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A new construction of Bol loops: The "odd" case

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Abstract

In [2], the authors presented a new construction of Bol loops as extensions of a loop B by $C_m \times C_n$, in the case that B contains a central element of order 2 and that m and n are even. In the final section of that paper, the authors remark that the assumption that m and n are even is important and promise to investigate this question in more detail in a future paper. This is that paper.

1. Introduction

A loop is *Moufang* if it satisfies the identity $(xy \cdot z)y = x(y \cdot zy)$ and (*right*) *Bol* if satisfies $(xy \cdot z)y = x(yz \cdot y)$. A loop is *left Bol* if it satisfies the reflection of the right Bol identity, namely, $y(z \cdot yx) = (y \cdot zy)x$. Throughout this paper, whether or not we say so explicitly, we assume that Bol loops are right Bol. In the recent literature (see [4] and many of the references in its bibliography), there has been much interest in Bol loops that satisfy the automorphic inverse property, which is expressed by the additional identity $(xy)^{-1} = x^{-1}y^{-1}$. Such loops are known by various names, including *Bruck loops*, *K-loops* and *gyrogroups*.

The theory of Moufang loops is now quite mature, but such is far from the case for Bol loops. For instance, whereas the 158 (nonassociative) Moufang loops of order less than 64 have been known for quite a while [3], our

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knowledge of the non-Moufang Bol loops of some relatively "small" orders (e.g., 24 and 30) is far from complete. For sure, Bol loops are much more abundant than Moufang loops. There are, for example, 2033 non-Moufang Bol loops of order 16 (constructed computationally by Eric Moorhouse [5]) but only five Moufang loops that are not groups. It is therefore always of interest to the subject whenever "new" ways to produce Bol loops are found.

Theorem 1.1. *Let m and n be positive integers, let B be a loop, let r, s, t, z and w be (not necessarily distinct) elements in $\mathcal{Z}(B)$, the centre of B, such that $r^m = r^n = s^2 = t^2 = 1$. Let $L = B \times C_m \times C_n$ with multiplication defined by*

$$[a, i, \pi][b, j, \rho] = [abr^{j\pi} s^{ij\rho} t^{j\pi\rho} z^p w^q, (i + j)^\star, (\pi + \rho)'],$$

where, for any integer i, $i^\star$ and i' denote the least nonnegative residues of i modulo m and n, respectively, $p = \frac{i+j-(i+j)^\star}{m}$ and $q = \frac{\pi+\rho-(\pi+\rho)'}{n}$. Then L is a loop which we denote $L(B, m, n, r, s, t, z, w)$.

Remark 1.2. In this paper, and in the proof of Theorem 1.1 in particular, we use frequently and implicitly that for any integers i and j,

1. $(i + j^*)^* = (i + j)^*$,
2. $i^* = j^*$ if and only if $m \mid (i - j)$ and
3. $(-i)^* = (-i^*)^*$,

with analogous properties holding for $'$.

Proof. Let $[a, i, \pi]$ and $[b, j, \rho]$ be elements of L. Let

$$k = (j - i)^\star, \quad \sigma = (\rho - \pi)',$$

$$p = \frac{i + k - (i + k)^\star}{m}, \quad q = \frac{\pi + \sigma - (\pi + \sigma)'}{n}$$

and $x = dr^{-k\pi} s^{-ik\sigma} t^{-k\pi\sigma} z^{-p} w^{-q}$, where d is the unique element of B such that $ad = b$. Then $[a, i, \pi][x, k, \sigma] = [b, j, \rho]$. Let $y = er^{-i\sigma} s^{-ik\pi} t^{-i\pi\sigma} z^{-p} w^{-q}$, where e is the unique element of B such that $ea = b$. Then $[y, k, \sigma][a, i, \pi] = [b, j, \rho]$. Thus, L is a quasigroup. Clearly, $[1, 0, 0]$ is an identity element, so L is a loop. $\square$

