Quotient hyper BCK-algebras

Arsham Borumand Saeid and Mohammad M. Zahedi

Abstract

In this note first we use the equivalence relation \( \sim_I \) which has been introduced in [1] and construct a quotient hyper BCK-algebra \( H/I \) from a hyper BCK-algebra \( H \) via a reflexive hyper BCK-ideal \( I \) of \( H \). Then we study the properties of this algebra, in particular we give some examples of this algebra. Finally we obtain some relationships between \( H/I \) and \( H \).

1. Introduction

The hyperalgebraic structure theory was introduced by F. Marty [7] in 1934. Imai and Iséki [4] in 1966 introduced the notion of a BCK-algebra. Recently [6] Jun, Borzooei and Zahedi et.al. applied the hyperstructure to BCK-algebras and introduced the concept of hyper BCK-algebra which is a generalization of BCK-algebra. Now, in this note we use the equivalence relation given in [1] and construct a quotient hyper BCK-algebra \( H/I \) via a hyper BCK-ideal \( I \), then we obtain some related results which have been mentioned in the abstract.

2. Preliminaries

Definition 2.1. Let \( H \) be a nonempty set and “\( \circ \)” be a hyperoperation on \( H \), that is “\( \circ \)” is a function from \( H \times H \) to \( \mathcal{P}^*(H) = \mathcal{P}(H) \setminus \{\emptyset\} \). Then \( H \) is called a hyper BCK-algebra if it contains a constant 0 and satisfies the following axioms:

\begin{itemize}
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  \item Keywords: hyper BCK-algebra, quotient hyper BCK-algebra, hyper BCK-ideal
\end{itemize}
(HK1) \((x \circ z) \circ (y \circ z) \ll x \circ y,\)

(HK2) \((x \circ y) \circ z = (x \circ z) \circ y,\)

(HK3) \(x \circ H \ll \{x\},\)

(HK4) \(x \ll y\) and \(y \ll x\) imply \(x = y,\)

for all \(x, y, z \in H\), where \(x \ll y\) is defined by \(0 \in x \circ y\) and for every \(A, B \subseteq H,\) \(A \ll B\) is defined by \(\forall a \in A, \exists b \in B\) such that \(a \ll b.\)

**Proposition 2.2.** [6] In any hyper \(BCK\)-algebra \(H\), for all \(x, y, z \in H\), the following statements hold:

\[
\begin{align*}
(i) & \quad 0 \circ 0 = \{0\}, \\
(ii) & \quad 0 \ll x, \\
(iii) & \quad x \ll x, \\
(iv) & \quad 0 \circ x = \{0\}, \\
(v) & \quad x \circ y \ll x, \\
(vi) & \quad x \circ 0 = \{x\}.
\end{align*}
\]

**Definition 2.3.** Let \(I\) be a nonempty subset of a hyper \(BCK\)-algebra \((H, \circ, 0)\) and \(0 \in I\). Then, \(I\) is called a hyper \(BCK\)-ideal of \(H\) if \(x \circ y \ll I\) and \(y \in I\) imply that \(x \in I\), for all \(x, y \in H\). If additionally \(x \circ x \subseteq I\) for all \(x \in H\), then \(I\) is called a reflexive hyper \(BCK\)-ideal.

**Lemma 2.4.** [5] Let \(A, B\) and \(I\) be subsets of \(H\).

\[
\begin{align*}
(i) & \quad \text{If } A \subseteq B \ll C, \text{ then } A \ll C. \\
(ii) & \quad \text{If } A \circ x \ll I \text{ for } x \in H, \text{ then } a \circ x \ll I \text{ for all } a \in A. \\
(iii) & \quad \text{If } I \text{ is a hyper } BCK\text{-ideal of } H \text{ and if } A \circ x \ll I \text{ for } x \in I, \text{ then } A \ll I. \\
(iv) & \quad \text{If } I \text{ is a reflexive hyper } BCK\text{-ideal of } H \text{ and let } A \text{ be a subset of } H. \text{ If } A \ll I, \text{ then } A \subseteq I.
\end{align*}
\]

**Definition 2.5.** [3] A hyper \(BCK\)-algebra \(H\) is said to be

- **weak positive implicative** if \((x \circ z) \circ (y \circ z) \subseteq (x \circ y) \circ z),

- **positive implicative** if \((x \circ z) \circ (y \circ z) = (x \circ y) \circ z),

- **implicative** if \(x \ll x \circ (y \circ x)

holds for all \(x, y, z \in H\).

