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# Necessary and sufficient conditions for the continuity of a pre-Haar system at a unit with singleton orbit

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**Abstract**

For developing an algebraic theory of functions on a locally compact groupoid, one needs an analogue of Haar measure on locally compact groups. This analogue is a system of measures, called Haar system, subject to suitable invariance and smoothness conditions called respectively "left invariance" and "continuity". Unlike the case of locally compact group, Haar system on groupoid need not exists. In this paper we shall consider a locally compact groupoid $G$, and we shall denote by $G_s^{(0)}$ the set of units with singleton orbit and by $G_s$ the reduction $G|_{G_s^{(0)}}$ of $G$ to $G_s^{(0)}$. We shall prove that if $G$ admits Haar systems, then the restriction of the range map at $G_s$ is an open map from $G_s$ to $G_s^{(0)}$. Conversely, we shall prove that if this map is open at every $x \in G_s$, then the continuity condition of a Haar system holds at every unit with singleton orbit.

## 1. Introduction

In order to establish notation for this paper we shall include some definitions that can be found in several places (e.g. [4], [5], [6], [7]).

**Definition 1.** A *groupoid* is a set $G$ endowed with a product map

$$(x, y) \to xy \;\left[: G^{(2)} \to G\right]$$

where $G^{(2)}$ is a subset of $G \times G$ called the *set of composable pairs*, and an *inverse map*

$$x \to x^{-1} \;[: G \to G]$$

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such that the following conditions hold:

(1) If $(x,y) \in G^{(2)}$ and $(y,z) \in G^{(2)}$, then $(xy,z) \in G^{(2)}$, $(x,yz) \in G^{(2)}$ and $(xy)\,z = x\,(yz)$,

(2) $\left(x^{-1}\right)^{-1} = x$ for all $x \in G$,

(3) $\left(x, x^{-1}\right) \in G^{(2)}$, and if $(z,x) \in G^{(2)}$, then $(zx)\,x^{-1} = z$, for each $x \in G$,

(4) $\left(x^{-1}, x\right) \in G^{(2)}$, and if $(x,y) \in G^{(2)}$, then $x^{-1}\,(xy) = y$, for each $x \in G$.

The maps $r$ and $d$ on $G$, defined by the formulae $r\,(x) = xx^{-1}$ and $d\,(x) = x^{-1}x$, are called the *range* and the *source maps*. It follows easily from the definition that they have a common image called the *unit space* of $G$, which is denoted $G^{(0)}$. Its elements are units in the sense that $xd\,(x) = r\,(x)\,x = x$. Units will usually be denoted by letters as $u$, $v$, $w$ while arbitrary elements will be denoted by $x$, $y$, $z$. It is useful to note that a pair $(x,y)$ lies in $G^{(2)}$ precisely when $d\,(x) = r\,(y)$, and that the cancellation laws hold (e.g. $xy = xz$ iff $y = z$). The fibres of the range and the source maps are denoted $G^u = r^{-1}\,(\{u\})$ and $G_v = d^{-1}\,(\{v\})$, respectively. More generally, given the subsets $A$, $B \subset G^{(0)}$, we define $G^A = r^{-1}\,(A)$, $G_B = d^{-1}\,(B)$ and $G^A_B = r^{-1}\,(A) \cap d^{-1}\,(B)$. The reduction of $G$ to $A \subset G^{(0)}$ is $G|A = G^A_A$. The relation $u \sim v$ iff $G^u_v \neq \emptyset$ is an equivalence relation on $G^{(0)}$. Its equivalence classes are called *orbits* and the orbit of a unit $u$ is denoted $[u]$. The quotient space for this equivalence relation is called the *orbit space* of $G$ and denoted $G^{(0)}/G$. A groupoid is called *transitive* iff it has a single orbit, or equivalently if the map $(r,d) : G \to G^{(0)} \times G^{(0)}, (r,d)\,(x) = (r\,(x)\,, d\,(x))$ is surjective. A groupoid is said *principal* if the map $(r,d) : G \to G^{(0)} \times G^{(0)}, (r,d)\,(x) = (r\,(x)\,, d\,(x))$ is injective.