Before continuing, we explain our rationale for the notation in Theorem 1.1. We think of the elements of the loop as being of the form $(au^i)v^\pi$

(hence the notation $[a, i, \pi]$), where $a \in B$, u generates C_m and v generates C_n, and where $u^m = z \in \mathcal{Z}(B)$ and $v^n = w \in \mathcal{Z}(B)$. Thus, for example, $u^{i+j} = z^p u^{(i+j)^\star}$, where $i + j = pm + (i + j)^\star$. We used a mix of Roman and Greek characters to indicate the source from which an exponent comes—Roman for the exponents of u, the second coordinate, and Greek for the exponents of v, the third coordinate. The elements r, s and t represent commutators and associators. Specifically, r represents the commutator of v and u, and the exponent $j\pi$ indicates that we are considering the commutator of v^π and u^j. Similarly, s and t, respectively, represent the associators (u, u, v) and (v, u, v), and the exponents $ij\rho$ on s and $j\pi\rho$ on t indicate that we are associating (u^i, u^j, v^ρ) and (v^π, u^j, v^ρ), respectively.

In [2], where m and n were assumed to be even, we showed that L is a right Bol loop if B is a right Bol loop and, moreover, that the conditions $r^m = r^n = s^2 = t^2 = 1$ are necessary to ensure this is the case. In this paper, we consider the possibility that m or n is odd and show that this condition implies $s = t = 1$. Thus the following theorem becomes relevant.

Theorem 1.3. *Let $L = L(B, m, n, r, s, t, z, w)$ be the loop constructed in Theorem 1.1 and assume that B is a right Bol loop. If $s = t = 1$, then L is a right Bol loop.*

Proof. Let $x_1 = [a, i, \pi]$, $x_2 = [b, j, \rho]$, $x_3 = [c, k, \sigma]$. Then

$$x_1 x_2 = [abr^{j\pi} z^{p_1} w^{q_1}, (i + j)^\star, (\pi + \rho)'],$$

where $i + j = p_1 m + (i + j)^\star$ and $\pi + \rho = q_1 n + (\pi + \rho)'$. Further,

$$(x_1 x_2)x_3 = [(ab \cdot c)r^{j\pi + k(\pi+\rho)'} z^{p_1+p_2} w^{q_1+q_2}, ((i + j)^* + k)^*, ((\pi + \rho)' + \sigma)'],$$

where

$$\begin{aligned} (i + j)^* + k &= p_2 m + ((i + j)^* + k)^* \\ \text{and} \qquad (\pi + \rho)' + \sigma &= q_2 n + ((\pi + \rho)' + \sigma)'. \end{aligned} \tag{1.1}$$

Now $((i + j)^* + k)^* = (i + j + k)^*$ and $((\pi + \rho)' + \sigma)' = (\pi + \rho + \sigma)'$. Then, making the substitutions $(i+j)^* = (i+j)-p_1 m$ and $(\pi+\rho)' = (\pi+\rho)-q_1 n$, equations (1.1) become

$$i + j + k = (p_1 + p_2)m + (i + j + k)^*$$

and

$$\pi + \rho + \sigma = (q_1 + q_2)n + (\pi + \rho + \sigma)'.$$

Similar observations and calculations give

$$[(x_1x_2)x_3]x_2 = [\{(ab{\cdot}c)b\}r^{\gamma_1}z^{p_1+p_2+p_3}w^{q_1+q_2+q_3}, (i+2j+k)^\star, (\pi+2\rho+\sigma)'],$$

where

$$i+2j+k = (p_1+p_2+p_3)m + (i+2j+k)^\star,$$
$$\pi+2\rho+\sigma = (q_1+q_2+q_3)n + (i+2j+k)'$$

and $\gamma_1 = j\pi + k(\pi+\rho)' + j(\pi+\rho+\sigma)'$.

