Zeroids and idempoids in AG-groupoids

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Abstract

Clifford and Miller (Amer. J. Math. 70, 1948) and Dawson (Acta Sci. Math. 27, 1966) have studied semigroups having left or right zeroids in a semigroup. In this paper, we have investigated AG-groupoids, and AG-groupoids with weak associative law, having zeroids or idempoids. Some interesting characteristics of these structures have been explored.

An Abel-Grassman’s groupoid [8], abbreviated as AG-groupoid, is a groupoid $G$ whose elements satisfy the left invertive law: $(ab)c = (cb)a$. It is also called a left almost semigroup [4, 5, 6, 7]. In [3], the same structure is called a left invertive groupoid. In this note we call it an AG-groupoid. It is a useful non-associative algebraic structure, midway between a groupoid and a commutative semigroup, with wide applications in the theory of flocks.

AG-groupoid is medial [5], that is, $(ab)(cd) = (ac)(bd)$ for all $a, b, c, d$ in $G$. It has been shown in [5] that if an AG-groupoid contains a left identity then it is unique. It has been proved also that an AG-groupoid with right identity is a commutative monoid, that is, a semigroup with identity element. An element $a_o$ of an AG-groupoid $G$ is called a left zero if $a_o a = a_o$ for all $a \in G$.

It has been shown in [5] that if $ab = cd$ then $ba = dc$ for all $a, b, c, d$ in an AG-groupoid with left identity. If for all $a, b, c$ in an AG-groupoid $G$, $ab = ac$ implies that $b = c$, then $G$ is called left cancellative. Similarly, if $ba = ca$ implies that $b = c$, then $G$ is called right cancellative. It is known [5] that every left cancellative AG-groupoid is right cancellative but the converse is not true. However, every right cancellative AG-groupoid with left identity is left cancellative.

Clifford and Miller [1] have defined an element $z_l$ as a left zeroid in a semigroup $G$ if for each element $x$ in $G$, there exists $a$ in $G$ such that $ax = z_l$. 

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A right zeroid is similarly defined. An element is a zeroid in $G$ if it is both left and right zeroid.

Dawson [2] has studied semigroups having left or right zeroid elements and investigated some of their properties. In this paper we introduce the concept of left idempoids in AG-groupoid and investigate some of their properties.

Next we prove the following result.

**Theorem 1.** An AG-groupoid $G$ is a semigroup if and only if $a(bc) = (cb)a$ for all $a, b, c \in G$.

**Proof.** Let $a(bc) = (cb)a$. Since $G$ is an AG-groupoid, $(ab)c = (cb)a$. As the right hand sides of the two equations are equal, we conclude that $(ab)c = a(bc)$. Thus $G$ is a semigroup.

Conversely, suppose that an AG-groupoid $G$ is a semigroup. This means that $(ab)c = (cb)a$ and $(ab)c = a(bc)$. Since the left hand sides of these equations are equal, we get $a(bc) = (cb)a$ for all $a, b, c \in G$.

An element $z_r$ of an AG-groupoid $G$ is called a right idempoid if, for each $x \in G$, there exists $a \in G$ such that $(xa)a = z_r$.

Note that $G$ contains a right idempoid because for any $x, y \in G$ there exists $a \in G$ such that $ax, ay \in G$. So $(ax)(ay) = (aa)(xy) = (aa)z = (za)a$, where $z = xy$ is an arbitrary element in $G$, implies that $G$ contains a right idempoid.

**Proposition 1.** An AG-groupoid $G$ is a semigroup if and only if $z_r = a(ax)$ is a right idempoid for some fixed $a$ and any $x \in G$.

**Proof.** The proof follows directly from Theorem 1.

**Theorem 2.** An AG-groupoid $G$ with $G^2 = G$ is a commutative semigroup if and only if $(ab)c = a(cb)$ for all $a, b, c \in G$.

**Proof.** Suppose $(ab)c = a(cb)$. Since $G$ is an AG-groupoid, $(cb)a = (ab)c$. Combining the two equations we obtain $(cb)a = a(cb)$ implying that $G$ is commutative. Thus $(ab)c = (cb)a = a(cb) = a(bc)$ shows that $G$ is a commutative semigroup.

The converse follows immediately.

**Corollary 1.** An AG-groupoid is a commutative semigroup if and only if $z_r = xa^2$ is a right idempoid for fixed $a \in G$ and any $z \in G$.

**Proof.** The proof follows immediately from Theorem 2.
Proposition 2. The square of every left zeroid in an AG-groupoid $G$ with an idempotent is a right idempoid.

Proof. Let $x$ be an idempotent and $z_l$ a left zeroid in $G$. Since $z_l$ is a left zeroid, there exists $a$ in $G$ such that $ax = z_l$. Therefore

$$z_lz_l = (ax)(ax) = (aa)(xx) = (aa)x = (xa)a = z_r,$$

which completes the proof.

Corollary 2. In an AG-groupoid $G$ there exists a left zeroid element.

Proof. If we define a mapping $l_a : G \to G$ by $(x)l_a = ax$ for all $x$ in $G$, then obviously these mappings are related to left zeroids in a natural way.

In the following we shall examine the necessary and sufficient conditions for $l_a$ to be an epimorphism, endomorphism, automorphism, monomorphism and anti-homomorphism.

Theorem 3. If in a left cancellative AG-groupoid $G$ we define for a fixed $a$ and some $x$, a mapping $l_a : x \mapsto ax$, from $G$ onto $G$, then the following statements are equivalent:

(i) $l_a$ is an epimorphism,
(ii) $a$ is an idempotent in $G$,
(iii) $l_a$ is an automorphism.

