the particular case that $x = \varepsilon$ in a pointed reflection quasigroup, γ is given by $\gamma(t) = y^t$, the t^{th}-power for the dyadic t.*

The proof of the theorem involves showing that $t \mapsto y^t$, as defined in the preceding remarks, is the unique $\bullet$-homomorphism sending 0 to ε and 1 to y, and then extending to the general case by pointing the reflection quasigroup at x.

Remark 3.3. A function $\beta : \mathbb{D} \to \mathbb{D}$ is a $\bullet$-homomophism if and only if it is of the form $\beta(t) = at + b$ for constants $a, b \in \mathbb{D}$. Such maps clearly preserve midpoints, hence are #-homomorphisms, and thus $\bullet$-homomorphisms. The uniqueness statement in the preceding theorem guarantees that these exhaust the $\bullet$-homomorphisms. Thus a reparametrization of a dyadic geodesic is again a dyadic geodesic if and only if the reparametrization is affine.

Definition 3.4. We define the *t-weighted mean* $x\#_t y = \gamma(t)$, where γ is the unique dyadic geodesic with $\gamma(0) = x$ and $\gamma(1) = y$.

The weighted means allow the simultaneous extension of the reflection operation and the mean operation to an all-inclusive, comprehensive setting. For a reflection quasigroup, we define

$$\Phi : \mathbb{D} \times X \times X \to X, \qquad \Phi(t, x, y) := x\#_t y.$$

We call pairs (X, Φ) arising in this way *dyadic symmetric sets*. (In our original paper [9] we defined dyadic symmetric sets to be what in this paper are called "reflection quasigroups," but the terminology there was motivated by the existence of the weighted means.)

Remark 3.5. Observe that (i) $x = x\#_0 y = \Phi(0, x, y)$, (ii) $y = x\#_1 y = \Phi(1, x, y)$, (iii) $y\#x = x\#y = x\#_{\frac{1}{2}} y = \Phi(1/2, x, y)$, (iv) $x \bullet y = x\#_{-1} y = \Phi(-1, x, y)$, and (v) $y \bullet x = x\#_2 y = \Phi(2, x, y)$. (The first two follow from the definition of the weighted mean and the others all follow from the fact that $t \mapsto x\#_t y$ is a $\bullet$- and #-homomophism from $\mathbb{D}$.) Thus for a reflection quasigroup the map Φ incorporates and extends the reflection and mean operations into a dyadic "module."

A triple (X, Φ, ε) is a *pointed dyadic symmetric set* if (X, Φ) is a dyadic symmetric set and ε is some distinguished element of X. In this case we have that $\Phi(t, \varepsilon, y) = \varepsilon\#_t y = y^t$.

Example 3.6. For a vector space V over a field with $2 \neq 0$, we have

$$\Phi : \mathbb{D} \times V \times V \to V, \qquad \Phi(t, v, w) = (1 - t)v + tw.$$

(Note that $m/2^n$ is defined in the field for all integers m, n.)

We list elementary properties of a dyadic symmetric set.

Proposition 3.7. *Let (X, Φ, ε) be a pointed dyadic symmetric set. Then the following properties hold for general elements of $\mathbb{D}$ and X:*

(*i*) *The map $\Phi_{x,y} : \mathbb{D} \to X$ defined by $\Phi_{x,y}(t) = x\#_t y$ is a $\bullet$- and $\#$-homomorphism; in particular, this holds for $t \mapsto x^t : \mathbb{D} \to X$.*

(*ii*) *$x\#_t(x\#_s y) = x\#_{ts}y$; in particular, $(x^s)^t = x^{st}$.*

(*iii*) *$x\#_t y = y\#_{1-t}x$.*

(*iv*) *$x \bullet (y\#_t z) = (x \bullet y)\#_t(x \bullet z)$; in particular, $(y\#_t z)^{-1} = y^{-1}\#_t z^{-1}$.*

(*v*) *The maps $x \mapsto x\#_t b$, $t \neq 1$, and $y \mapsto a\#_t y$, $t \neq 0$, are bijective.*