Similarly,

$$x_2x_3 = [bcr^{k\rho}z^{p_4}w^{q_4}, (j+k)^\star, (\rho+\sigma)'],$$

where $j+k = p_4m + (j+k)^\star$ and $\rho+\sigma = q_4n + (\rho+\sigma)'$,

$$(x_2x_3)x_2 = [(bc\cdot b)r^{k\rho+j(\rho+\sigma)'}z^{p_4+p_5}w^{q_4+q_5}, 2j+k, 2\rho+\sigma],$$

where $2j+k = (p_4+p_5)m + (2j+k)^\star$ and $2\rho+\sigma = (q_4+q_5)n + (2\rho+\sigma)'$, and

$$x_1[(x_2x_3)x_2] = [a\{bc\cdot b\}r^{\gamma_2}z^{p_4+p_5+p_6}w^{q_4+q_5+q_6}, (i+2j+k)^\star, (\pi+2\rho+\sigma)'],$$

where

$$i+2j+k = (p_4+p_5+p_6)m + (i+2j+k)^\star,$$
$$\pi+2\rho+\sigma = (q_4+q_5+q_6)n + (\pi+2\rho+\sigma)',$$

and $\gamma_2 = k\rho + j(\rho+\sigma)' + (2j+k)^*\pi$. It follows that $p_1+p_2+p_3 = p_4+p_5+p_6$ and $q_1+q_2+q_3 = q_4+q_5+q_6$. Also, $r^{d^*} = r^{d'} = r^d$ for any integer d because $r^m = r^n = 1$, so $r^{\gamma_1} = r^{j\pi+k(\pi+\rho)+j(\pi+\rho+\sigma)} = r^{2j\pi+k\pi+k\rho+j\rho+j\sigma} = r^{\gamma_2}$. We have shown that the right Bol identity holds in L. □

2. Properties of the loop $L(B,m,n,r,s,t,z,w)$

Let $L = L(B,m,n,r,s,t,z,w)$ be a loop as in Theorem 1.1.

Theorem 2.1. *Regardless of the values of m and n, if d is in the centre of B, then $[d,0,0]$ is in the centre of L.*

Proof. Let $x_1 = [a,i,\pi]$, $x_2 = [b,j,\rho]$ and $x_3 = [c,k,\sigma]$. Then

$$x_1x_2 = [a,i,\pi][b,j,\rho] = [abr^{j\pi}s^{ij\rho}t^{j\pi\rho}z^pw^q, (i+j)^\star, (\pi+\rho)']$$

and

$$x_2x_1 = [b,j,\rho][a,i,\pi] = [bar^{i\rho}s^{ij\pi}t^{i\pi\rho}z^pw^q, (i+j)^\star, (\pi+\rho)'].$$

Setting $j=\rho=0$, it is clear that if b commutes with every element of B, then $[b,0,0]$ commutes with every element of L.
Similarly,

$$\begin{aligned}(x_1x_2)x_3 &= ([a,i,\pi][b,j,\rho])[c,k,\sigma]\\ &= [abr^{j\pi}s^{ij\rho}t^{j\pi\rho}z^{p_1}w^{q_1},(i+j)^*,(\pi+\rho)'][c,k,\sigma]\\ &= [(ab\cdot c)r^{j\pi+k(\pi+\rho)'}s^{ij\rho+(i+j)^*k\sigma}t^{j\pi\rho+k(\pi+\rho)'\sigma}z^{p_1+p_2}w^{q_1+q_2},\\ &\qquad (i+j+k)^*,(\pi+\rho+\sigma)']\end{aligned}$$

and

$$\begin{aligned}x_1(x_2x_3) &= [a,i,\pi]([b,j,\rho][c,k,\sigma])\\ &= [a,i,\pi][bcr^{k\rho}s^{jk\sigma}t^{k\rho\sigma}z^{p_3}w^{q_3},(j+k)^\star,(\rho+\sigma)']\\ &= [(a\cdot bc)r^{k\rho+(j+k)^*\pi}s^{jk\sigma+i(j+k)^*(\rho+\sigma)'}t^{k\rho\sigma+(j+k)^*\pi(\rho+\sigma)'}z^{p_3+p_4}w^{q_3+q_4},\\ &\qquad (i+j+k)^*,(\pi+\rho+\sigma)'],\end{aligned}$$

