Quasigroups and Related Systems **11** (2004), 71 – 78

# Quotient groups induced by fuzzy subgroups

*Yong Lin Liu*

**Abstract**

We construct a quotient group induced by a fuzzy normal subgroup and prove the corresponding isomorphism theorems. Obtained results are used to the characterization of selected classes of quotient groups.

## 1. Introduction

In [16] L. A. Zadeh introduced the concept of fuzzy sets and fuzzy set operations. A. Rosenfeld [14] applied this concept to the theory of groupoids and groups. The various constructions of fuzzy quotient groups and fuzzy subgroup isomorphisms have been investigated by several researchers (see e.g. [1, 3, 6, 9, 11, 13]). In this paper we give a new method of construction of quotient groups by fuzzy normal subgroups and apply this construction to the characterization of selected classes of quotient groups.

## 2. Preliminaries

A *fuzzy subset* of a group $G$, i.e. a function $\mu$ from $G$ into [0,1], is called a *fuzzy subgroup* of $G$ if

($F_1$) $\mu(xy) \geqslant \min\{\mu(x), \mu(y)\}$ for all $x, y \in G$, and

($F_2$) $\mu(x^{-1}) \geqslant \mu(x)$ for all $x \in G$,

or, equivalently, if $\mu(xy^{-1}) \geqslant \min\{\mu(x), \mu(y)\}$ for all $x, y \in G$.

A fuzzy subgroup $\mu$ of a group $G$ is called *normal* if for all $x, y \in G$ it satisfies one of the following equivalent conditions (cf. [15]):

2000 Mathematics Subject Classification: 20E10, 94D05
Keywords: Fuzzy (normal) subgroup, fuzzy isomorphism theorem

(F$_3$) $\mu(xyx^{-1}) \geqslant \mu(y)$,
(F$_4$) $\mu(xyx^{-1}) = \mu(y)$,
(F$_5$) $\mu(xy) = \mu(yx)$.

It is not difficult to see that for all fuzzy subgroups $\mu$ of a group $G$ and all $x, y \in G$

(*i*) $\mu(e) \geqslant \mu(x)$,
(*ii*) $\mu(x^{-1}) = \mu(x)$,
(*iii*) $\mu(xy^{-1}) = \mu(e)$ implies $\mu(x) = \mu(y)$.

Fuzzy subgroups of $G$ can be characterized by the collection of *levels*, i.e. sets of the form $\mu_t = \{g \in G \,|\, \mu(g) \geqslant t\}$, where $t \in [0,1]$. Namely, as it is proved in [15], *a fuzzy subset $\mu$ of a group $G$ is a fuzzy (normal) subgroup of $G$ if and only if for all $t \in [0,1]$, $\mu_t$ is either empty or a (normal) subgroup of $G$.*

The *image* $f(\eta)$ of a fuzzy subset $\eta$ of $G$ and *preimage* $f^{-1}(\mu)$ of a fuzzy subset $\mu$ of $G'$ and a map $f : G \to G'$ are defined as

$$f(\eta)(y) = \begin{cases} \sup\limits_{x \in f^{-1}(y)} \eta(x) & \text{if } f^{-1}(y) \neq \emptyset, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$f^{-1}(\mu)(x) = \mu(f(x)), \quad x \in G.$$

It is not difficult to see that $f(\eta)$ and $f^{-1}(\mu)$ are fuzzy subsets.

## 3. Quotient groups induced by fuzzy subgroups

Let $\mu$ be a fuzzy normal subgroup of a group $G$. For any $x, y \in G$, define a binary relation $\sim$ on $G$ by

$$x \sim y \iff \mu(xy^{-1}) = \mu(e),$$

where $e$ is the unit of $G$.

**Lemma 1.** *$\sim$ is a congruence of $G$.*

*Proof.* The reflexivity and symmetry are obvious. To prove the transitivity let $x \sim y$ and $y \sim z$. Then $\mu(xy^{-1}) = \mu(yz^{-1}) = \mu(e)$ and $\mu(xz^{-1}) = \mu(xy^{-1}yz^{-1}) \geqslant \min\{\mu(xy^{-1}), \mu(yz^{-1})\} = \mu(e)$. Hence $\mu(xz^{-1}) = \mu(e)$, which proves that $\sim$ is an equivalence relation.

