Characterization of division $\mu$-LA-semigroups

Qaiser Mushtaq and Khalid Mahmood

Abstract

Let $G$ be a left almost semigroup (LA-semigroup), also known as Abel-Grassman's groupoid and a left invertive groupoid. In this paper we have shown that $G$ is a division $\mu$-LA-semigroup if and only if it has a linear form. Characterization of division $\mu$-LA-semigroups is also done by using permutations.

1. Introduction

A left almost semigroup [2], abbreviated as LA-semigroup, is an algebraic structure midway between a groupoid and a commutative semigroup. Although the structure is non-associative and non-commutative, nevertheless, it possesses many interesting properties which we usually find in associative and commutative algebraic structures.

Kazim and Naseerudin have introduced the concept of an LA-semigroup and have investigated some basic but important characteristics of this structure in [2]. They have generalized some useful results of semigroup theory. Relationships between LA-semigroups and quasigroups, semigroups, loops, monoids, and groups have been established.

Later, Mushtaq and others in [1], [5], [6], [7], [8], and [10] have studied the structure further and added many results to the theory of LA-semigroups. Holgate [1], has called the same structure as left invertive groupoid. It is also known as Abel-Grassman's groupoid or AG-groupoid. In this paper we shall call it LA-semigroup.

Kepka [4] has done extensive study of quasigroups satisfying some weak forms of the medial law. In this paper we have extended some of his results to LA-semigroups.

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A groupoid $G = (G, \cdot)$ is called a left almost semigroup, abbreviated as LA-semigroup, if its elements satisfy the left invertive law: $(ab)c = (cb)a$. Examples of LA-semigroups can be found in [5] and [6].

An element $e \in G$ is called a left identity if $ea = a$ for all $a \in G$. An element $a' \in G$ is called a left inverse of $a$ if $G$ contains left identity $e$ and $a'a = e$. As in the case of semigroups, both $e$ and $a'$ are unique [5]. In [5] it is proved also that if $G$ contains a left identity then $ab = cd$ implies $ba = dc$ for all $a, b, c, d \in G$. As in the case of semigroups, an element $a$ of an LA-semigroup $G$ is called left cancellative if $ab = ac$ implies $b = c$. Similarly, it is right cancellative if $ba = ca$ implies $b = c$. If it is both left and right, it is called cancellative.

It is also known [2] that in an LA-semigroup $G$, the medial law: $(ab)(cd) = (ac)(bd)$ holds for every $a, b, c, d \in G$. An LA-semigroup with a left identity is called an LA-monoid. In [9] an LA-monoid with a left inverse is called an LA-group. Because in an LA-group every left inverse is a right inverse, therefore, we can re-define an LA-group as follows: An LA-monoid $G$ is called a left almost group, abbreviated as LA-group, if it contains inverses.

Suppose that $(G, \cdot)$ is a commutative group. Then it is easy to see that $(G, \cdot)$, where $a \ast b = ba^{-1}$, is an example of an LA-group.

Let $G$ be an LA-semigroup and $a \in G$. A mapping $L_a : G \rightarrow G$, defined by $L_a(x) = ax$, is called the left translation by $a$. Similarly a mapping $R_a : G \rightarrow G$, defined by $R_a(x) = xa$ is called the right translation by $a$. An LA-semigroup $G$ is called a division LA-semigroup if the mappings $L_a$ and $R_a$ are onto for all $a \in G$.

An LA-semigroup $G$ is called a $\mu$-LA-semigroup if there are two mappings $\alpha, \beta$ of the set $G$ onto $G$ and an LA-monoid $(G, \circ)$ such that $ab = \alpha(a) \circ \beta(b)$ for all $a, b \in G$. Note that if we take $\alpha, \beta$ to be identity maps and $(G, \circ) = (G, \cdot)$, then an LA-monoid $(G, \cdot)$ is trivially a $\mu$-LA-monoid.