where p_1,p_2,p_3,p_4 are defined by the equations

$$\begin{array}{rcl} i+j &=& p_1m+(i+j)^*,\\ i+j+k &=& (p_1+p_2)m+(i+j+k)^*,\\ j+k &=& p_3m+(j+k)^*,\\ i+j+k &=& (p_3+p_4)m+(i+j+k)^* \end{array} \tag{2.1}$$

and q_1,q_2,q_3,q_4 by the equations

$$\begin{array}{rcl} \pi+\rho &=& q_1n+(\pi+\rho)',\\ \pi+\rho+\sigma &=& (q_1+q_2)n+(\pi+\rho+\sigma)',\\ \rho+\sigma &=& q_3n+(\rho+\sigma)',\\ \pi+\rho+\sigma &=& (q_3+q_4)n+(\pi+\rho+\sigma)'. \end{array} \tag{2.2}$$

(Consequently, $p_1+p_2=p_3+p_4$ and $q_1+q_2=q_3+q_4$.) Setting $k=\sigma=0$, it is clear then that, if $(ab)c=a(bc)$ for every $a,b\in B$, then $([a,i,\pi][b,j,\rho])[c,0,0]=[a,i,\pi]([b,j,\rho][c,0,0])$ for every $[a,i,\pi]$ and $[b,j,\rho]$ in L. In other words, if c is in the right nucleus of B, then $[c,0,0]$ is in the right nucleus of L. Similarly, if c is the middle (respectively left) nucleus of B, then $[c,0,0]$ is in the middle (respectively left) nucleus of L. So if c is in the nucleus of B, then $[c,0,0]$ is in the nucleus of L. This completes the proof. □

Corollary 2.2. *If B is centrally nilpotent of class at most 2, then so is L.*

Proof. If B is centrally nilpotent of class at most 2, then all commutators and associators in B are central. But it is clear from the proof of Theorem 2.1 that any commutator or associator in L is of the form $[d, 0, 0]$, $d \in \mathcal{Z}(B)$, the centre of B. Therefore, by Theorem 2.1, all commutators and associators in L are central, so L is centrally nilpotent of class at most 2. □

Corollary 2.3. *Let M be the loop $L(\mathcal{Z}, m, n, r, s, t, z, w)$, $\mathcal{Z} = \mathcal{Z}(B)$. Assume at least one of r, s, t is different from 1. Then M is centrally nilpotent of class 2.*

Proof. Let $u = [1, 1, 0]$ and $v = [1, 0, 1]$. Then u and v are in M. Setting $i = \rho = 1$ and $j = \pi = 0$ in the expressions for x_1x_2 and x_2x_1 in Theorem 2.1 gives $(u, v) = [r^{-1}, 0, 0]$. Setting $i = j = \sigma = 1$ and $k = \pi = \rho = 0$ in the expressions for $(x_1x_2)x_3$ and $x_1(x_2x_3)$ gives $(u, u, v) = [s^{-1}, 0, 0]$ and setting $i = \rho = k = 0$ and $\pi = j = \sigma = 1$ gives $(v, u, v) = [t^{-1}, 0, 0]$. Since $\mathcal{Z}$ is an abelian group, M is centrally nilpotent of class at most 2 by Corollary 2.2. Since (u, v), (u, u, v) and (v, u, v) are not all the identity element, $[1, 0, 0]$, M is not an abelian group. Therefore, M is centrally nilpotent of class 2. □

In the proof of the theorem that follows, we use the facts that if x and y are elements of a centrally nilpotent Bol loop of class 2 and if k is any integer, then $(x, x, y)^k = (x^k, x, y)$ and $(x, y, x) = (x, x, y)^{-1}$ [1, Corollary 2.7 and Lemma 2.2].