**Definition 2.6.** [3] A nonempty subset \(I\) of a hyper \(BCK\)-algebra \(H\) containing \(0\) is called

- **a weak implicative hyper }BCK\text{-ideal** if for all } x, y, z \in H

\((x \circ z) \circ (y \circ x) \subseteq I \text{ and } z \in I \text{ imply } x \in I, \)

- **an implicative hyper }BCK\text{-ideal** if for all } x, y, z \in H

\((x \circ z) \circ (y \circ x) \ll I \text{ and } z \in I \text{ imply } x \in I. \)

**Definition 2.7.** [6] Let \(H\) be a hyper \(BCK\)-algebra. Define the set
\(\nabla(a, b) := \{x \in H \mid 0 \in (x \circ a) \circ b\}.\) If for any \(a, b \in H,\) the set \(\nabla(a, b)\) has the greatest element, then we say that \(H\) satisfies the hyper condition.
Proposition 2.8. [1] Let I be a reflexive hyper BCK-ideal of H and let
\[ x \sim_I y \text{ if and only if } x \circ y \subseteq I \text{ and } y \circ x \subseteq I. \]

Then \( \sim_I \) is an equivalence relation on H.

Proposition 2.9. [1] Let A, B are subsets of H, and I a reflexive hyper BCK-ideal of H. Then we define \( A \sim_I B \) if and only if \( \forall a \in A, \exists b \in B \text{ in which } a \sim_I b \), and \( \forall b \in B, \exists a \in A \text{ in which } a \sim_I b \). Then relation \( \sim_I \) is an equivalence relation on \( \mathcal{P}^*(H) \).

3. Quotient hyper BCK-algebras

From now on H is a hyper BCK-algebra and I is a reflexive hyper BCK-ideal of H, unless otherwise is stated.

Lemma 3.1. Let A, B \( \in \mathcal{P}^*(H) \), and I be a hyper BCK-ideal of H. Then \( A \circ B \ll I \) and \( B \circ A \ll I \) imply that \( A \sim_I B \).

Proof. For all \( a \in A \) and \( b \in B \) we have \( b \circ a \subseteq B \circ A \) and \( a \circ b \subseteq A \circ B \). Since \( A \circ B \ll I \) and \( B \circ A \ll I \), then we have \( b \circ a \ll I \), and \( a \circ b \ll I \).

Since I is reflexive then \( a \sim_I b \), which implies that \( A \sim_I B \). \( \square \)

Theorem 3.2. The relation \( \sim_I \) is a congruence relation on H.

Proof. By considering Proposition 2.8, it is enough to show that If \( x \sim_I y \) and \( u \sim_I v \), then \( x \circ u \sim_I y \circ v \). Since \( x \sim_I y \), we have \( x \circ y \ll I \) and \( y \circ x \ll I \). So \( (x \circ v) \circ (y \circ v) \ll x \circ y \) and \( x \circ y \ll I \) imply that \( (x \circ v) \circ (y \circ v) \ll I \). Similarly \( (y \circ v) \circ (x \circ v) \ll I \). Therefore by Lemma 3.1 \( x \circ v \sim_I y \circ v \).

Also we have \( (x \circ u) \circ (v \circ u) \ll x \circ v \). Then for all \( t \in x \circ u \) and \( r \in v \circ u \) we have \( t \circ r \subseteq (x \circ u) \circ (v \circ u) \). Therefore for all \( s \in t \circ r \) there exists \( a \in x \circ v \) such that \( s \ll a \), hence \( (s \circ a) \cap I \neq \emptyset \). Since \( s \circ a \subseteq (t \circ r) \circ a \), then \( ((t \circ r) \circ a) \cap I \neq \emptyset \). By Lemma 2.4 we have \( (t \circ r) \circ a \ll I \). Thus \( (t \circ a) \circ r \ll I \) and \( r \in I \), which implies that \( t \circ a \ll I \). Since \( t \in x \circ u \) and \( r \in v \circ u \) we can get that \( (x \circ u) \circ (x \circ v) \ll I \). Similarly \( (x \circ v) \circ (x \circ u) \ll I \).