Now, if $x \sim y$, then $\mu(xy^{-1}) = \mu(e)$. Thus for all $z \in G$ we have $\mu((xz)(yz)^{-1}) = \mu(xzz^{-1}y^{-1}) = \mu(xy^{-1}) = \mu(e)$. Hence $xz \sim yz$. Since $\mu$ is a fuzzy normal subgroup, we have $\mu((zx)(zy)^{-1}) = \mu(zxy^{-1}z^{-1}) = \mu(z^{-1}zxy^{-1}) = \mu(xy^{-1}) = \mu(e)$. This gives $zx \sim zy$.

Using these facts it is not difficult to see that $\sim$ is a congruence. □

The equivalence class containing $x$ is denoted by $\mu_x$. $G/\mu$ denotes the corresponding quotient set.

**Proposition 1.** *If $\mu$ is a fuzzy normal subgroup of a group $G$, then $G/\mu$ is a group with the operation $\mu_x\mu_y = \mu_{xy}$.* □

**Example.** Let $G$ be the additive group of all integers and let $\mu(x) = t_1$ if $2|x$, and $\mu(x) = t_0$ if $2 \not| x$, where $0 \leqslant t_0 < t_1 \leqslant 1$. Then $\mu$ is a fuzzy normal subgroup of $G$ and $G/\mu = \{\mu_0, \mu_1\}$ is a quotient group induced by $\mu$. □

**Lemma 2.** [13] *If $f : G \to G'$ is an epimorphism of groups and $\mu$ a fuzzy normal subgroup of $G$, then $f(\mu)$ is a fuzzy normal subgroup of $G'$.*

Basing on this Lemma and Proposition 4.2 in [7] we can proved

**Lemma 3.** *Let $f : G \to G'$ be a homomorphism of groups, $\mu$ a fuzzy subgroup of $G$ and $\nu$ a fuzzy subgroup of $G'$.*

*(i) If $f$ is an epimorphism, then $f(f^{-1}(\nu)) = \nu$.*

*(ii) If $\mu$ is a constant on $ker f$, then $f^{-1}(f(\mu)) = \mu$.* □

Let $G_\mu = \mu_{\mu(0)} = \{x \in G | \mu(x) = \mu(0)\}$. It is obvious that if $\mu$ is a fuzzy (normal) subgroup of $G$, then $G_\mu$ is a (normal) subgroup of $G$.

**Theorem 1.** *Let $f : G \to G'$ be an epimorphism of groups and $\mu$ a fuzzy normal subgroup of $G$ with $ker f \subseteq G_\mu$. Then $G/\mu \cong G'/f(\mu)$.*

*Proof.* By Proposition 1 and Lemma 2, $G/\mu$ and $G'/f(\mu)$ are groups.

Let $\eta : G/\mu \to G'/f(\mu)$, where $\eta(\mu_x) = (f(\mu))_{f(x)}$. If $\mu_x = \mu_y$, then $\mu(xy^{-1}) = \mu(e)$. Since $ker f \subseteq G_\mu$, then $\mu$ is a constant on $ker f$, and by Lemma 3 $(ii)$ we have $f^{-1}(f(\mu)) = \mu$. Thus $(f^{-1}(f(\mu)))(xy^{-1}) = (f^{-1}(f(\mu)))(e)$, i.e. $f(\mu)(f(xy^{-1})) = f(\mu)(f(e))$, then $f(\mu)(f(x)(f(y))^{-1}) = f(\mu)(e')$, and so $(f(\mu))_{f(x)} = (f(\mu))_{f(y)}$. Hence $\eta$ is well-defined.

It is also a homomorphism because $\eta(\mu_x\mu_y) = \eta(\mu_{xy}) = (f(\mu))_{f(xy)} = (f(\mu))_{f(x)f(y)} = (f(\mu))_{f(x)}(f(\mu))_{f(y)} = \eta(\mu_x)\eta(\mu_y)$. Since $f$ is an epimorphism, for any $(f(\mu))_y \in G'/f(\mu)$, there exists $x \in G$ such that $f(x) = y$. So $\eta(\mu_x) = (f(\mu))_{f(x)} = (f(\mu))_y$, which means that $\eta$ is an epimorphism.