Let $u = [1, 1, 0]$ and $v = [1, 0, 1]$ be as above. It is not hard to show by induction that, for $k < m$, $u^k = [1, k, 0]$, and that $u^m = [1, m-1, 0][1, 1, 0] = [z, 0, 0] \in \mathcal{Z}(L)$. Similarly, $v^n = [w, 0, 0] \in \mathcal{Z}(L)$.

Theorem 2.4. *If $L(B, m, n, r, s, t, z, w)$ is a right Bol loop, then $s^m = t^n = 1$.*

Proof. Let M be as in Corollary 2.3, $u = [1, 1, 0]$ and $v = [1, 0, 1]$. Since M is a centrally nilpotent Bol loop of class 2, we have $[s^{-m}, 0, 0] = (u, u, v)^m = (u^m, u, v) = [1, 0, 0]$ (since $u^m \in \mathcal{Z}(L)$) and, similarly, $[t^{-n}, 0, 0] = (v, u, v)^n = (v, v, u)^{-n} = (v^{-n}, v, u) = [1, 0, 0]$. □

Corollary 2.5. *Suppose $L(B, m, n, r, s, t, z, w)$ is a right Bol loop. If m is odd, then $s = 1$ and if n is odd, then $t = 1$.*

Proof. This follows from $s^m = s^2 = 1$ and $t^n = t^2 = 1$. □

Corollary 2.6. *Let $x_1 = [a, i, \pi]$, $x_2 = [b, j, \rho]$ and $x_3 = [c, k, \sigma]$ be elements of the loop $L = L(B, m, n, r, s, t, z, w)$. If L is a right Bol loop, then the commutator of x_1 and x_2, and the associator of x_1, x_2, and x_3 are as follows:*

$$(x_1, x_2) = [(a, b)r^{j\pi - i\rho}s^{ij(\pi+\rho)}t^{\pi\rho(i+j)}, 0, 0] \tag{2.3}$$

$$(x_1, x_2, x_3) = [(a, b, c)s^{ij\rho+ik\sigma+i(j+k)(\rho+\sigma)'}t^{j\pi\rho+k\pi\sigma+(j+k)^*\pi(\rho+\sigma)}, 0, 0]. \tag{2.4}$$

Proof. The expression for the commutator follows from the expressions for x_1x_2 and x_2x_1 given in the proof of Theorem 2.1 using the fact that $s^2 = t^2 = 1$. Now $r^m = 1 = s^m$, so $r^{k^*} = r^k$ and $s^{k^*} = s^k$ for any integer k, and $r^n = 1 = t^n$, so $r^{k'} = r^k$ and $t^{k'} = t^k$ for any integer k. With reference again to the proof of Theorem 2.1, it follows that

$$\begin{aligned}(x_1x_2)x_3 = [(ab \cdot c)&r^{j\pi+k\pi+k\rho}s^{ij\rho+ik\sigma+jk\sigma}t^{j\pi\rho+k\pi\sigma+k\rho\sigma}z^{p_1+p_2}w^{q_1+q_2},\\ &(i+j+k)^*, (\pi+\rho+\sigma)']\end{aligned}$$

and

$$\begin{aligned}&x_1(x_2x_3)\\ &\quad= [(a \cdot bc)r^{k\rho+j\pi+k\pi}s^{jk\sigma+i(j+k)(\rho+\sigma)'}t^{k\rho\sigma+(j+k)^*\pi(\rho+\sigma)}z^{p_1+p_2}w^{q_1+q_2},\\ &\qquad\quad (i+j+k)^*, (\pi+\rho+\sigma)'].\end{aligned}$$

So the associator (x_1, x_2, x_3) is as stated. □

3. The case $s = t = 1$

In the remaining sections of this paper, it turns out to be the case that $s = t = 1$, so we find it convenient to summarize here facts about loops of the form $L(B, m, n, r, 1, 1, z, w)$.