Proof. Part (i) follows from the definition of $x\#_t y$. Parts $(ii) - (iv)$ follow from Theorem 3.2 (the uniqueness of the defining homomorphism) and the observation that affine maps $f(t) = at + b$ are $\bullet$-homomorphisms on $\mathbb{D}$, and isomorphisms for $a \neq 0$. For (ii) the two sides of the equality, thought of as maps of t, are $\bullet$-homomorphisms (the right hand side involves composition with $f(t) = st$) sending 0 to x and 1 to $x\#_s y$, and hence agree everywhere. For part (iii) the homomorphisms again agree at 0 and 1. For (iv) the left hand composition with S_x is a $\bullet$-homomorphism as a consequence of the distributive law and the two sides again agree at 0 and 1 (the second part is the special case $x = \varepsilon$). For (v), we may assume, by changing the pointing if necessary, that $a = \varepsilon$. Then $a\#_t y = \varepsilon\#_t y = y^t$. Since we saw at the beginning of the section that the map $y \mapsto y^t$, $t \neq 0$, is some appropropriate composition of the bijections S_ε, L_ε and $L^{\#}_\varepsilon$, we conclude that $y \mapsto y^t = a\#_t y$ is bijective. The bijectivity of $x \mapsto x\#_t b$ then follows from (iii). □

Lemma 3.8. *Let $(X, \bullet)$ be a reflection quasigroup, and let $m \in X$. Then $X_m := \{(a, b) \in X \times X | a\#b = m\}$ is a subquasigroup of the product quasigroup $X \times X$.*

Proof. Let $(a, b), (c, d) \in X$, i.e., $a\#b = m = c\#d$. By the defining equation of $\#$, $m \bullet a = (a\#b) \bullet a = b$ and similarly $m \bullet c = d$. Thus

$$m \bullet (a \bullet c) = (m \bullet a) \bullet (m \bullet c) = b \bullet d,$$

and hence $m = (a \bullet c)\#(b \bullet d)$. It follows from the limited medial property that $m = (a\#c)\#(b\#d)$. □

The preceding lemma allows us to derive an extended version of limited mediality in the context of dyadic symmetric sets.

Corollary 3.9. *Let $(X, \bullet)$ be a reflection quasigroup and suppose that $a\#c = m = b\#d$. Then $(a\#_t b)\#(c\#_t d) = m$ for all t.*

Proof. We note that $(a, c), (b, d) \in X_m$ as defined in Lemma 3.8. Since by that lemma X_m is a subquasigroup of $X \times X$ and hence a reflection quasigroup, there exists a dyadic geodesic $t \mapsto (a, c)\#_t(b, d)$ in X_m. Since the operations are defined coordinatewise, this map is a $\bullet$-homomorphism in each coordinate, and by uniqueness of this homomorphism we must have $(a, c)\#_t(b, d) = (a\#_t b, c\#_t d)$ for each t. Since the image is in X_m for each m, the corollary follows. □

Recall from Theorem 2.8 that the quasigroup $(X, \#)$ derived from a reflection quasigroup can be characterized as a quasigroup satisfying

(1) (idempotency) $a\#a = a$;

(2) (commutativity) $a\#b = b\#a$;

(3) (limited mediality) If $a\#c = m = b\#d$, then $(a\#b)\#(c\#d) = m$.

In addition we have seen that the map $\gamma(t) = a\#_t b$ is a $\#$-homomorphism from $\mathbb{D}$ to X carrying 0 to a and 1 to b. Thus

(4) $(a\#_r b)\#(a\#_s b) = \gamma(r)\#\gamma(s) = \gamma((r + s)/2) = a\#_{(r+s)/2}b$.

We can obtain a version of Theorem 2.8 for dyadic symmetric sets by showing that all of these properties generalize to the setting of weighted means and that these generalized properties characterize dyadic symmetric sets. The preceding properties are obtained from the corresponding ones in the following theorem by specializing to the case $t = 1/2$.