Moreover, $(f(\mu))_{f(x)} = (f(\mu))_{f(y)} \Rightarrow f(\mu)(f(x)(f(y))^{-1}) = f(\mu)(e') \Rightarrow f(\mu)(f(xy^{-1})) = f(\mu)(f(e)) \Rightarrow (f^{-1}(f(\mu)))(xy^{-1}) = (f^{-1}(f(\mu)))(e) \Rightarrow \mu(xy^{-1}) = \mu(e) \Rightarrow \mu_x = \mu_y$, which proves that $\eta$ is an isomorphism.

Hence $G/\mu \cong G'/f(\mu)$. $\square$

**Corollary 1.** *Let $f : G \to G'$ be an epimorphism of groups and $\nu$ a fuzzy normal subgroup of $G'$. Then $G/f^{-1}(\nu) \cong G'/\nu$.*

*Proof.* Since $f^{-1}(\nu)$ is a fuzzy normal subgroup (cf. [12]), $G/f^{-1}(\nu)$ and $G'/\nu$ are groups. Moreover, by Lemma 3, we have $\nu = f(f^{-1}(\nu))$.

If $x \in kerf$, then $f(x) = e' = f(e)$, and so $\nu(f(x)) = \nu(f(e))$, i.e. $f^{-1}(\nu)(x) = f^{-1}(\nu)(e)$. Hence $x \in G_{f^{-1}(\nu)}$, i.e. $kerf \subseteq G_{f^{-1}(\nu)}$.

Theorem 1 completes the proof. $\square$

**Proposition 2.** *Let $\chi_S$ be a characteristic function of a subset $S$ of a group $G$. Then $\chi_S$ is a fuzzy normal subgroup of $G$ if and only if $S$ is a normal subgroup of $G$.*

*Proof.* If $x, y \in S$, where $S$ is a normal subgroup of $G$, then $\chi_S(xy^{-1}) = \chi_S(x) = \chi_S(y) = 1$. Hence $\chi_S(xy^{-1}) = \min\{\chi_S(x), \chi_S(y)\}$. If at least one of $x$ and $y$ is not in $S$, then at least one of $\chi_S(x)$ and $\chi_S(y)$ is 0. Therefore $\chi_S(xy^{-1}) \geqslant \min\{\chi_S(x), \chi_S(y)\}$. Hence $\chi_S$ is a fuzzy subgroup of $G$. Moreover, for any $x, y \in G$, if $y \in S$, then $xyx^{-1} \in S$ and $\chi_S(xyx^{-1}) = 1 = \chi_S(y)$. If $y \notin S$, then $\chi_S(y) = 0$, so $\chi_S(xyx^{-1}) \geqslant \chi_S(y)$. Hence $\chi_S$ is a fuzzy normal subgroup of $G$.

Conversely, if $\chi_S$ be a fuzzy normal subgroup of $G$, then for any $x, y \in S$, we have $\chi_S(xy^{-1}) \geqslant \min\{\chi_S(x), \chi_S(y)\} = 1$. Thus $\chi_S(xy^{-1}) = 1$ and $xy^{-1} \in S$. Similarly for any $y \in S$, $x \in G$ we have $\chi_S(xyx^{-1}) \geqslant \chi_S(y) = 1$. Hence $\chi_S(xyx^{-1}) = 1$ and $xyx^{-1} \in S$. This proves that $S$ is a normal subgroup of $G$. $\square$

**Corollary 2.** *$G/\chi_{kerf} \cong G'$ for any epimorphism $f : G \to G'$ of groups.*

*Proof.* It follows from the fact that $\chi_{\{e\}} f = \chi_{kerf}$ and $G'/\chi_{\{e\}} \cong G'$. $\square$

Let $N$ be a normal subgroup of a group $G$. Recall that a quotient group $G/N$ induced by a normal subgroup $N$ is determined by an equivalent relation $\sim$, where $x \sim y$ is defined by $xy^{-1} \in N$. For no confusion, we write $x \sim y(N)$ if $x$ is equivalent to $y$ with respect to $N$, and $x \sim y(\chi_N)$ if $x$ is equivalent to $y$ with respect to the fuzzy normal subgroup $\chi_N$.