Then by Lemma 3.1 we can see that \( x \circ v \sim_I x \circ u \).

Since \( \sim_I \) is an equivalence relation on \( \mathcal{P}^*(H) \), then \( x \circ v \sim_I y \circ v \) and \( x \circ v \sim_I x \circ u \) imply that \( x \circ u \sim_I y \circ v \). \( \square \)

Suppose \( I \) is a reflexive hyper BCK-ideal of \( (H, \circ, 0) \). Denote the equivalence classes of \( x \) by \( C_x \).

Lemma 3.3. In any hyper BCK-algebra \( H \) we have \( I = C_0 \).
Proof. Let $x \in I$. Since $x \in x \circ 0$, we have $(x \circ 0) \cap I \neq \emptyset$. Then $x \circ 0 \subseteq I$ and since $0 \circ x = 0$ hence $0 \circ x \subseteq I$. Then $0 \sim_I x$ therefore $x \in C_0$. Conversely let $x \in C_0$ hence $x \sim_I 0$ which means that $x \circ 0 \subseteq I$. Since $x \in x \circ 0$ then we have $x \in I$. 

Denote $H/I = \{C_x : x \in H\}$ and define $C_x * C_y = \{C_t \mid t \in x \circ y\}$. If $C_x = C_x'$ and $C_y = C_y'$, then $C_x * C_y = C_x' * C_y'$. Indeed, if $C_x = C_x'$ and $C_y = C_y'$ then $x \sim_I x'$ and $y \sim_I y'$, we can conclude that $x \circ y \sim_I x' \circ y'$ since $\sim_I$ is a congruence relation. Now let $C_t \in C_x * C_y$ then $t \in x \circ y$. Then there exist $r \in x' \circ y'$ such that $t \sim_I r$ hence $C_t = C_r$. Therefore $C_x * C_y \subseteq C_x' * C_y'$, and similarly $C_x' * C_y' \subseteq C_x * C_y'$. Hence $*$ is well-defined.

On $H/I$ we define $\ll$ putting: $C_x \ll C_y$ if and only if $C_0 \subseteq C_x * C_y$. Observe that: $x \ll y \Rightarrow 0 \in x \circ y \Rightarrow C_0 \subseteq C_x * C_y \Rightarrow C_x \ll C_y$.

**Theorem 3.4.** Let $(H, \circ, 0)$ be a hyper BCK-algebra and let $I$ be a reflexive hyper BCK-ideal of $H$. Then $(H/I, *, C_0)$ is a hyper BCK-algebra.

**Proof.** (HK1): Since $H$ is a hyper BCK-algebra, we have $(x \circ z) \circ (y \circ z) \ll (x \circ y)$. So for all $t \in a \circ b \subseteq (x \circ z) \circ (y \circ z)$ there exists $s \in (x \circ y)$ such that $t \ll s$. Therefore $C_t \ll C_s$, where $C_t \in C_a \circ C_b \subseteq (C_x * C_z) * (C_y * C_z)$ and $C_s \in C_x * C_y$, hence $(C_x * C_z) * (C_y * C_z) \ll C_x * C_y$.

(HK2): We must show that $(C_x * C_y) * C_z = (C_x * C_z) * C_y$. Let $C_t \in (C_x * C_y) * C_z$. Then $t \in a \circ z \subseteq (x \circ z) \circ y = (x \circ y) \circ z$, which means that $C_t \in (C_x * C_z) * C_y$. Hence $(C_x * C_y) * C_z \subseteq (C_x * C_z) * C_y$. Similarly $(C_x * C_z) * C_y \subseteq (C_x * C_y) * C_z$.

(HK3): $C_x * \{C_t \mid t \in H\} = \{C_x * C_t \mid t \in H\} = \bigcup_{t \in H} \{C_y \mid y \in x \circ t\}$. By Proposition 2.2 for all $y \in x \circ t$ we have $y \ll x$. So $C_y \ll C_x$, therefore $\{C_y \mid y \in x \circ t\} \ll C_x$. Thus $\bigcup_{t \in H} \{C_y \mid y \in x \circ t\} \ll C_x$. Therefore $C_x \circ H/I \ll C_x$.