Theorem 3.10. *In a reflection quasigroup $(X, \bullet)$ the weighted means satisfy the following properties for all $a, b, c, d \in X$, all $r, s, t \in \mathbb{D}$:*

(0) $a\#_0 b = a$, $a\#_1 b = b$;

(1) (*idempotency*) $a\#_t a = a$;

(2) (*commutivity*) $a\#_t b = b\#_{1-t} a$;

(3) (*limited mediality*) *If* $a\#c = m = b\#d$, *then* $(a\#_t b)\#(c\#_t d) = m$;

(4) (*affine change of parameter*) $(a\#_r b)\#_t(a\#_s b) = a\#_{(1-t)r+ts} b$;

(5) (*exponential law*) $a\#_r(a\#_s b) = a\#_{rs} b$;

(6) (*cancellativity*) $a\#_t b = a\#_t c$ *for* $t \neq 0$ *implies* $b = c$.

Conversely if for $\Phi : \mathbb{D} \times X \times X \to X$, $a\#_t b := \Phi(t, a, b)$ *satisfies items* $(0) - (6)$, *then* $a\#_t b$ *is the* t*-weighted mean for the reflection quasigroup* X *with operations* $a \bullet b := a\#_{-1} b$ *and* $a\#b := a\#_{1/2} b$.

Proof. Property (0) holds by Remark 3.5. The unique dyadic geodesic carrying 0 and 1 to a is the constant map to a, so (1) is satisfied. Properties (2) and (5) we have already established in Proposition 3.7 and property (3) in Corollary 3.9. For (4) the left hand side is a $\bullet$-homomorphism in t, and the right hand side is also, since it is the composition of the dyadic geodesic $t \mapsto a\#_t b$ with the affine map on $\mathbb{D}$ sending t to $(1-t)r + ts$ (see Remark 3.3). Since they both sent 0 to $a\#_r b$ and 1 to $a\#_s b$, by uniqueness they agree. Property (6) follows from Proposition 3.7.

Conversely suppose that items (1) through (6) are satisfied. We set $a\#b := a\#_{1/2} b$. As we remarked before the theorem, properties $(1) - (3)$ ensure that the corresponding properties of Theorem 2.8 are satisfied by $\#$. Note that the equation $a\#x = b$ has solution $x = a\#_2 b$ since $a\#_{1/2}(a\#_2 b) = a\#_1 b = b$ by Properties (5) and (0). The uniqueness follows from (6), and then by commutivity $(X, \#)$ is a quasigroup. Thus by Theorem 2.8, $(X, \bullet) := (X, \#_r)$ is a reflection quasigroup with $\#$ as its mean. From

$$(a\#_2 b)\#a = a\#_{1/2}(a\#_2 b) = a\#_1 b = b,$$

we conclude that $b\#_r a = a\#_2 b = b\#_{-1} a$. Thus $a \bullet b = a\#_{-1} b$. Finally it follows from Property (4) with $t = 1/2$ that $t \mapsto a\#_t b$ is a $\#$-homomorphism, and hence a $\bullet$-homomorphism. Thus it is the unique dyadic geodesic carrying 0 to a and b to 1. $\square$

4. Solving equations involving means

In this section we apply the machinery of weighted means that we have developed to the solution of equations involving the mean operation. There is no attempt here to develop a comprehensive theory–only to illustrate how the need for such machinery arises and how it can be employed. In [12] we have studied lower degree symmetric matrix equations in some detail. There again the mean was a crucial tool, although it did not appear directly in the equations in the setting.

Throughout this section we assume that $(X, \bullet)$ is a dyadic symmetric set and $a\#b$ is the associated geometric mean operation on X. *We also assume that the weighted mean extends to all real numbers t and that the properties of the weighted mean developed in the last section remain valid in this context.* Many of the typical examples (positive definite matrices, positive elements of a C^*-algebra), indeed most topological examples, satisfy this requirement. We have studied in some detail the topological setting for the theory of dyadic symmetric sets and the extension of the weighted mean to all real parameters in [11].