**Examples** structures which fit naturally into the study of groupoids:

**1.** *Groups*: A group $G$ is a groupoid with $G^{(2)} = G \times G$ and $G^{(0)} = \{e\}$ (the unit element).

**2.** *Spaces*. A space $X$ is a groupoid letting $X^{(2)} = \{(x,x) \in G \times G\} = diag(X)$, $xx = x$, and $x^{-1} = x$.

**3.** *Equivalence relations*. Let $R \subset X \times X$ be an equivalence relation on the set $X$. Let $R^{(2)} = \{((x_1,y_1)\,, (x_2,y_2)) \in R \times R : y_1 = x_2\}$. With product $(x,y)\,(y,z) = (x,z)$ and $(x,y)^{-1} = (y,x)\,, R$ is a principal groupoid. $R^{(0)}$ may be identified with $X$. Two extreme cases deserve to be single out. If $R = X \times X$, then $R$ is called the *trivial groupoid* on $X$, while if $R = diag\,(X)\,,$ then $R$ is called the *co-trivial groupoid* on $X$ (and may be identified with the groupoid in Example 2).

**4.** *Transformation groups.* Let $\Gamma$ be a group acting on a set $X$ such that for $x \in X$ and $g \in \Gamma$, $xg$ denotes the transform of $x$ by $g$. Let $G = X \times \Gamma$, $G^{(2)} = \{((x, g), (y, h)) : y = xg\}$. With the product $(x, g)(xg, h) = (x, gh)$ and the inverse $(x, g)^{-1} = \left(xg, g^{-1}\right)$ $G$ becomes a groupoid. The unit space of $G$ may be identified with $X$.

**Definition 2.** A *topological groupoid* consists of a groupoid $G$ and a topology compatible with the groupoid structure:

(1) $x \to x^{-1}$ $[: G \to G]$ is continuous.

(2) $(x, y)$ $\left[: G^{(2)} \to G\right]$ is continuous where $G^{(2)}$ has the induced topology from $G \times G$.

If $G$ is a topological groupoid, then $r$ and $d$ are identification maps, and $x \to x^{-1}$ is a homeomorphism. If $G$ is Hausdorff, $G^{(0)}$ is closed, and if $G^{(0)}$ is Hausdorff, $G^{(2)}$ is closed in $G \times G$.

We are exclusively concerned with topological groupoids which are locally compact and Hausdorff. It was shown in [6] that measured groupoids (in the sense of Definition 2.3. from [4]) may be assume to have locally compact topologies, with no loss in generality.

There are several generalizations of the classical Haar measure associated with a locally compact topological group to the setting of a locally compact topological groupoid (see [14], [9], [10], [11], [4], [7]). Now in general use is the definition adopted by Jean Renault in [7]:

**Definition 3.** A *Haar system* on a locally compact groupoid $G$ is a family of positive Radon measures on $G$, $\left\{\nu^u,\, u \in G^{(0)}\right\}$, having the following properties:

1) For all $u \in G^{(0)}$, $supp(\nu^u) = G^u$.

2) For all $f \in C_c(G)$

$$u \to \int f(x)\, d\nu^u(x) \quad \left[: G^{(0)} \to \mathbf{C}\right]$$

is continuous.

3) For all $f \in C_c(G)$ and all $x \in G$,

$$\int f(y)\, d\nu^{r(x)}(y) = \int f(xy)\, d\nu^{d(x)}(y)$$

The system of measures $\left\{\nu^u,\, u \in G^{(0)}\right\}$ will be called *Borel Haar system* if it has the properties 1), 3) and

$2'$) For all $f \geqslant 0$ Borel on $G$,

$$u \to \int f(x)\, d\nu^u(x) \quad \left[: G^{(0)} \to \bar{\mathbf{R}}\right]$$

is a real-extended Borel map, where the Borel sets of a topological spaces $G$ and $G^{(0)}$ are taken to be the $\sigma$-algebra generated by the open sets.