(HK4): Let $C_x \ll C_y$ and $C_y \ll C_x$. We must show that $C_x = C_y$. Since $C_x \ll C_y$ then $C_0 \subseteq C_x * C_y$. So there exists a $t \in x \circ y$ such that $t \sim_I 0$. Therefore $t \circ 0 \ll I$, thus $t \in I$. Hence $(x \circ y) \cap I \neq \emptyset$. Now, since $I$ is a reflexive hyper BCK-ideal we conclude that $x \circ y \subseteq I$. Similarly $y \circ x \subseteq I$. Thus $x \sim_I y$ which means that $C_x = C_y$.

**Theorem 3.5.** If $H$ is a bounded hyper BCK-algebra with the greatest element 1, then $(H/I, *, C_0)$ is also a bounded hyper BCK-algebra with the greatest element $C_1$. 

Proof. It is enough to prove that $C_1$ is the greatest element of $H/I$. For any $x \in H$, since $0 \in x \circ 1$ then $C_0 \subseteq C_x * C_1$. This means that $C_1$ is the greatest element of $H/I$. \hfill \Box

The inverse of the above theorem does not hold.

**Example 3.6.** Let $H = \{0, 1, 2\}$. Then the following table shows a hyper BCK-algebra structure on $H$, which is not bounded.

<table>
<thead>
<tr>
<th>$\circ$</th>
<th>0</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>${0}$</td>
<td>${0}$</td>
<td>${0}$</td>
</tr>
<tr>
<td>1</td>
<td>${1}$</td>
<td>${0}$</td>
<td>${1}$</td>
</tr>
<tr>
<td>2</td>
<td>${2}$</td>
<td>${2}$</td>
<td>${0, 2}$</td>
</tr>
</tbody>
</table>

Then $I = \{0, 2\}$ is a reflexive hyper BCK-ideal of $H$. Now construct the quotient hyper BCK-algebra $H/I$ via $I$. Because

$C_0 = I = \{0, 2\} = C_2 = \{y \mid y \sim_I 2\}$, \quad $C_1 = \{y \mid y \sim_I 1\} = \{1\}$,

then $H/I = \{C_0, C_1\}$ and

<table>
<thead>
<tr>
<th>$*$</th>
<th>$C_0$</th>
<th>$C_1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C_0$</td>
<td>$C_0$</td>
<td>$C_0$</td>
</tr>
<tr>
<td>$C_1$</td>
<td>$C_1$</td>
<td>$C_0$</td>
</tr>
</tbody>
</table>

We can check that $(H/I, *, C_0)$ is a bounded hyper BCK-algebra. \hfill \Box

**Theorem 3.7.** If $J$ is a reflexive hyper BCK-ideal of $H$ and $I \subseteq J$, then:

(a) $I$ is a hyper BCK-ideal of the hyper BCK-subalgebra $J$ of $H$,

(b) the quotient hyper BCK-algebra $J/I$ is a hyper BCK-ideal of $H/I$.

Proof. (a) At first we show that $J$ is a hyper BCK-subalgebra of $H$. To show this let $x, y \in J$ we must show that $x \circ y \subseteq J$. Since $x \circ y \ll x$, then for all $a \in x \circ y$ we have $a \ll x$. Hence $0 \in a \circ x$. Thus $(a \circ x) \cap I \neq \emptyset$, since $I$ is reflexive then $a \circ x \subseteq I$ and therefore $a \circ x \subseteq J$. Now $x \in J$ implies that $a \in J$, thus $x \circ y \subseteq J$. Hence $J$ is a hyper BCK-subalgebra of $H$. It is clear that $I$ is hyper BCK-ideal of the hyper BCK-subalgebra of $J$.