Theorem 4.1. *The geometric mean $x = a\#b$ is the unique solution of*

$$(x\#a)\#(x\#b) = x. \tag{4.2}$$

Proof. Let $x = a\#b$. Then $x = x\#x = a\#b = (x\#a)\#(x\#b)$ by the limited medial property and hence $a\#b$ is a solution of the equation (4.2).

Conversely, suppose that $(x\#a)\#(x\#b) = x$. Then

$$x\#b = x \bullet (x\#a) = (x \bullet x)\#(x \bullet a) = x\#(x \bullet a)$$

and by the cancellative law $b = x \bullet a$. Therefore, $x = a\#b$. □

Theorem 4.2. *The weighted mean $a\#_{\frac{4}{3}}b$ is the unique solution of the equation*

$$x\#(x\#a) = b.$$

Furthermore, $a\#_{\frac{2}{3}}b$ is the unique solution of the equation

$$(a\#x)\#b = x.$$

Proof. Applying the left translation $a\#_{4/3}(\cdot)$ to

$$b = x\#(x\#a) = x\#_{1/2}(x\#_{1/2}a) = x\#_{1/4}a = a\#_{3/4}x,$$

we obtain $a\#_{4/3}b = a\#_{4/3}(a\#_{3/4}x) = x$. Conversely if $x = a\#_{4/3}b$, then

$$x\#(x\#a) = x\#_{1/4}a = a\#_{3/4}x = a\#_{3/4}(a\#_{4/3}b) = a\#_1 b = b.$$

Next, consider the equation $(a\#x)\#b = b\#(a\#x) = x$. By the defining property of the mean, this is equivalent to

$$b = x \bullet (a\#x) = (x \bullet a)\#(x \bullet x) = (x \bullet a)\#x. \tag{4.3}$$

Setting $y = x \bullet a$, we have $x = y\#a$ and thus (4.3) becomes

$$y\#(y\#a) = b.$$

This has the unique solution $y = a\#_{\frac{4}{3}}b$ and hence

$$x = y\#a = (a\#_{4/3}b)\#a = a\#_{1/2}(a\#_{4/3}b) = a\#_{\frac{2}{3}}b. \qquad \square$$

Remark 4.3. We observe that for a reflection quasigroup $(X, \bullet)$ possessing general weighted means, the two quasigroups $(X, \bullet)$ and $(X, \#)$ are *orthogonal* in the sense that given $a, b \in X$, the simultaneous equations

$$\begin{cases} a &= x \bullet y \\ b &= x\#y \end{cases}$$

have a unique solution. Indeed since $a = x \bullet y$ is equivalent to $x = a\#y$, the second equation reduces to

$$b = y\#(y\#a).$$

By Theorem 4.2, we can uniquely solve the system by

$$y = a\#_{\frac{4}{3}}b, \quad x = a\#(a\#_{\frac{4}{3}}b) = a\#_{\frac{2}{3}}b.$$

Theorem 4.4. *Let $(X, \bullet, \varepsilon)$ be a pointed dyadic symmetric set and $a, b \in X$. Then the simultaneous equations*

$$\begin{cases} c \bullet a &= b\#x \\ c \bullet x &= a\#x \end{cases}$$

have unique solutions in x and c, namely $x = a\#b$ and $c = a\#(b\#(a\#b)) = (a\#b)\#(a\#(a\#b)) = a\#_{3/8}b$. In particular, the geometric mean $x = a\#b$ is the unique solution of

$$a\#(b\#x) = x\#(x\#a).$$

Proof. The second equation is equivalent to

$$x = c \bullet (c \bullet x) = c \bullet (a\#x) = (c \bullet a)\#(c \bullet x).$$

By the defining equation of the geometric mean, this is equivalent to

$$c \bullet a = x \bullet (c \bullet x).$$

Combining this with the two equations of the theorem, we have

$$b\#x = c \bullet a = x \bullet (c \bullet x) = x \bullet (a\#x) = (x \bullet a)\#(x \bullet x) = (x \bullet a)\#x.$$

From the cancellative property of the mean operation, $b = x \bullet a$ and therefore $x = a\#b$ by the mean defining equation. The characterizations for c in order that the original equations be satisfied for $x = a\#b$ then follow directly from those equations. By properties of the weighted mean in the previous section, these both reduce to $a\#_{3/8}b$, and hence are equal. Their equality ensures that $a\#b$ is a solution of the equation in the last assertion of the theorem.