**Lemma 4.** *If $N$ is a normal subgroup of a group $G$, then $x \sim y(N)$ if and only if $x \sim y(\chi_N)$.* $\square$

**Corollary 3.** *Let $f : G \to G'$ be an epimorphism of groups and $N$ be a normal subgroup of $G$ such that $kerf \subseteq N$. Then $G/\chi_N \cong G'/\chi_{f(N)}$.*

*Proof.* By Proposition 2, $\chi_N$ and $\chi_{f(N)}$ are fuzzy normal subgroups of $G$ and $G'$, respectively. Putting $\mu = \chi_N$ in Theorem 1, we obtain $G_\mu =$

$G_{\chi_N} = N \supseteq kerf$. Since $f$ is an epimorphism, for any $x' \in G'$, there exists $x \in G$ such that $x' = f(x)$. If $x' \in f(N)$, then $x \in N$, which by Lemma 3 $(ii)$ gives $f(\mu)(x') = f(\chi_N)(x') = f(\chi_N)(f(x)) = \chi_N(x) = 1 = \chi_{f(N)}(x')$. If $x' \notin f(N)$, then $x \notin N$ and $f(\mu)(x') = f(\chi_N)(x') = \chi_N(x) = 0 = \chi_{f(N)}(x')$. Hence $G/\chi_N \cong G'/\chi_{f(N)}$. □

Observe that by Lemma 4, we obtain $G/\chi_N \cong G/N$ and $G'/\chi_{f(N)} \cong G'/f(N)$. This together with Corollary 3 implies the First Isomorphism Theorem for groups.

Moreover, if $f : G \to G'$ is an epimorphism of groups and $K$ is a normal subgroup of $G'$, then, by Proposition 2, we see that $\chi_{f^{-1}(K)}$ and $\chi_K$ are fuzzy normal subgroups of $G$ and $G'$, respectively.

Putting $\nu = \chi_K$, we have $f^{-1}(\nu) = f^{-1}(\chi_K) = \chi_{f^{-1}(K)}$. Indeed, if $x \in f^{-1}(K)$, then $f(x) \in K$, $f^{-1}(\chi_K)(x) = \chi_K f(x) = 1 = \chi_{f^{-1}(K)}(x)$. If $x \notin f^{-1}(K)$, then $f(x) \notin K$, $f^{-1}(\chi_K)(x) = \chi_K f(x) = 0 = \chi_{f^{-1}(K)}(x)$.

Thus for $\nu = \chi_K$, as a consequence of Corollary 1, we obtain

**Corollary 4.** *If $f : G \to G'$ is an epimorphism of groups and $K$ is a normal subgroup of $G'$, then $G/\chi_{f^{-1}(K)} \cong G'/\chi_K$.*

**Lemma 5.** *If $N$ is a normal subgroup and $\mu$ is a fuzzy normal subgroup of a group $G$, then $\mu$ restricted to $N$ is a fuzzy normal subgroup of $N$ and $N/\mu$ is a normal subgroup of $G/\mu$.*

*Proof.* Indeed, if $\mu_a, \mu_b \in N/\mu$, where $a, b \in N$, then $\mu_a(\mu_b)^{-1} = \mu_a\mu_{b^{-1}} = \mu_{ab^{-1}} \in N/\mu$. If $\mu_a \in N/\mu$, $\mu_x \in G/\mu$, where $a \in N$ and $x \in G$, then $xax^{-1} \in N$ and $\mu_x\mu_a(\mu_x)^{-1} = \mu_x\mu_a\mu_{x^{-1}} = \mu_{xax^{-1}} \in N/\mu$. Thus $N/\mu$ is a normal subgroup of $G/\mu$. □

**Theorem 2.** *If $\mu$ and $\nu$ are two fuzzy normal subgroups of a group $G$ such that $\mu(e) = \nu(e)$, then $G_\mu G_\nu/\nu \cong G_\mu/(\mu \cap \nu)$.*

*Proof.* By Lemma 5, $\nu$ is a fuzzy normal subgroup of $G_\mu G_\nu$. By [11] $\mu\cap\nu$ is a fuzzy normal subgroup of $G_\mu$. Thus $G_\mu G_\nu/\nu$ and $G_\mu/(\mu \cap \nu)$ are groups.

For any $x \in G_\mu G_\nu$, $x = ab$, where $a \in G_\mu$ and $b \in G_\nu$, we define $g : G_\mu G_\nu/\nu \to G_\mu/(\mu \cap \nu)$ putting $g(\nu_x) = (\mu \cap \nu)_a$.