Let $G$ be a division $\mu$-LA-semigroup. Then $((G, \circ), \alpha, \psi, g)$ is said to be a right linear form of $G$ if $(G, \circ)$ is an LA-group, $\alpha$ a mapping of $G$ onto $G$, $\psi$ an endomorphism of $(G, \circ)$, $g \in G$ and $ab = \alpha(a) \circ (g \circ \psi(b))$ for all $a, b \in G$. Similarly $((G, \circ), \psi, \alpha, g)$ is said to be a left linear form of $G$ if $ab = \psi(a) \circ (g \circ \alpha(b))$ for all $a, b \in G$. If $\varphi = \alpha$ is an endomorphism of $G$, then $((G, \circ), \varphi, \psi, g)$ is a called a linear form of $G$. 
2. Division LA-semigroups

Having set the terminology and given the basic definitions we are now in a position to prove the following results.

**Proposition 2.1.** Every LA-group is a division µ-LA-group.

*Proof.* Let $\mathcal{G}$ be an LA-group and $L_a$ its left translation. Then

$$ab = (ea)b = (ba)e$$

yields

$$L_a((xe)a^{-1}) = a((xe)a^{-1}) = (((xe)a^{-1})a)e = ((aa^{-1})(xe))e = (e(xe))e = (xe)e = (ex) = ex.$$  

Thus for every $x \in G$ there exists $(xe)a^{-1} \in G$ such that $L_a((xe)a^{-1}) = x$. Hence $L_a$ is onto. Also $R_a$ is onto because $R_a(xa^{-1}) = (xa^{-1})a = (aa^{-1})x = ex = x$ for every $x \in G$. Hence $\mathcal{G}$ is a division LA-group.

Thus, the observation that every LA-monoid is trivially a µ-LA-monoid, and Theorem 9 in [3], imply that $\mathcal{G}$ is in fact a division µ-LA-group. \hfill $\square$

Let $C(G, \circ)$ denote the centre of LA-semigroup $(G, \circ)$.

**Theorem 2.2.** If $\mathcal{G}$ is an LA-semigroup, then the following statements are equivalent:

(i) $\mathcal{G}$ is a division µ-LA-semigroup,

(ii) $\mathcal{G}$ has a linear form $((G, \circ), \varphi, \psi, g)$ such that $\varphi\psi(a) \circ g = g \circ \psi\varphi(a)$ for every $a \in G$. In this case $C(G, \circ) = G$.

*Proof.* (i) $\Rightarrow$ (ii). Since $\mathcal{G}$ is a division µ-LA-semigroup satisfying the medial law, by Theorem 15 in [3], $((G, \circ), \varphi, \psi, g)$ is the linear form of $\mathcal{G}$ such that $\varphi\psi(a) \circ h = h \circ \psi\varphi(a)$ for all $a \in G$, where $h = \psi\varphi(x) \circ g$ for some $x \in G$. But by Theorem 15 in [3], we can assume that $x$ is the left identity of $(G, \circ)$. Thus $h = x \circ g = g$.

(ii) $\Rightarrow$ (i). Since $\mathcal{G}$ has a linear form $((G, \circ), \varphi, \psi, g)$, therefore by the definition, $\mathcal{G}$ is a division µ-LA-semigroup and so $ab = \varphi(a) \circ (g \circ \psi(b))$ for all $a, b \in G$, where $(G, \circ)$ is an LA-group. If $e$ is the left identity in $(G, \circ)$, then this last equation can be written as $\varphi(a) \circ (e \circ \psi(b)) = \varphi(a) \circ \psi(b)$, which implies that $\mathcal{G}$ is a division µ-LA-semigroup.

Let $x \in C(G, \circ)$. We wish to show that $x \in G$. Let $a, b, c \in G$, then

$$(ax)(bc) = (\varphi(a) \circ (g \circ \psi(x))) (\varphi(b) \circ (g \circ \psi(c)))$$

$$= \varphi(\varphi(a) \circ (g \circ \psi(x))) \circ (g \circ \psi(\varphi(b) \circ (g \circ \psi(c))))$$

$$= \varphi^2(a) \circ (\varphi(g) \circ \varphi\psi(x)) \circ (g \circ (\psi\varphi(b) \circ (\psi(g) \circ \psi^2(c)))).$$
Since \((G, \circ)\) is an LA-group, we can apply the medial and the left invertive laws (which hold in \((G, \circ)\)) to the above identity. Hence

\[(ax)(bc) = (\varphi^2(a) \circ g) \circ ((\psi_1 \circ (\psi_1(g) \circ \psi_2(c))) \circ (\varphi(g) \circ \varphi_2(b))).\]