First note that when $s = t = 1$, the multiplication rule in L becomes

$$[a, i, \pi][b, j, \rho] = [abr^{j\pi}z^pw^q, (i+j)^\star, (\pi+\rho)']$$

with p and q defined by

$$i + j = pm + (i+j)^* \quad \text{and} \quad \pi + \rho = qn + (\pi+\rho)'. \tag{3.1}$$

Let $x_1 = [a,i,\pi]$ and $x_2 = [b,j,\rho]$. Setting $s = t = 1$ in (2.3), the commutator of x_1 and x_2 is

$$(x_1, x_2) = [(a,b)r^{j\pi - i\rho}, 0, 0]. \tag{3.2}$$

In particular, L is commutative if and only if B is commutative and $r = 1$.

With $x_3 = [c,k,\sigma]$ (and x_1, x_2 as above), setting $s = t = 1$ in (2.4) shows that the associator of x_1, x_2 and x_3 is

$$(x_1, x_2, x_3) = [(a,b,c), 0, 0]. \tag{3.3}$$

In particular, L is associative if and only if B is associative.

With x_1, x_2, x_3 as above, we have

$$(x_1 x_2 \cdot x_3)x_2 = [[(ab\cdot c)b]r^{j\pi + k(\pi+\rho)' + j(\pi+\rho+\sigma)'}, (i+2j+k)^*, (\pi + 2\rho + \sigma)']$$

and

$$x_1(x_2 x_3 \cdot x_2) = [[a(bc\cdot b)]r^{k\rho + j(\rho+\sigma)' + (2j+k)^*\pi}, (i+2j+k)^*, (\pi + 2\rho + \sigma)'].$$

Since $r^m = r^n = 1$, the $'$s and *s can be dropped in the exponents of r. The exponents then become the same and it is clear that L is Moufang if and only if B is Moufang.

Can L be a Bruck loop? It is a straightforward computation to see that, for $[a,i,\pi] \in L$,

$$[a,i,\pi]^{-1} = [a^{-1}r^{i\pi}z^p w^q, (-i)^\star, (-\pi)'], \tag{3.4}$$

where $p = \left\{ \begin{array}{rl} 0 & \text{if } i = 0 \\ -1 & \text{otherwise} \end{array} \right\}$ and $q = \left\{ \begin{array}{rl} 0 & \text{if } \pi = 0 \\ -1 & \text{otherwise} \end{array} \right\}$.

Let $x = [1,0,1]$ and $y = [1,1,0]$. Then $xy = [r,1,1]$ and $(xy)^{-1} = [z^{-1}w^{-1}, m-1, n-1]$. Also $x^{-1} = [w^{-1}, 0, n-1]$ and $y^{-1} = [z^{-1}, m-1, 0]$, so $x^{-1}y^{-1} = [z^{-1}w^{-1}r, m-1, n-1]$. Thus, if L is a Bruck loop, then $r = 1$, the subloop $\overline{B} = \{[b,0,0] \mid b \in B\} \cong B$ is Bruck, and the multiplication rule becomes

$$[a,i,\pi][b,j,\rho] = [abz^p w^q, (i+j)^\star, (\pi+\rho)'], \tag{3.5}$$

with p and q defined by equations (3.1).

Conversely, suppose B is a Bruck loop and $r = 1$ so that multiplication in L is defined by (3.5). Let $x = [a,i,\pi]$ and $y = [b,j,\rho]$.