Unlike the case of locally compact group, Haar system on groupoid need not exists. The continuity assumption 2) has topological consequences for $G$. It entails that the range map $r : G \to G^{(0)}$, and hence the domain map $d : G \to G^{(0)}$ is an open (Proposition I.4 [13]). So "the range map is an open map" is a necessary condition for the existence of Haar systems. A. K. Seda has established sufficient conditions for the existence of Haar systems. He has proved that if for all $u \in G^{(0)}$, the map $r_u : G_u \to G^{(0)}$, $r_u(x) = r(x)$ is open, then the continuity assumption 2) follows from the left invariance assumption 3) (Theorem 2, p.430 [10]). Thus he has proved that locally transitive groupoids admit Haar system. At the opposite case of totally intransitive groupoids, Renault has established necessary and sufficient conditions. More precisely, Renault has proved that a locally compact group bundle (a groupoid with the property that $r(x) = d(x)$ for all $x$) admits a Haar system if and only if $r$ is open (Lemma 1.3, p.6 [8]).

In this paper we shall study the continuity of a pre-Haar system at the units $u$ with singleton orbits (this means $[u] = \{u\}$). We shall establish necessary and sufficient conditions. When all units of the groupoid are with singleton orbits we shall re-obtain the result of Renault Lemma 1.3, p.6 [8].

## 2. The existence of a pre-Haar system

**Definition 4.** A (*left*) *pre-Haar system* on $G$ is a family of (positive) Radon measures on $G$, $\{\nu^u,\ u \in G^{(0)}\}$,with the following properties:

1) $\nu^u$ concentrated on $G^u$ for all $u \in G^{(0)}$;
2) $\int f(y)\, d\nu^{r(x)}(y) = \int f(xy)\, d\nu^{d(x)}(y)$ for all $x \in G$ and $f \in C_c(G)$
3) $\sup\{\nu^u(K),\ u \in G^{(0)}\} < \infty$ for each compact set $K \subset G$.

**Definition 5.** The pre-Haar system $\{\nu^u,\ u \in G^{(0)}\}$ is said *continuous at* $u_0$ if for all $f \in C_c(G)$, the map

$$u \to \int f(x)\, d\nu^u(x) \quad \left[: G^{(0)} \to \mathbf{C}\right]$$

is continuous is continuous at $u_0$.

The pre-Haar system is said *continuous* if it is continuous at every unit (or equivalently if it is a Haar system).

In [2] we have shown that the continuity of a pre-Haar system is equivalent with the continuity of a family of homomorphisms associated to the pre-Haar system.

**Notation 6.** Let $\{\nu^u,\ u \in G^{(0)}\}$ be a pre-Haar system on the locally compact groupoid $G$. For each $f \in C_c(G)$ let us denote by $F_f : G \to \mathbf{C}$ the map defined by

$$F_f(x) = \int f(y)\, d\nu^{d(x)}(y) - \int f(y)\, d\nu^{r(x)}(y) \quad (\forall)\ x \in G$$

For each $f$, $F_f$ is a homomorphism of groupoids:

$$F_f(xy) = F_f(x) - F_f(y) \text{ for all } (x, y) \in G^{(2)}.$$

$\{F_f\}_{f \in C_c(G)}$ will be called the *family of homomorphisms associated with the pre-Haar system.*

We state Lemma 4.2 p.40 [2]:

**Lemma 7.** *Let $G$ be a locally compact groupoid whose unit space $G^{(0)}$ is paracompact, let $\left\{\nu^u,\ u \in G^{(0)}\right\}$ be a pre-Haar system on $G$ and let $\{F_f\}_f$ the family of associated homomorphisms. Then for each $f \in C_c(G)$ and each $\varepsilon > 0$ there is $W_\varepsilon$, a conditionally compact neighborhood of $G^{(0)}$, such that:*

$$|F_f(x)| < \varepsilon \text{ for all } x \in W_\varepsilon$$

We shall show how to construct a pre-Haar system. Let $G$ be a locally compact second countable groupoid. In Section 1 of [8] Jean Renault constructs a Borel Haar system for $G'$. One way to do this is to choose a function $F_0$ continuous with conditionally support which is nonnegative and equal to 1 at each $u \in G^{(0)}$. Then for each $u \in G^{(0)}$ choose a left Haar measure $\beta_u^u$ on $G_u^u$ so the integral of $F_0$ with respect to $\beta_u^u$ is 1.