Since \((\varphi_1 \circ g) \circ (g \circ \psi_1(b)) = (\psi_1 \circ (g \circ \psi_1(a)))\), therefore

\[(ax)(bc) = (\varphi^2(a) \circ g) \circ ((\psi_1 \circ g) \circ (\psi_1(g) \circ \psi_2(c))) \circ \varphi_2(x)).\]

Applying the medial law again, we get

\[(ax)(bc) = (\varphi^2(a) \circ g) \circ ((\psi_1 \circ g) \circ (\psi_1(g) \circ \psi_2(c))) \circ \varphi_2(x)) = (\varphi^2(a) \circ g) \circ ((\varphi_1 \circ \varphi_2(b)) \circ ((\psi_1(g) \circ (\psi_1(g) \circ \psi_2(c)))).\]

Thus \(x \in G\), and so \(C(G, \circ) \subseteq G\).

Conversely, let \(y \in G\). Then

\[(\varphi_1(y) \circ \varphi_2(a)) \circ g = (\psi_1(y) \circ g) \circ \varphi_2(a).\]

Since \(\varphi_1(y) \circ g = g \circ \psi_1(a),\) therefore the above identity gives

\[(g \circ \psi_1(a)) \circ \psi_1(y) = (g \circ \varphi_2(y)) \circ \varphi_2(a),\]

i.e.

\[(\psi_1(y) \circ \varphi_2(a)) \circ g = (\varphi_1(y) \circ \varphi_2(a)) \circ g.\]

Since \((G, \circ)\) is cancellative, \(\psi_1(y) \circ \varphi_2(a) = \psi_1(y) \circ \varphi_2(y)\). But \(\psi_2 = \varphi_2\), by Theorem 16 in [3]. So \(\psi_1(y) \circ \varphi_2(a) = \psi_1(y) \circ \psi_1(a)\). Thus \(\psi_1(y) \in C(G, \circ)\). This together with the fact that \(\psi_2 : G \to G\) is a homomorphism, imply \(y \in G\). Hence \(G \subseteq C(G, \circ)\), and in consequence \(G = C(G, \circ)\).

**Corollary 2.3.** A division \(\mu\)-LA-semigroup \(G\) is commutative if it has a linear form \((G, +, \varphi, \psi, g)\) such that \((G, +)\) is a commutative group and \(\psi_1 = \varphi_2\).

**Proof.** If a division \(\mu\)-LA-semigroup \(G\) has a linear form as above, then \(\varphi_1(a) + g = \varphi_1(a) + g = g + \psi_1(a)\). Therefore \(G = C(G, \circ)\) by Theorem 2.2.

**Theorem 2.4.** For any division \(\mu\)-LA-semigroup \(G\) there are mappings \(\alpha, \beta\) of \(G\) onto \(G\) such that \(\alpha(a) \beta(b) = \alpha(b) \beta(a)\) for every \(a, b \in G\).

**Proof.** Since \(G\) is a division \(\mu\)-LA-semigroup, therefore \(\alpha = L_c\) and \(\beta = R_c\) are onto mappings (for all \(c \in G\), and \(\alpha(a) \beta(b) = L_c(a) R_c(b) = (ca)(bc) = (bc)(ca) = (cb)(ac) = L_c(b) R_c(a) = \alpha(b) \beta(a)\).
**Theorem 2.5.** A division $\mu$-LA-semigroup $G$ is commutative if and only if the mapping $a \mapsto aa$ is an endomorphism of $G$.

*Proof.* If $a \mapsto aa$ is an endomorphism of $G$. Then $(ab)(ab) = (aa)(bb)$ for every $a, b \in G$, because $G$ is medial, and so $G = C(G, \circ)$ by Theorem 2.2.

Conversely, if $G$ is commutative, then $(ab)(ab) = (aa)(bb)$ implies that the mapping $a \mapsto aa$ is an endomorphism of $G$.

**Proposition 2.6.** The mapping $a \mapsto aa$ is an endomorphism of $G$ if $G$ is an LA-semigroup.

*Proof.* The proof is a trivial consequence of the medial law.

Note here that the converse is not true because there are medial groupoids, which are not LA-semigroups.