Proof. Suppose (i) holds. Then there exists $x$ in $G$ such that for some fixed $a$, $ax = y$, in $G$. This implies that for some $x$ in $G$ and a fixed $a$ in $G$, there exists an element $y$ in $G$ such that $y = (x)l_a$. Now $(a)l_ay = (a)l_a(x)l_a = (aa)(ax)$ and $(a)l_a(x)l_a = (ax)l_a = a(ax) = ay$ imply that $(a)l_a = a$, that is, $a$ is an idempotent in $G$. Hence (i) implies (ii).

Also $(x)l_a(y)l_a = (ax)(ay) = (aa)(xy) = a(xy)$ because $a$ is idempotent. This implies that $(x)l_a(y)l_a = (xy)l_a$, which further implies that $l_a$ is an endomorphism. In order to show that $l_a$ in an automorphism it is sufficient to show that $l_a$ is one-to-one. But this is obvious since $(x)l_a = (y)l_a$ and $ax = ay$ implies that $x = y$ by virtue of left cancellation. Thus (ii) implies (iii).

Since $l_a$ is an automorphism, (iii) implies (i).

Theorem 4. In an AG-groupoid $G$ the following statements are equivalent:

(i) $G$ has a right zero,
(ii) $l_a : x \mapsto ax$ an automorphism and $G$ has an idempotent element,
(iii) $G$ has a zero.
Proof. If \( x \) is a right zero of \( G \), then \( ax = x \) for some \( a \in G \). But \( x = ax = (x)l_a \) for every \( x \) in \( G \). This implies that \( l_a \) is the identity mapping, which is an automorphism and, in particular, \( a = (a)l_a \). It follows that \( a = aa \), that is, \( a \) is an idempotent. Thus (i) implies (ii).

Further, for any \( x \) and some \( a \) in \( G \), we have \( a(xx) = (xx)l_a = xx \) and \( (xx)a = (ax)x = (x)l_ax = xx \). This implies that \( a(xx) = (xx)a = xx \), showing that \( xx \) is a zero in \( G \). Hence (ii) implies (iii).

(iii) obviously implies (i). \( \square \)

**Theorem 5.** If \( (G)l_a = \{(x)l_a : x \in G\} \), where \( a \) is a fixed idempotent of an AG-groupoid \( G \), then \( (G)l_a \) is an AG-groupoid with an idempotent \( a \).

**Proof.** Let \( (x)l_a \), \( (y)l_a \) belong to \( (G)l_a \). Then

\[
(x)l_a(y)l_a = (ax)(ay) = (aa)(xy) = a(xy)(xy)l_a.
\]

This implies that \( (x)l_a(y)l_a \in (G)l_a \). Now

\[
(x)l_a(y)l_a(z)l_a = ((ax)(ay))(az) = ((az)(ay))(ax) = (z)l_a(y)(x)l_a.
\]

Hence \( (G)l_a \) is an AG-groupoid. \( \square \)

**Theorem 6.** If \( (G)l_a = \{(x)l_a : x \in G\} \), where \( a \) is a fixed element of a right cancellative AG-groupoid \( G \), then \( l_a \) is an endomorphism if and only if \( a \) is an idempotent of \( G \).

**Proof.** Let \( l_a \) be an endomorphism. Then \( (xx') = (x)l_a(x')l_a \). Hence

\[
a(xx') = (ax)(ax') = (aa)(xx')
\]

imply that \( a = aa \).

Conversely, if \( a = aa \) then

\[
(x)l_a(x')l_a = (ax)(ax') = (aa)(xx') = a(xx')(x)l_a,
\]

which completes our proof. \( \square \)

**Theorem 7.** If \( G \) is an AG-groupoid with an idempotent \( a \) and \( l_a \) is an anti-homomorphism, then \( a \) commutes with every element of \( G \).

**Proof.** Let \( x \) be an arbitrary element of \( G \). Then there exists \( x' \in G \) such that \( (x')l_a = x \). Consider \( xa \) for any \( x \) and some idempotent \( a \) in \( G \). Then

\[
xa = x(aa) = x(a)l_a = (x')l_a(a)l_a = (ax')l_a(a)l_a = a(ax') = a(x')l_a = ax.
\]

This implies that \( a \) commutes with every \( x \) in \( G \). \( \square \)
Theorem 8. In a right cancellative AG-groupoid $G$ with an idempotent $a$, if $l_a : x \mapsto ax$ is an anti-homomorphism, then the following statements are equivalent:

(i) $l_a$ is an anti-epimorphism,

(ii) $G$ is a commutative monoid,

(iii) $l_a$ is an anti-automorphism.

Proof. Suppose (i) holds. Then for a fixed $a \in G$, there exist $x$ and $y$ in $G$ such that, $y = ax = (x)l_a$. Now

$$ya = y(aa) = (x)l_a(a)l_a = (ax)l_a = a(ax) = a(x)l_a = ay$$

because $l_a$ is an anti-epimorphism.

Further $ay = (aa)y = (ya)a$, which implies that $ya = (ya)a$. So $y = ya = ay$. Hence $a$ is the identity of $G$. But an AG-groupoid with right identity is a commutative monoid by a result in [5]. Hence (i) implies (ii).

Now, since $a$ is the identity in $G$, then for any $x$ in $G$, we have $ax = x$ implying that $(x)l_a = x$ and so $l_a$ is the identity mapping. This implies that $l_a$ is an anti-automorphism. It follows that (ii) implies (iii).

Also, (iii) implies (i), follows immediately since an anti-automorphism must necessarily be an anti-epimorphism. \qed

References