(b) We can check that $J/I \subseteq H/I$. If $C_x * C_y \ll J/I$ and $C_y \in J/I$, then for any $t \in x \circ y$, there exists $s \in J$ such that $C_t \ll C_s$. Thus $C_0 \subseteq C_t * C_s$. So $C_0 = C_r$ for some $r \in t \circ s$. Therefore $0 \sim_I r$ and this implies that $0 \circ r \subseteq I$ and $r \circ 0 \subseteq I$. Hence $r \in I$, which means that $(t \circ s) \cap I \neq \emptyset$. Since $I$ is reflexive, then $t \circ s \subseteq I$. Now $t \circ s \subseteq J$, and $s \in J$ implies that $t \in J$. Thus $x \circ y \ll J$. Since $y \in J$, so $x \in J$, thus $C_x \subseteq J/I$. Hence $J/I$ is a hyper BCK-ideal of $H/I$. \hfill \Box
Theorem 3.8. If \( L \) is a hyper BCK-ideal of \( H/I \), then \( J = \{ x \mid C_x \in L \} \) is a hyper BCK-ideal of \( H \) and moreover \( I \subseteq J \). Furthermore \( L = J/I \).

Proof. Since \( I = C_0 \subseteq L \), then \( 0 \in J \). Let \( x \circ y \ll J \) and \( y \in J \). Then for any \( t \in x \circ y \) there exists \( s \in J \) such that \( t \ll s \). Hence \( C_t \ll C_s \), which implies that \( C_x \ast C_y \ll L \). Since \( y \in J \), we get that \( C_y \in L \), thus \( C_x \in L \). Therefore \( x \in J \), hence \( J \) is a hyper BCK-ideal of \( H \). Let \( x \in I = C_0 \). Then \( x \sim_I 0 \), thus \( C_x = C_0 \) and hence \( C_x \in L \). Therefore \( x \in J \), that is \( I \subseteq J \). Clearly \( L = J/I \). □

Theorem 3.9. If \( I \) is a hyper BCK-ideal of \( H \), then there is a bijection from the set \( \mathcal{I}(H, I) \) of all hyper BCK-ideals of \( H \) containing \( I \) to the set \( \mathcal{I}(H/I) \) of all hyper BCK-ideals of \( H/I \).

Proof. Define \( f : \mathcal{I}(H, I) \to \mathcal{I}(H/I) \) by \( f(J) = J/I \). By Theorem 3.7(b) \( f \) is well-defined, also Theorems 3.8 implies that \( f \) is onto. Let \( A, B \in \mathcal{I}(H, I) \) and \( A \neq B \). Without loss of generality, we may assume that there is an \( x \in (B \setminus A) \). If \( f(A) = f(B) \), then \( C_x \in f(B) = B/I \) and \( C_x \in f(A) = A/I \). Thus there exists \( y \in A \) such that \( C_x = C_y \) so \( x \sim_I y \), that is \( x \circ y \ll I \) and \( y \circ x \ll I \). Since \( I \subseteq A \) we have \( x \circ y \ll A \). Thus \( y \in A \) implies that \( x \in A \), which is a contradiction. So \( f \) is one-to-one. □

Theorem 3.10. Let \( I \) be a hyper BCK-ideal of \( H \). Then there exists a canonical surjective homomorphism \( \varphi : H \to H/I \) by \( \varphi(x) = C_x \), and \( \ker \varphi = I \), where \( \ker \varphi = \varphi^{-1}(C_0) \).

Proof. It is clear that \( \varphi \) is well-defined. Let \( x, y \in H \). Then \( \varphi(x \circ y) = \{ \varphi(t) \mid t \in x \circ y \} = \{ C_t \mid t \in x \circ y \} = C_x \ast C_y = \varphi(x) \ast \varphi(y) \). Hence \( \varphi \) is homomorphism. Clearly \( \varphi \) is onto. We have \( \ker \varphi = \{ x \in H \mid \varphi(x) = C_0 \} = \{ x \in H \mid C_x = C_0 = I \} = \{ x \in H \mid x \in I \} = I \). □

Theorem 3.11. Let \( f : H_1 \to H_2 \) be a homomorphism of hyper BCK-algebras, and let \( I \) be a hyper BCK-ideal of \( H_1 \) such that \( I \subseteq \ker f \). Then there exists a unique homomorphism \( \bar{f} : H_1/I \to H_2 \) such that \( \bar{f}(C_x) = f(x) \) for all \( x \in H_1 \), \( \text{Im}(ar{f}) = \text{Im}(f) \) and \( \ker \bar{f} = \ker f/I \). Moreover \( \bar{f} \) is an isomorphism if and only if \( f \) is surjective and \( I = \ker f \).