An LA-semigroup $G$ is called *idempotent* if $aa = a$ for all $a \in G$. An LA-semigroup $G$ in which $aa = bb$ for all $a, b \in G$ is called *unipotent*.

**Proposition 2.7.** Let $G$ be a left cancellative LA-semigroup. Then:

(i) $\alpha$ and $\psi$ are permutations of $G$, if $((G, \circ), \alpha, \psi, g)$ is a right linear form of $G$.

(ii) $\varphi$ and $\beta$ are permutations of $G$, if $((G, \circ), \varphi, \beta, g)$ is a left linear form of $G$.

*Proof.* (i) Since $((G, \circ), \alpha, \psi, g)$ is a right linear form of a left cancellative LA-semigroup $G$, therefore $\alpha$ is a mapping from $G$ onto $G$ and $\psi$ is an endomorphism of $G$. We prove that $\alpha$ and $\psi$ are one-to-one.

Let $\alpha(a) = (aj) \circ g^{-1} = R_j(a) \circ g^{-1}$. If $\alpha(a) = \alpha(b)$, then $R_j(a) \circ g^{-1} = R_j(b) \circ g^{-1}$. Since $(G, \circ)$ is cancellative, therefore $R_j(a) = R_j(b)$, which by Theorem 2.6 from [5], implies $a = b$. Hence $\alpha$ is one-to-one.

Let $\psi(a) = L_y(a)$, where $y = \alpha^{-1}(g^{-1})$. Since $\alpha(a) = R_j(a) \circ g^{-1}$, therefore $\alpha(y) = R_j(y) \circ g^{-1}$. But $\alpha(y) = g^{-1}$ implies $g^{-1} = R_j(y) \circ g^{-1}$, i.e. $y = \alpha^{-1}(R_j(y) \circ g^{-1}) = \alpha^{-1}(g^{-1})$. Now $\psi(a) = L_y(a) = \alpha^{-1}(R_j(y) \circ g^{-1})a$. If $\psi(a) = \psi(b)$, then $\alpha^{-1}(yj \circ g^{-1})a = \alpha^{-1}(yj \circ g^{-1})b$. Since $\alpha$ is one-to-one, therefore $(yj \circ g^{-1})a = (yj \circ g^{-1})b$, which by Theorem 2.6 from [5] implies $a = b$. Thus $\psi$ is one-to-one.

(ii) Analogously as (i).

**Theorem 2.8.** Let $G$ be an LA-semigroup. Then the following conditions are equivalent:

(i) $G$ is a division $\mu$-LA-semigroup,

(ii) $G$ has a linear form $((G, +), \sigma, \psi, g)$ such that $(G, +)$ is a commutative group and $\sigma(\psi(a) + g) = \sigma(g) + \psi\sigma(a)$. 
Proof. Since a division LA-semigroup $G$ is medial, by Theorem 16 in [3], $G$ has a linear form $((G, +), \sigma, \psi, g)$ such that $(G, +)$ is a commutative group and $\sigma \psi = \psi \sigma$. Thus $\sigma(\psi(a) + g) = \sigma(g) + \sigma \psi(a) = \sigma(g) + \psi \sigma(a)$ because $\sigma$ is an endomorphism.

Conversely, if an LA-semigroup $G$ has a linear form as in (ii), then $ab = \sigma(a) + g + \psi(b)$, which for $g = 0$ shows that $G$ is a division $\mu$-LA-semigroup. □

Theorem 2.9. Let an LA-semigroup $G$ has a linear form $((G, \circ), \varphi, \psi, g)$. Then $G$ is a commutative group, if $\varphi, \psi$ are central automorphism of $(G, \circ)$ and $\varphi \psi = \psi \varphi$.

Proof. If $\varphi, \psi$ are central automorphisms of $(G, \circ)$ such $\varphi \psi = \psi \varphi$, then $\varphi(a), \psi(a) \in C(G, \circ)$ for every $a \in G$. Thus $\varphi \psi(a) \in C(G, \circ)$ and $\varphi \psi(a) \circ g = g \circ \psi \varphi(a)$ for every $g \in G$. Theorem 2.2 completes the proof. □

References


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Department of Mathematics, Quaid-i-Azam University, Islamabad, Pakistan
e-mail: qmushtaq@apollo.net.pk