If $\nu_x = \nu_y$, where $y = a_1b_1$, $a_1 \in G_\mu$ and $b_1 \in G_\nu$, then

$$\nu(ab(a_1b_1)^{-1}) = \nu(abb_1^{-1}a_1^{-1}) = \nu(a_1^{-1}abb_1^{-1}) = \nu(a_1^{-1}a(b_1b^{-1})^{-1}) = \nu(e).$$

Hence $\nu(a_1^{-1}a) = \nu(b_1b^{-1}) = \nu(e)$. Thus

$$\begin{aligned}(\mu \cap \nu)(aa_1^{-1}) &= \min\{\mu(aa_1^{-1}), \nu(aa_1^{-1})\} = \min\{\mu(e), \nu((a_1^{-1}a)^{-1})\} \\ &= \min\{\mu(e), \nu(e)\} = (\mu \cap \nu)(e),\end{aligned}$$

i.e. $(\mu\cap\nu)_a = (\mu\cap\nu)_{a_1}$. Hence $g$ is well-defined.

If $\nu_x, \nu_y \in G_\mu G_\nu/\nu$, where $x = ab$, $y = a_1b_1$, $a, a_1 \in G_\mu$ and $b, b_1 \in G_\nu$, then $xy = aba_1b_1$. Since $G_\mu$ is normal, $ba_1b_1 \in G_\mu$. Hence $g(\nu_x\nu_y) = g(\nu_{xy}) = (\mu\cap\nu)_{a(ba_1b_1)} = (\mu\cap\nu)_a(\mu\cap\nu)_{ba_1b_1}$ and $(\mu\cap\nu)((ba_1b_1)a_1^{-1}) = \min\{\,\mu(ba_1b_1a_1^{-1}),\ \nu(ba_1b_1a_1^{-1})\,\} = \min\{\,\mu((ba_1b_1)a_1^{-1}),\ \nu(b(a_1b_1a_1^{-1}))\,\} = \min\{\,\mu(e),\ \nu(e)\,\} = (\mu\cap\nu)(e)$. Hence $(\mu\cap\nu)_{ba_1b_1} = (\mu\cap\nu)_{a_1}$, i.e. $g(\nu_x\nu_y) = (\mu\cap\nu)_a(\mu\cap\nu)_{a_1} = g(\nu_x)g(\nu_y)$, which shows that $g$ is a homomorphism.

It is also endomorphism since for $(\mu\cap\nu)_a \in G_\mu/(\mu\cap\nu)$ and $b \in G_\nu$, we have $x = ab \in G_\mu G_\nu$ and $g(\nu_x) = (\mu\cap\nu)_a$.

Moreover, if $x, y \in G_\mu G_\nu$, where $x = ab$, $y = a_1b_1$, $a, a_1 \in G_\mu$, $b, b_1 \in G_\nu$, and $(\mu\cap\nu)_a = (\mu\cap\nu)_{a_1}$, then $(\mu\cap\nu)(aa_1^{-1}) = (\mu\cap\nu)(e)$, i.e, $\min\{\mu(aa_1^{-1}), \nu(aa_1^{-1})\} = \min\{\mu(e), \nu(e)\}$. But $\mu(e) = \nu(e)$ and $\mu(aa_1^{-1}) = \mu(e)$ imply $\nu(aa_1^{-1}) = \nu(e)$. Therefore $\nu(xy^{-1}) = \nu(ab(a_1b_1)^{-1}) = \nu(abb_1^{-1}a_1^{-1}) = \nu(a_1^{-1}abb_1^{-1}) \geqslant \min\{\nu(a_1^{-1}a), \nu(bb_1^{-1})\} = \min\{\nu((aa_1^{-1})^{-1}), \nu(bb_1^{-1})\} = \min\{\nu(e), \nu(e)\} = \nu(e)$. Thus $\nu_x = \nu_y$.