Proof. Let \( C_x = C_{x'} \). Then \( x \sim_I x' \), which implies that \( x \circ x' \subseteq I \) and \( x' \circ x \subseteq I \). Thus there exists \( t \in (x \circ x') \setminus I \). Then \( 0 = f(t) \in f(x \circ x') = f(x) \circ f(x') \), hence \( f(x) \ll f(x') \). Similarly \( f(x') \ll f(x) \), therefore \( \bar{f} \) is well-defined.

We have \( \bar{f}(C_x \ast C_y) = \bar{f}(\{ C_t \mid t \in x \circ y \}) = \{ \bar{f}(C_t) \mid t \in x \circ y \} = \{ f(t) \mid t \in x \circ y \} = f(x \circ y) = f(x) \circ f(y) = \bar{f}(C_x) \ast \bar{f}(C_y) \). Then \( \bar{f} \) is a
homomorphism. On the other hand

\[ C_x \in \ker \tilde{f} \iff \tilde{f}(C_x) = 0 \iff f(x) = 0 \iff x \in \ker f. \]

Note that \( \tilde{f} \) is unique, since it is completely determined by \( f \). Finally it is clear that \( \tilde{f} \) is surjective if and only if \( f \) is surjective. \( \square \)

**Theorem 3.12.** Let \( f : H_1 \longrightarrow H_2 \) be a homomorphism of hyper BCK-algebras. Then \( H_1/\ker f \cong \operatorname{Im}(f). \)

**Theorem 3.13.** Let \( I, J \) be hyper BCK-ideals of \( H \). Then there is a (natural) homomorphism of hyper BCK-algebras between \( I/(I \cap J) \) and \( <I \cup J>/J \), where \( <I \cup J>/J \) is the hyper BCK-ideal generated by \( I \cup J \).

**Proof.** Define \( \varphi : I \rightarrow <I \cup J>/J \) by \( \varphi(x) = C_x^J \), where \( C_x^J \) is the equivalence classes \( C_x \) via the hyper BCK-ideal \( J \). If \( x_1 = x_2 \), then it is clear that \( C_{x_1}^J = C_{x_2}^J \), which means that \( \varphi \) is well-defined. Also we have

\[ \varphi(x \circ y) = \{ \varphi(t) \mid t \in x \circ y \} = \{ C_t^I \mid t \in x \circ y \} = C_x^J \ast C_y^J = \varphi(x) \ast \varphi(y). \]

So that \( \varphi \) is a homomorphism. Moreover

\[ \ker \varphi = \{ x \in I \mid \varphi(x) = C_0^J \} = \{ x \in I \mid C_x^J = C_0^J = J \} = \{ x \in I \mid x \in J \} = I \cap J. \]

Thus by Theorem 3.12 the proof is completed. \( \square \)

**Open Problem 1.** Under what condition(s) is the defined homomorphism in Theorem 3.11 an isomorphism?

**Theorem 3.14.** Let \( I, J \) be hyper BCK-ideals of \( H \) such that \( I \subseteq J \). Then \( (H/I)/(J/I) \cong H/J \).

**Proof.** It is clear that \( J/I \subseteq H/I \). Define \( f : H/I \longrightarrow H/J \) by \( C_x^I \rightarrow C_x^J \), where \( C_x^I \in H/I \) and \( C_x^J \in H/J \).

If \( C_x^I = C_y^J \), then \( x \sim_I y \) which implies that \( x \circ y \subseteq I \) and \( y \circ x \subseteq I \). Since \( I \subseteq J \) hence \( x \circ y \subseteq J \) and \( y \circ x \subseteq J \). Thus \( x \sim_J y \) then \( C_x^J = C_y^J \) which means that \( f \) is well-defined.

\[ f(C_x^I \ast C_y^J) = f(\{ C_t^I \mid t \in x \circ y \}) = \{ C_t^J \mid t \in x \circ y \} = C_x^J \ast C_y^J = f(C_x^I) \ast f(C_y^J). \]

Clearly \( f \) is onto and

\[ \ker f = \{ C_x^I \in H/I \mid f(C_x^I) = C_0^J \} = \{ C_x^I \in H/I \mid C_x^J = C_0^J \} \]

\[ = \{ C_x^I \in H/I \mid x \in J \} = J/I. \]