Renault defines $\beta_v^u = x\beta_v^v$ if $x \in G_v^u$ (where $x\beta_v^v(f) = \int f(xy)\, d\beta_v^v(y)$ as usual). If $z$ is another element in $G_v^u$, then $x^{-1}z \in G_v^v$, and since $\beta_v^v$ is a left Haar measure on $G_v^v$, it follows that $\beta_v^u$ is independent of the choice of $x$. If $K$ is a compact subset of $G$, then $\sup\limits_{u,v} \beta_v^u(K) < \infty$.

For constructing a pre-Haar system it is enough to choose a family of probability measure on $G^{(0)}$ indexed on the orbit space $\left\{\mu^{\dot{u}},\ \dot{u}\in G^{(0)}/G\right\}$ such that $supp\left(\mu^{\dot{u}}\right)=[u]$. We define

$$\int f(y)\,d\nu^{u}(y)=\int f(y)\,d\beta_{v}^{u}(y)\,d\mu^{\dot{u}}(y)$$

for all continuous function $f$ on $G$ with compact support.

It is not hard to see that $\{\nu^{u},\ u\in G^{(0)}\}$ is a pre-Haar system. Applying a result of Federer and Morse [3], it follows that the map

$$(r,d):G\to(r,d)(G)$$

has Borel section $\sigma$. If we define $h(x)=F_0\left(\sigma(r(x),d(x))^{-1}x\right)$, then we obtain a Borel function with the property that

$$\int h(x)\,\nu^{u}(y)=1 \text{ for all } u.$$

Another construction of pre-Haar system can be found in [1].

## 3. The continuity of a pre-Haar system

**Lemma 8.** *Let $G$ be a locally compact groupoid with the range map $r$ open. Then the set of units with singleton orbit is a closed subset of the unit space.*

*Proof.* Let $(u_i)_i$ be net of units with singleton orbits. Let us assume that $(u_i)_i$ converges to $u$ and let $x$ in $G$ with $r(x)=u$. We shall prove that $r(x)=d(x)$, and it will follows that $[u]=\{u\}$. Since $r$ is an open map, eventually passing to a subnet, we may assume that there is a net $(x_i)_i$ in $G$ that converges to $x$, such that $r(x_i)=u_i$. Since each $u_i$ is with singleton orbit, $d(x_i)=r(x_i)=u_i$. Hence

$$d(x)=\lim d(x_i)=\lim u_i=u.$$

Thus the set of units with singleton orbit is a closed subset of the unit space. □

**Notation 9.** Let $G$ be a locally compact groupoid with the range map $r$ open. Let us denote by $G_s^{(0)}$ the set of units with singleton orbit. According to the preceding lemma $G_s^{(0)}$ is a closed subset of the unit space. Let us denote by $G_s$ the reduction $G|_{G_s^{(0)}}$ of $G$ to $G_s^{(0)}$. Then $G_s$ is a closed subgroupoid of $G'$.

**Lemma 10.** *Let $G$ be a locally compact groupoid that admits a Haar system. Then*

$$r_s : G_s \to G_s^{(0)}, \quad r_s(x) = r(x)$$

*is an open map.*

*Proof.* Let $\{\nu^u,\ u \in G^{(0)}\}$ be a Haar system on $G$. Let $x_0 \in G_s$ and let $U$ be a nonempty compact neighborhood of $x_0$ in $G_s$. Choose a nonnegative continuous function, $f$ on $G_s$ , with $f(x_0) > 0$ and $supp\,(f) \subset U$. Let $V$ be an open neighborhood of $G_s$. Let $\tilde{f}$ be a continuous function extending $f$ to $G$ with $supp\left(\tilde{f}\right) \subset V$. Let $W$ the set of units $u$ with the property that $\nu^u(f) > 0$. Then $W$ is an open neighborhood of $u_0 = r(x_0)$ contained in $r(U) \cup r(V - G_s)$ . Since $W \cap G_s^{(0)} \subset r(U) \cap G_s^{(0)}$, it follows that $r(U)$ is a neighborhood of $u_0$ in $G_s^{(0)}$. □

**Definition 11.** Let $u$ be a unit with singleton orbit. We shall say that the *restriction of $r$ to $G_s$ is open at $x \in G_u^u$*, if it sends every open neighborhood of $x$ to a open neighborhood of $u$ in $G^{(0)}$.

