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# Fuzzy congruences on groups

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**Abstract**

In this paper we define a fuzzy congruence on a group which is a new notion and consider their fundamental properties. We show that there is a lattice isomorphism between the set of fuzzy normal subgroups of a group and the set of fuzzy congruences on this group.

## 1. Introduction

While there are many papers about fuzzy group theory, we can not find papers about *fuzzy congruences* on groups. In the theory of crisp group theory, there exist close relationships between normal subgroups and congruences. It is a natural question to extend these relationships to the case of fuzzy group theory. In this paper we define fuzzy congruences on groups and fuzzy quotient groups by fuzzy congruences and investigate their properties. In [4], Rosenfeld defined fuzzy subgroupoids and proved that a homomorphic image of a fuzzy subgroupoid with the *sup property* was a fuzzy subgroupoid, and hence that a homomorphic image of a fuzzy subgroup with the sup property was a fuzzy subgroup. This theorem needs the sup property. But in this paper we can show the theorem without the sup property, that is, a homomorphic image of a fuzzy subgroup is a fuzzy subgroup. And in [3], Mukherjee and Bhattacharya showed that if $\bar{A}$ is a fuzzy subgroup of a *finite* group $G$ is such that all the level subgroups of $G$ are normal subgroups then $\bar{A}$ is a fuzzy normal subgroup. We can also prove the theorem without finiteness using the *transfer principle* which is a fundamental tool we have developped here.

In this paper we show that

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1. The lattice $FNS(G)$ of all fuzzy normal subgroups of a group $G$ is isomorphic to the lattice $FCon(G)$ of all fuzzy congruences on $G$;

2. $FNS_\alpha(G)$ forms a modular lattice for every $\alpha \in [0,1]$;

3. Let $G$ and $G'$ be groups and $f : G \to G'$ be a homomorphism. If $\bar{A}$ is a fuzzy (normal) subgroup of $G$ then $f[\bar{A}]$ is a fuzzy (normal) subgroup of $G'$;

4. Let $G$, $G'$ and $f$ be as above. If $\bar{A}$ is a fuzzy normal subgroup of $G'$ then $G/f^{-1}(\bar{A}) \cong f(G)/\bar{A}$.

Of course, these results are not completely new, but these are obtained by the crisp group theory with the so-called transfer principle. This means that the transfer principle is a very important tool to investigate the fuzzy theory.

## 2. Fuzzy groups

Let $G$ be any group. By a *fuzzy subgroup* $\bar{A}$ of $G$ we mean a function $\bar{A} : G \to [0,1]$ such that

$$\bar{A}(xy^{-1}) \geqslant \bar{A}(x) \wedge \bar{A}(y) = \min\{\, \bar{A}(x),\, \bar{A}(y)\,\}$$

for all $x, y \in G$. Moreover a *fuzzy normal subgroup* $\bar{A}$ of $G$ is defined as a fuzzy subgroup satisfying the condition

$$\bar{A}(xy) \geqslant \bar{A}(yx).$$

For the sake of simplicity, we denote by $FS(G)$ ($FNS(G)$) the class of all fuzzy (normal) subgroups of $G$.

For every fuzzy subgroup $\bar{A}$ of $G$ we have the following (see [3, 4]):

**Proposition 1.** *Let $G$ be a group with the unit element $e$ and $\bar{A}$ be a fuzzy subgroup of $G$. For all $x, y \in G$,*

1. $\bar{A}(x) \leqslant \bar{A}(e)$,
2. $\bar{A}(x) = \bar{A}(x^{-1})$,
3. $\bar{A}(xy^{-1}) = \bar{A}(e)$ *implies* $\bar{A}(x) = \bar{A}(y)$.

As to the converse problem whether $\bar{A}(x) = \bar{A}(y)$ implies $\bar{A}(xy^{-1}) = \bar{A}(e)$, we have a counter-example. Let $G = \{e, a, b, c\}$ be a *Klein's* group defined by the following table.

| | $e$ | $a$ | $b$ | $c$ |
|---|---|---|---|---|
| $e$ | $e$ | $a$ | $b$ | $c$ |
| $a$ | $a$ | $e$ | $c$ | $b$ |
| $b$ | $b$ | $c$ | $e$ | $a$ |
| $c$ | $c$ | $b$ | $a$ | $e$ |

Let $\bar{A}$ be a map from $G$ to $[0,1]$ as $\bar{A}(e) = 1$, $\bar{A}(a) = 0.5$, $\bar{A}(b) = \bar{A}(c) = 0$. We see that $\bar{A}$ is a fuzzy normal subgroup of $G$. Then, while $\bar{A}(b) = \bar{A}(c) = 0$, we have $\bar{A}(bc^{-1}) = \bar{A}(bc) = \bar{A}(a) = 0.5 \neq 1 = \bar{A}(e)$. Thus the converse problem above does not hold.