Now by Theorem 3.12 the proof is completed. \( \square \)
4. Some result in quotient hyper \( BCK \)-algebras

Let \( C_a, C_b \in H/I \). Then according to Definition 2.7 we have
\[
\nabla(C_a, C_b) := \{ C_x \in H/I \mid C_0 \in (C_x \ast C_a) \ast C_b \}.
\]
Obviously \( C_0, C_a, C_b \in \nabla(C_a, C_b) \), \( \nabla(C_0, C_0) = \{ C_0 \} \) and \( \nabla(C_a, C_b) = \nabla(C_b, C_a) \) for all \( C_a, C_b \in H/I \).

**Theorem 4.1.** If \( H \) satisfies the hyper condition, then \( H/I \) so is.

**Proof.** If \( x \in \nabla(a, b) \), then we have \( x \circ a \ll b \). Thus for all \( t \in x \circ a, t \ll b \). Therefore \( C_t \ll C_b \), thus \( C_x \ast C_a \ll C_b \). Hence \( C_x \in \nabla(C_a, C_b) \). Since \( \nabla(a, b) \) has the greatest element, then by Theorem 3.5, \( \nabla(C_a, C_b) \) has the greatest element too. \( \square \)

**Remark 4.2.** The converse of the above theorem is not correct in general. Let \( H = \{0, 1, 2\} \) and
\[
\begin{array}{c|ccc}
\circ & 0 & 1 & 2 \\
\hline
0 & \{0\} & \{0\} & \{0\} \\
1 & \{1\} & \{0, 1\} & \{1\} \\
2 & \{2\} & \{2\} & \{0\}
\end{array}
\]

Then \( I = \{0, 1\} \) is a reflexive hyper \( BCK \)-ideal of a hyper \( BCK \)-algebra \((H, o, 0)\) and the elements of the quotient hyper \( BCK \)-algebra \( H/I \) are as follows: \( C_0 = I = \{0, 1\} = C_1 = \{y \mid y \sim I 1\}, \; C_2 = \{y \mid y \sim I 2\} = \{2\} \). Hence \( H/I = \{C_0, C_2\} \) and
\[
\begin{array}{c|cc}
* & C_0 & C_2 \\
\hline
C_0 & C_0 & C_0 \\
C_2 & C_2 & C_0
\end{array}
\]

It can be checked that the quotient hyper \( BCK \)-algebra \( H/I \) satisfies the hyper condition, but \( H \) does not satisfy the hyper condition, since \( \nabla(1, 2) = \{0, 1, 2\} \), \( 1 \ll 2 \) and \( 2 \ll 1 \). \( \square \)

**Theorem 4.3.** If \( H \) is an implicative hyper \( BCK \)-algebra, then so is \( H/I \).

**Proof.** The proof is easy. \( \square \)

Note that the converse of the above theorem is not correct in general.

**Example 4.4.** The set \( H = \{0, 1, 2\} \) with the operation
\[
\begin{array}{c|ccc}
\circ & 0 & 1 & 2 \\
\hline
0 & \{0\} & \{0\} & \{0\} \\
1 & \{1\} & \{0\} & \{0\} \\
2 & \{2\} & \{1\} & \{0, 1\}
\end{array}
\]

is a hyper \( BCK \)-algebra. \( I = \{0, 1\} \) is a reflexive hyper \( BCK \)-ideal such that \( C_0 = I = \{0, 1\} = C_1 = \{y \mid y \sim I 1\}, \; C_2 = \{y \mid y \sim I 2\} = \{2\} \) and
<table>
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<tr>
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<th>$C_0$</th>
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We can check that $H/I = \{C_0,C_2\}$ is an implicatve hyper BCK-algebra, while the hyper BCK-algebra $H$ is not, since $1 \nleq 1 \circ (2 \circ 1)$. \hfill \Box

**Theorem 4.5.** If $H$ is a (weak) positive implicatve hyper BCK algebra, then so is $H/I$.

**Proof.** Let $H$ be a positive implicatve hyper BCK-algebra. Then we have

$$C_t \in (C_x * C_z) * (C_y * C_z) \iff C_t = C_s \text{ for some } s \in (x \circ z) \circ (y \circ x), \ t \sim_I s
$$

$$\iff C_t = C_s \text{ for some } s \in (x \circ y) \circ z, \ s \sim_I t
$$

$$\iff C_t \in (C_x * C_y) * C_z.$$ 

The other case is similar. \hfill \Box

Note that Example 4.4 shows that the converse of the above theorem is not correct in general. Since $H/I$ is positive implicatve while $H$ is not, since $(2 \circ 2) \circ (2 \circ 2) = \{0,1\} \neq \{0\} = (2 \circ 2) \circ 2$.