**Lemma 12.** *Let $u$ be a unit with singleton orbit. If the restriction of $r$ to $G_s$ is open at $x \in G_u^u$, then $G_sW$ is a neighborhood of $x$ in $G$, for each neighborhood $W$ of $G^{(0)}$.*

*Proof.* Let $x$ be an element of $G_u^u$. Let $V$ be an open neighborhood of $G^{(0)}$ contained in $W$. Let us prove that $G_sW$ contains $x$ in its interior. Let $(x_i)_i$ be a net converging to $x$ . Since $r$ sends every open neighborhood of $x$, to an open neighborhood of $u$, eventually passing to a subnet we may assume that there is a net $(z_i)_i$ in $G_s$ that converges to $x$, such that $r(x_i) = r(z_i)$. The net $\left(z_i^{-1}x_i\right)_i$ converges to $d(x)$, so for $i$ large enough $z_i^{-1}x_i$ belongs to $W$. Consequently, $x_i = z_i t_i$ with $z_i \in G'$ and $t_i = z_i^{-1}x_i$ in $W$. Hence $G_sW$ is a neighborhood of $x$. □

**Theorem 13.** *Let $G$ be a locally compact groupoid whose unit space $G^{(0)}$ is paracompact and whose range map $r$ is open. Let $\{\nu^u,\ u \in G^{(0)}\}$ be pre-Haar system. Let $u$ be a unit with singleton orbit $[u]$. Assume that if $W$ is an open neighborhood of $G^{(0)}$, then each $x \in G_s$ is in the interior of $G_sW$.*

*If there exists a function $h : G \to [0,1]$, universally measurable on each transitivity component $G|_{[w]}$, with $\nu^w(h) = 1$ for all $w \in G^{(0)}$, then the pre-Haar system is continuous at $u$.*

*Proof.* To prove the continuity of the pre-Haar system at $u$ we shall use the same argument as in Lemma 1.3 p.6 [8]. Let $\mathcal{B}$ be the linear space of the bounded sequences of real numbers. Let $s \mapsto Lim\,(s)\,[: \mathcal{B} \to \mathbf{R}]$ be a linear map with the following properties:

1) If $s = (s_i)_i$ and $s_i \geqslant 0$, then $Lim\,(s) \geqslant 0$;

2) $Lim\,(1, 1, ..., 1...) = 1$;

3) $Lim\,(s_1, s_2, s_3, ...) = Lim\,(s_1, s_1, s_2, s_2, s_3, s_3...)$;

4) If $s, t \in \mathcal{B}$ and $\lim_n (s_n - t_n) = 0$, then $Lim\,(s) = Lim\,(t)$;

It is not hard to show that $\lim_n s_n = a$ implies that $Lim\,(s_1, s_2, ...) = a$. Also if $Lim\,(s') = a$ for every subsequence $s'$ of $s$, then $\lim_n s_n = a$.

Let $\{F_f\}_{f \in C_c(G)}$ be the family of homomorphisms associated with the pre-Haar system.

Let $(u_i)_i$ a sequence converging to $u$. For each continuous function with compact support, $f : G \to \mathbf{R}$, we set

$$\mu\,(f) = Lim\left(i \mapsto \int f\,(y)\,d\nu^{u_i}\,(y)\right)$$

$\mu$ is a positive linear functional on the space of continuous functions with compact support. We claim that $\mu\,(f)$ depends only on the restriction on $f$ on $G_u^u$. Suppose that $f$ and $g$ coincide on $G_u^u$. We denote by $K$ the compact set

$$(supp\,(f) \cup supp\,(g)) \cap r^{-1}\,(\{u_i, i = 1, 2, ...\} \cup \{u\})$$

Then we have

$$\left|\int f\,(y)\,d\nu^{u_i}\,(y) - \int g\,(y)\,d\nu^{u_i}\,(y)\right| \leqslant \int |f\,(y) - g\,(y)|\,d\nu^{u_i}\,(y) \leqslant$$