Then

$$xy = [a,i,\pi][b,j,\rho] = [abz^{p_1}w^{q_1}, (i+j)^\star, (\pi+\rho)'],$$

with

$$i+j = p_1 m + (i+j)^* \quad \text{and} \quad \pi+\rho = q_1 n + (\pi+\rho)'.$$

so

$$(xy)^{-1} = [(ab)^{-1}z^{-p_1+p_2}x^{-q_1+q_2}, (-(i+j)^*)^*, (-(\pi+\rho)')']$$

where

$$p_2 = \begin{cases} 0 & \text{if } (i+j)^* = 0 \\ -1 & \text{otherwise} \end{cases} \qquad \text{and} \qquad q_2 = \begin{cases} 0 & \text{if } (\pi+\rho)^* = 0 \\ -1 & \text{otherwise.} \end{cases}$$

Now $x^{-1} = [a^{-1}z^{p_3}w^{q_3}, (-i)^*, (-\pi)']$ where

$$p_3 = \begin{cases} 0 & \text{if } i = 0 \\ -1 & \text{otherwise} \end{cases} \qquad \text{and} \qquad q_3 = \begin{cases} 0 & \text{if } \pi = 0 \\ -1 & \text{otherwise} \end{cases}$$

and $y^{-1} = [b^{-1}z^{p_4}w^{q_4}, (-j)^*, (-\rho)']$ where

$$p_4 = \begin{cases} 0 & \text{if } j = 0 \\ -1 & \text{otherwise} \end{cases} \qquad \text{and} \qquad q_4 = \begin{cases} 0 & \text{if } \rho = 0 \\ -1 & \text{otherwise} \end{cases}$$

so

$$x^{-1}y^{-1} = [a^{-1}b^{-1}z^{p_3+p_4+p_5}w^{q_3+q_4+q_5}, ((-i)^* + (-j)^*)^*, ((-\pi)' + (-\rho')'],$$

where

$$\begin{aligned} (-i)^* + (-j)^* &= p_5 m + ((-i)^* + (-j)^*)^*, \\ (-\pi)' + (-\rho)' &= q_5 n + ((-\pi)' + (-\rho)')'. \end{aligned}$$

Now $(-(i+j)^*)^* = (-(i+j))^* = (-i-j)^*$ and $((-i)^* + (-j)^*)^* = ((-i)+(-j))^* = (-i-j)^*$, so the second components of $(xy)^{-1}$ and $x^{-1}y^{-1}$ are equal. Similarly, so are the third. That the first components are also equal follows from the table below which shows that $-p_1+p_2 = p_3+p_4+p_5$

in all cases (and a similar table showing that $-q_1+q_2=q_3+q_4+q_5$ in all cases as well).

i	j	$(i+j)^*$	p_1	p_2	p_3	p_4	p_5
0	0	0	0	0	0	0	0
0	$\neq 0$	$\neq 0$	0	-1	0	-1	0
$\neq 0$	0	$\neq 0$	0	-1	-1	0	0
$\neq 0$	$\neq 0$	0	1	0	-1	-1	1
$\neq 0$	$\neq 0$	$\neq 0$	$\begin{cases} 0 & i+j<m \\ 1 & i+j>m \end{cases}$	-1	-1	-1	$\begin{cases} 1 & i+j<m \\ 0 & i+j>m \end{cases}$

Thus L is a Bruck loop. In summary, L is Bruck if and only if B is Bruck and $r=1$.

Now let $\mathrm{Comm}(S)$, $\mathrm{Ass}(S)$ and S' respectively denote the subloop of a loop S generated by all commutators, the subloop of S generated by all associators, and the commutator/associator subloop of S. Let $C(S)$ and $N(S)$ denote, respectively, the centrum and the nucleus. Then

1. $\mathrm{Comm}(L)=\langle \mathrm{Comm}(B), r\rangle$,
2. $\mathrm{Ass}(L)=\mathrm{Ass}(B)$,
3. $L'=\langle B', r\rangle$,
4. $N(L)=\{[c,k,\sigma] \mid c\in N(B)\}$, (In particular, the nucleus of L can never be trivial.)
5 $C(L)=\{[a,i,\pi] \mid a\in C(B), r^i=r^\pi=1\}$,
6. $\mathcal{Z}(L)=\{[a,i,\pi] \mid a\in \mathcal{Z}(B),\ r^i=r^\pi=1\}$,
7. $N(L)=\mathcal{Z}(L)$ if and only if $N(B)=\mathcal{Z}(B)$ and $r=1$, and
8. $N(L)\lhd L$ if and only if $N(B)\lhd B$.