**Theorem 4.6.** Let $I$ and $J$ be reflexive hyper BCK-ideals of $H$ and $I \subseteq J$. If $J$ is a weak implicatve hyper BCK-ideal of $H$, then $J/I$ is a weak implicatve hyper BCK-ideal of $H/I$.

**Proof.** Let $J$ be a weak implicatve hyper BCK-ideal of $H$ and $(C_x * C_z) * (C_y * C_x) \subseteq J/I$ and $C_z \in J/I$. Then for all $C_s \in (C_x * C_z) * (C_y * C_x)$ where $s \in (x \circ z) \circ (y \circ x)$, we have $C_s \in J/I$. Thus $s \sim_I r$, for some $r \in J$. So $s \circ r \subseteq I$, hence $s \circ r \subseteq J$. Consequently $r \in J$ implies that $s \in J$. Thus $(x \circ z) \circ (y \circ x) \subseteq J$, and from $C_z \in J/I$ we can conclude that $z \in J$. Since $J$ is a weak implicatve hyper BCK-ideal, then we get that $x \in J$. Hence $C_x \in J/I$, which means that $J/I$ is a weak implicatve hyper BCK-ideal of $H/I$. \hfill \Box

**Open Problem 2.** Does the converse of the above theorem true?

**Theorem 4.7.** Let $I \subseteq J$ be reflexive hyper BCK-ideals of $H$. Then $J/I$ is an implicatve hyper BCK-ideal of $H/I$ if and only if $J$ is an implicatve hyper BCK-ideal of $H$.

**Proof.** Let $J$ be an implicatve hyper BCK-ideal and $C_x * (C_y * C_x) \ll J/I$. Then for all $C_t \in C_x * (C_y * C_x)$ there exists $C_r \in J/I$ such that $C_t \ll C_r$, where $t \sim_I s$, $s \in x \circ (y \circ x)$ and $r \in J$. Since $C_t \ll C_r$ then $C_0 \in C_t * C_r$, hence there exists $u \in t \circ r$ such that $0 \sim_I u$. Thus $u \circ 0 \subseteq I$, therefore $u \in I$. Then $(t \circ r) \cap I \neq \emptyset$ which means that $t \circ r \cap J \neq \emptyset$. Therefore $r \in J$ implies that $t \in J$. Since $t \sim_I s$ thus $s \circ t \subseteq I$ and hence $s \circ t \subseteq J$. Thus $t \in J$
implies that $s \in J$, hence $x \circ (y \circ x) \ll J$. Since $J$ is an implicating hyper BCK-ideal by Theorem 3.6 of [3] we can get that $x \in J$. Hence $C_x \in J/I$. Now Theorem 3.6 [3] implies that $J/I$ is an implicating hyper BCK-ideal of $H/I$.

Conversely, let $J/I$ be an implicating hyper BCK-ideal of $H/I$ and $x \circ (y \circ x) \ll J$. Then for all $t \in x \circ (y \circ x)$ there exists $r \in J$ such that $t \ll r$. Thus $C_t \ll C_r$, and we can conclude that $C_x \ast (C_y \ast C_x) \ll J/I$. Since $J/I$ is an implicating hyper BCK-ideal of $H$, then $C_x \in J/I$, we can get that $x \in J$. Therefore $J$ is an implicating hyper BCK-ideal of $H$, by Theorem 3.6 of [3]. 

\[ \square \]

References


A. B. Saeid
Department of Mathematics
Islamic Azad University of Kerman
Kerman
Iran
e-mail: arsham@iaukt.ac.ir

M. M. Zahedi
Department of Mathematics
Shahid Bahonar University of Kerman
Kerman
Iran
e-mail: zahedi_mm@mail.uk.ac.ir

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