$$\leqslant \sup_{y \in G^{u_i}} |f\,(y) - g\,(y)|\,\nu^{u_i}\,(K)$$

One observes that $\sup_{y \in G^{u_i}} |f\,(y) - g\,(y)|\,\nu^{u_i}\,(K)$ converges to 0. Therefore $\mu\,(f) = \mu\,(g)$. Next we show that $\mu$ is left invariant on $G_u^u$, and consequently, a Haar measure on $G_u^u$.Let $x \in G_u^u$. Because $r : G \to G^{(0)}$ is an open map there exists a sequence $(x_i)_i$ converging to $x$ such that $r\,(x_i) = u_i$.

Let $f, g : G \to \mathbf{R}$ two continuous function with compact support such that $g\,(y) = f\left(x^{-1}y\right)$ for all $y \in G^u$.

Then we have

$$\left|\int f\,(y)\,d\nu^{u_i}\,(y) - \int g\,(y)\,d\nu^{u_i}\,(y)\right| =$$

$$\leqslant \left|\int f d\nu^{r(x_i)} - \int f d\nu^{d(x_i)}\right| + \left|\int f(y)\, d\nu^{d(x_i)}(y) - \int g(x_i y)\, d\nu^{d(x_i)}(y)\right|$$
$$\leqslant |F_f(x_i)| + \sup_{y\in G^{d(x_i)}} |f(y) - g(x_i y)|\, \nu^{d(x_i)}(K_1),$$

where $K_1$ is the compact set

$$(supp\,(f) \cup \{x, x_i, i = 1, 2, ..\}\, supp\,(g)) \cap r^{-1}(\{d(x_i), i = 1, 2, ...\} \cup \{u\}).$$

The sequence $i \mapsto \sup_{y\in G^{d(x_i)}} |f(y) - g(x_i y)|\, \nu^{d(x_i)}(K_1)$ converges to 0.

Let $\varepsilon > 0$ and let $W$ be a neighborhood of $G^{(0)}$ such that

$$|F_f(y)| < \frac{\varepsilon}{2} \quad \text{for all } y \in W.$$

Since $G_s W$ is a neighborhood of $x$, and $(x_i)_i$ converges to $x$, we may assume that $x_i$ belongs to $G_s W$ for large $i$. Thus there is $z_i \in G_s$ and $y_i \in W$ such that $x_i = z_i y_i$ Consequently, we have

$$|F_f(x_i)| = |F_f(z_i y_i)| = |F_f(z_i) + F_f(y_i)| = |F_f(y_i)| < \frac{\varepsilon}{2}.$$

and this imply

$$\left|\int f(y)\, d\nu^{u_i}(y) - \int g(y)\, d\nu^{u_i}(y)\right| \leqslant \varepsilon \quad \text{for large } i$$

Therefore $\mu(f) = \mu(g)$ and hence $\mu$ and $\nu^u$ are Haar measures on $G_u^u$ and $\mu(h) = 1 = \nu^u(h)$. From uniqueness of Haar measure on $G_u^u$ it follows that $\mu = \nu^u$. This means that $i \mapsto \int f(y)\, d\nu^{u_i}(y)$ converges to $\int f(y)\, d\nu^u(y)$ for every continuous function with compact support $f$. □

**Theorem 14.** *Let $G$ be a locally compact groupoid with paracompact unit space and open range map. If $G$ admits a Haar system then*

$$r_s : G_s \to G_s^{(0)}, \quad r_s(x) = r(x)$$

*is an open map. And conversely, if the restriction of $r$ to $G_s$ is open at any $x \in G_s$, then there is a pre-Haar system on $G$ that is continuous at any unit in $G_s^{(0)}$.*

*Proof.* It follows from Theorem 13 and Lemma 10. □

**Remark 15.** If $G$ is a locally compact group bundle, then from the preceding theorem we obtain the result of J. Renault about the existence of Haar systems Lemma 1.3 p.6 [8].

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