Suppose that $a\#(b\#x) = x\#(x\#a)$. Setting $c = b\#x$ and $d = x\#a$, we have from the limited medial property that

$$a\#c = x\#d = (a\#x)\#(c\#d) = d\#(c\#d).$$

From the cancellation property, $x = c\#d = (x\#b)\#(x\#a)$. By Theorem 4.1, $x = a\#b$. Thus the solution in the last equation of the theorem is unique. □

References

[1] **T. Ando**: *Topics on Operator Inequalities*, Lecture Notes Hokkaido Univ., Sapporo, 1978.

[2] **T. Ando, C.-K. Li, R. Mathias**: *The geometric mean*, Linear Algebra Appl. **385** (2004), 305 – 334.

[3] **M. Aschbacher**: *Near subgroups of finite groups*, J. Group Theory **1** (1998), 113 – 129.

[4] **M. Fiedler and V. Pták**: *A new positive definite geometric mean of two positive definite matrices*, Linear Algebra Appl. **251** (1997), 1 – 20.

[5] **D. Joyce**: *A classifying invariant of knots, the knot quandle*, J. Pure Appl. Alg. **23** (1982), 37 – 66.

[6] **M. Kano, H. Nagao, and N. Nobusawa**: *On finite homogeneous symmetric sets*, Osaka J. Math. **13** (1976), 399 – 406.

[7] **F. Kubo and T. Ando**: *Means of positive linear operators*, Math. Ann. **246** (1980), 205 – 224.

[8] **J. D. Lawson and Y. Lim**: *The geometric mean, matrices, metrics, and more*, Amer. Math. Monthly **108** (2001), 797 – 812.

[9] **J. D. Lawson and Y. Lim**: *Symmetric sets with midpoints and algebraically equivalent theories*, Results in Mathematics **46** (2004), 37 – 56.

[10] **J. D. Lawson and Y. Lim**: *Means on dyadic symmetric sets and polar decompositions,* Abh. Math. Sem. Univ. Hamburg **74** (2004), 135 – 150.

[11] **J. D. Lawson and Y. Lim**: *Symmetric spaces with convex metrics*, to appear Forum Math.

[12] **J. D. Lawson and Y. Lim**: *Solving symmetric matrix word equations via symmetric space machinary*, Linear Algebra and Appl. **414** (2006), 560 – 569.

[13] **Y. Lim**: *Geometric means on symmetric cones*, Arch. der Math. **75** (2000), 39 – 45.

[14] **O. Loos**: Symmetric spaces, I: General Theory, Benjamin, New York, Amsterdam, 1969.

[15] **N. Nobusawa**: *On symmetric structure of a finite set*, Osaka J. Math. **11** (1974), 569 – 575.

[16] **N. Nobusawa**: *Orthogonal groups and symmetric sets*, Osaka J. Math. **20** (1983), 5 – 8.

[17] **R. Pierce**: *Symmetric groupoids*, Osaka J. Math. **15** (1978), 51 – 76.

[18] **R. Pierce**: *Symmetric groupoids II*, Osaka J. Math. **16** (1979), 317 – 348.

[19] **D. Stanovský**: *Left distributive left quasigroups*, Ph.D. thesis, Charles University, Prague, 2004.

[20] **M. Takasaki**: *Abstraction of symmetric transformations* (in Japanese), Tohaku Math. J. **49** (1942/3), 145 – 207.

Received November 6, 2005

Jimmie Lawson
Department of Mathematics
Louisiana State University
Baton Rouge, LA 70803
U.S.A.
e-mail: lawson@math.lsu.edu

Yongdo Lim
Department of Mathematics
Kyungpook National Universit
Taegu 702-701
Korea
e-mail: ylim@knu.ac.kr