Properties (1)–(7) follow directly from the expressions for commutator and associator in (2.3) and (2.4). Property (8) require a bit more computation but essentially follows from the multiplication law and Property (4).

4. The case that m or n is odd

Suppose $m=2k+1$, $k\geq 1$, is odd. Then $s=1$ by Corollary 2.5.

If n is odd, then $t=1$ using Corollary 2.5 a second time. Suppose $n=2\ell$, $\ell\geq 1$, is even. Let $g=\gcd(m,n)$ and let $x_1=[1,0,1]$, $x_2=[1,k,0]$

and $x_3 = [1, 1, 1]$. With q defined by $2 = qn + 2'$, we find

$$\begin{aligned}(x_1x_2 \cdot x_3)x_2 &= \{([1, 0, 1][1, k, 0])[1, 1, 1]\}[1, k, 0]\\ &= ([r^k, k, 1][1, 1, 1])[1, k, 0]\\ &= [r^{k+1}tw^q, k + 1, 2'][1, k, 0] = [r^{k+1+2'k}tzw^q, 0, 2']\end{aligned}$$

whereas

$$\begin{aligned}x_1(x_2x_3 \cdot x_2) &= [1, 0, 1]\{([1, k, 0][1, 1, 1])[1, k, 0]\}\\ &= [1, 0, 1]([1, k + 1, 1][1, k, 0])\\ &= [1, 0, 1][r^k z, 0, 1] = [r^k zw^q, 0, 2'],\end{aligned}$$

so $r^{k+1+2'k}t = r^k$, giving $r^{2'k+1}t = 1$. Since $r^m = r^n = 1$, $r^g = 1$ so, if $n = 2$, then $g = 1$. This implies $r = 1$ and $t = 1$. If $n \neq 2$, then $2' = 2$ and the equation $r^{2'k+1}t = 1$ reads $r^m t = 1$ so, again, $t = 1$. Thus, if n is even, we again have $t = 1$ and, if $n = 2$, $r = 1$ as well.

We summarize.

Theorem 4.1. *Let m and n be positive integers with either m or n odd. Let B be a loop satisfying the right Bol identity. If $m \neq 2$ and $n \neq 2$, then $L = L(B, m, n, r, s, t, z, w)$ is a Bol loop if and only if $s = t = 1$. If $m = 2$ or $n = 2$, then L is a right Bol loop if and only if $r = s = t = 1$.*

Remark 4.2. Let $u = [1, 1, 0]$ and $v = [1, 0, 1]$. Since $u^m = [z, 0, 0]$ and $v^n = [w, 0, 0]$ are central, in the case that $s = t = r = 1$, L is just the quotient of the direct product $B \times C_m \times C_n$ by the normal (central) subgroup generated by $u^m z^{-1}$ and $v^n w^{-1}$, where $z = [z, 0, 0]$ and $w = [w, 0, 0]$.

Acknowledgement. We would like to thank the referee for noting that our construction may be viewed as a special case of the following, more general, construction: Let B be a loop, let G be an abelian group, and let ζ be a map from $G \times G \to Z(B)$. Define multiplication on $L = B \times G$ by $(a, \alpha) \circ (b, \beta) = (ab\zeta(\alpha, \beta), \alpha + \beta)$. If we place appropriate conditions on ζ, then L becomes a Bol loop, and many of our proofs and calculations become less messy.

Nevertheless, we prefer our notation for two reasons—it is the notation that was used in [2] and it makes more clear how the multiplication in L depends only on the multiplication in B, the elements $z = u^m$ and $w = v^n$, the commutator $r = (v, u)$ and the associators $s = (u, u, v)$ and $t = (v, u, v)$.

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