Hence $G_\mu G_\nu/\nu \cong G_\mu/(\mu\cap\nu)$. □

**Corollary 5.** *Let $N$, $K$ be two normal subgroups of a group $G$. Then $NK/\chi_K \cong N/\chi_{N\cap K}$.*

*Proof.* By Proposition 2, $\chi_N$ and $\chi_K$ are fuzzy normal subgroups of $G$. Putting $\mu = \chi_N$ and $\nu = \chi_K$ in Theorem 2, we obtain $G_\mu = N$, $G_\nu = K$, $\mu\cap\nu = \chi_N \cap \chi_K = \chi_{N\cap K}$ and $\mu(e) = 1 = \nu(e)$. Hence $NK/\chi_K \cong N/\chi_{N\cap K}$. □

Since $NK/\chi_K \cong NK/K$ and $N/\chi_{N\cap K} \cong N/N\cap K$, as a consequence of the above two lemmas we obtain the Second Isomorphism Theorem of groups. The Third Isomorphism Theorem is a consequence of the following

**Theorem 3.** *Let $\mu$ and $\nu$ be two fuzzy normal subgroups of a group $G$ with $\nu \leqslant \mu$ and $\nu(e) = \mu(e)$. Then $(G/\nu)/(G_\mu/\nu) \cong G/\mu$.*

*Proof.* By Lemma 5, $G_\mu/\nu$ is a normal subgroup of $G/\nu$.

Putting $f(\nu_x) = \mu_x$ for all $x \in G$, we define $f : G/\nu \to G/\mu$ such that $\nu(xy^{-1}) = \nu(e) = \mu(e)$ for all $\nu_x = \nu_y$. Because $\nu \leqslant \mu$, we have $\mu(xy^{-1}) \geqslant \nu(xy^{-1}) = \mu(e)$, and so $\mu(xy^{-1}) = \mu(e)$, i.e. $\mu_x = \mu_y$, which means that $f$ is well-defined. Since $f(\nu_x\nu_y) = f(\nu_{xy}) = \mu_{xy} = \mu_x\mu_y = f(\nu_x)f(\nu_y)$, $f$ is a homomorphism. By the definition, it is an epimorphism, too. But $ker f = \{\nu_x \in G/\nu \mid f(\nu_x) = \mu_e\} = \{\nu_x \in G/\nu \mid \mu_x = \mu_e\} = \{\nu_x \in G/\nu \mid \mu(x) = \mu(e)\} = \{\nu_x \in G/\nu \mid x \in G_\mu\} = G_\mu/\nu$. Thus $ker f = G_\mu/\nu$ and $(G/\nu)/(G_\mu/\nu) \cong G/\mu$. □

**Corollary 6.** $(G/\chi_K)/(N/\chi_K) \cong G/\chi_N$ *for any normal subgroups $N \subseteq K$ of a group $G$.* □

Finally we consider *fuzzy abelian subgroups*, i.e. fuzzy subgroups $\mu$ of a group $G$ satisfying the identity $\mu(xyx^{-1}y^{-1}) = \mu(e)$.

**Proposition 3.** *A fuzzy subgroup $\mu$ of a group $G$ is abelian if and only if $G/\mu$ is abelian.*

*Proof.* If $\mu$ is a fuzzy abelian subgroup, then $\mu(xyx^{-1}y^{-1}) = \mu(e)$, and hence $\mu(xy) = \mu(yx)$. Thus $\mu$ is fuzzy normal. Since $\mu(xy(yx)^{-1}) = \mu(xyx^{-1}y^{-1}) = \mu(e)$, we have $\mu_{xy} = \mu_{yx}$, i.e. $\mu_x\mu_y = \mu_y\mu_x$. Hence $G/\mu$ is an abelian group.

Conversely, if $G/\mu$ is abelian, then $\mu_{xy} = \mu_{yx}$ and $\mu(xy(yx)^{-1}) = \mu(e)$. So $\mu(xyx^{-1}y^{-1}) = \mu(e)$. □

Let $\mu$ be a fuzzy subgroup of a group $G$. The smallest positive integer $n$ (if it exists) such that $\mu(x^n) = \mu(e)$ is called the *fuzzy order* of $x$ with respect to $\mu$ and is denoted by $FO_\mu(x)$ (cf. [4]). If $FO_\mu(x)$ is finite for every $x \in G$, then $\mu$ is called *fuzzy torsion*. In the case when for all $x \in G$ $FO_\mu(x)$ is a power of a prime number $p$, we say that $\mu$ is a *fuzzy $p$-subgroup* of $G$.

**Proposition 4.** *A fuzzy normal subgroup $\mu$ of a group $G$ is a fuzzy $p$-subgroup if and only if $G/\mu$ is a $p$-group.*

*Proof.* If $\mu$ is a fuzzy $p$-subgroup of $G$, then for any $\mu_x \in G/\mu$ there is a nonnegative integer $s$ such that $\mu(x^{p^s}) = \mu(e)$, i.e. $\mu_{x^{p^s}} = \mu_e$. Hence $(\mu_x)^{p^s} = \mu_e$. Conversely, if $G/\mu$ is a $p$-group of $G$, then for any $x \in G$ and some nonnegative integer $t$ we have $(\mu_x)^{p^t} = \mu_e$, i.e. $\mu_{x^{p^t}} = \mu_e$. Thus $\mu(x^{p^t}) = \mu(e)$, which completers the proof. □

**Proposition 5.** *A fuzzy subgroup $\mu$ of an abelian group $G$ is fuzzy torsion if and only if $G/\mu$ is torsion.*

*Proof.* Because $G$ is an abelian group, $\mu$ is normal. Let $G/\mu$ be torsion. For any $x \in G$, there is a positive integer $n$ such that $(\mu_x)^n = \mu_e$, i.e. $\mu_{x^n} = \mu_e$, and so $\mu(x^n) = \mu(e)$. Hence $FO_\mu(x)$ is finite and $\mu$ is fuzzy torsion.

The converse is obvious. □

## References

[1] **N. Ajmal**: *Homomorphism of fuzzy groups, correspondence theorem and fuzzy quotient groups*, Fuzzy Sets and Systems **61** (1994), 329 – 339.

[2] **P. S. Das**: *Fuzzy groups and level subgroups*, J. Math. Anal. Appl. **84** (1981), 264 – 269.

[3] **J. X. Fang**: *Fuzzy homomorphism and fuzzy isomorphism*, Fuzzy Sets and Systems **63** (1994), 237 − 242.

[4] **J. G. Kim**: *Some characterizations of fuzzy subgroups*, Fuzzy Sets and Systems **87** (1997), 243 − 249.

[5] **J. G. Kim and H. D. Kim**: *A characterization of fuzzy subgroups of some abelian groups*, Inform. Sci. **80** (1994), 243 − 252.

[6] **I. J. Kumar, P. K. Saxena and P. Yadav**: *Fuzzy normal subgroups and fuzzy quotients*, Fuzzy Sets and Systems **46** (1992), 121 − 132.

[7] **H. V. Kumbhojkar and M. S. Bapat**: *Correspondence theorem for fuzzy ideals*, Fuzzy Sets and Systems **41** (1991), 213 − 219.

[8] **Y. L. Liu, J. Meng and X. L. Xin**: *Quotient rings induced via fuzzy ideals*, Korean J. Comput & Appl. Math. **8** (2001), 631 − 643.

[9] **B. B. Makamba**: *Direct products and isomorphism of fuzzy subgroups*, Inform. Sci. **65** (1992), 33 − 43.

[10] **N. N. Morsi and S. E. B. Yehia**: *Fuzzy-quotient groups*, Inform. Sci. **81** (1994), 177 − 191.

[11] **N. P. Mukherjee and P. Bhattacharya**: *Fuzzy groups: some group-theoretic analogs*, Inform. Sci. **39** (1986), 247 − 268.

[12] **N. P. Mukherjee and P. Bhattacharya**: *Fuzzy normal subgroups and fuzzy cosets*, Inform. Sci. **34** (1984), 225 − 239.

[13] **S. Ray**: *Isomorphic fuzzy groups*, Fuzzy Sets and Systems **50** (1992), 201 − 207.

[14] **A. Rosenfeld**: *Fuzzy groups*, J. Math. Anal. Appl. **35** (1971), 512 − 517.

[15] **W. M. Wu**: *Normal fuzzy subgroups*, Fuzzy Math. **1** (1981), 21 − 30.

[16] **L. A. Zadeh**: *Fuzzy sets*, Inform. Control **8** (1965), 338 − 353.

Department of Applied Mathematics
Xidian University
Xi Zan 710071, Shaanxi
P.R.China
and
Department of Mathematics
Nanping Teachers College
Nanping 153000
Fujian
P.R.China
e-mail: ylliun@163.net

Received July 23, 2002