## 3. Transfer principle

To express the *transfer principle* in group theory exactly we shall first of all define several terms in the theory of groups.

Let $G$ be any group, $V$ be a countable set $\{x, y, z, \ldots\}$ of syntactic variables which range over the elements of $G$. A *term* of $G$ is defined recursively:

(1) The unit element $e \in G$ is a *term*;

(2) Each variable of $V$ is a *term*;

(3) If $s$ and $t$ are *terms*, then $st$ is a *term*.

Thus, for example, $x$, $ex$, and $x(ey)$ are terms of $G$. We use the symbols $a, b, c, \ldots$ as elements of $G$ and $x, y, z, \ldots$ as variables of $V$. We have to distinguish the elements of $G$ and the terms of $G$.

Let $A$ be a subset of $G$ which satisfies the following property $\mathcal{P}$ expressed by a first-order formula:

$$\mathcal{P} : \forall x \ldots \forall y (t_1(x, \ldots, y) \in A \wedge \ldots \wedge t_n(x, \ldots, y) \in A \to t(x, \ldots, y) \in A),$$

where $t_1(x, \ldots, y), \ldots, t_n(x, \ldots, y)$, and $t(x, \ldots, y)$ are terms of $G$ constructed by variables $x, \ldots, y$. We say that the subset $A$ satisfies the property $\mathcal{P}$ if, for all elements $a, \ldots, b \in G$, $t(a, \ldots, b) \in A$ whenever $t_1(a, \ldots, b), \ldots,$ $t_n(a, \ldots, b) \in A$. For the subset $A$ which satisfies $\mathcal{P}$ we define a fuzzy subset $\bar{A}$ which satisfies the following property $\bar{\mathcal{P}}$:

$$\bar{\mathcal{P}} : \forall x \ldots \forall y \ (\, \bar{A}(t(x, \ldots, y)) \geqslant \bar{A}(t_1(x, \ldots, y)) \wedge \ldots \wedge \bar{A}(t_n(x, \ldots, y)) \,).$$

For any $\alpha \in [0,1]$, we put

$$U(\bar{A} : \alpha) = \{x \in G \mid \bar{A}(x) \geqslant \alpha\}.$$

The closed interval $[0,1]$ has a lattice structure with the usual order, so we have $\alpha \wedge \beta = \min\{\alpha, \beta\}$ for $\alpha, \beta \in [0,1]$.

Now the following is established. We call the theorem the *transfer principle.*

**Theorem 1. (Transfer principle)** *A fuzzy subset $\bar{A}$ satisfies property $\bar{\mathcal{P}}$ if and only if for any $\alpha \in [0,1]$ if $U(\bar{A} : \alpha) \neq \emptyset$ then the crisp set $U(\bar{A} : \alpha)$ satisfies the property $\mathcal{P}$.*

We simply denote this result by

$$\bar{A} : \bar{\mathcal{P}} \Longleftrightarrow \forall \alpha \ (\ U(\bar{A} : \alpha) \neq \emptyset \Longrightarrow U(\bar{A} : \alpha) : \mathcal{P}\ )$$

*Proof.* $\Rightarrow$: Suppose that $\bar{A}$ satisfies $\bar{\mathcal{P}}$. If there is $\alpha \in [0,1]$ such that $U(\bar{A} : \alpha) \neq \emptyset$ but $U(\bar{A} : \alpha)$ does not satisfy $\mathcal{P}$, then we have $t_1(a, \ldots, b) \in U(\bar{A} : \alpha)$, $\ldots$, $t_n(a, \ldots, b) \in U(\bar{A} : \alpha)$ but $t(a, \ldots, b) \notin U(\bar{A} : \alpha)$ for some $a, \ldots, b \in G$. Since $\bar{A}(t_1(a, \ldots, b)) \geqslant \alpha$, $\ldots$, $\bar{A}(t_n(a, \ldots, b)) \geqslant \alpha$ but $\bar{A}(t(a, \ldots, b)) \ngeqslant \alpha$, it follows that

$$\bar{A}(t(a, \ldots, b)) \ngeqslant \bar{A}(t_1(a, \ldots, b)) \wedge \ldots \wedge \bar{A}(t_n(a, \ldots, b))$$

for some $a, \ldots, b \in G$. This means that $\bar{A}$ does not satisfy $\bar{\mathcal{P}}$. This is a contradiction.

$\Leftarrow$: Suppose that $\bar{A}$ does not satisfy $\bar{\mathcal{P}}$. Since

$$\bar{A}(t(a, \ldots, b)) \ngeqslant \bar{A}(t_1(a, \ldots, b)) \wedge \ldots \wedge \bar{A}(t_n(a, \ldots, b))$$

for some $a, \ldots, b \in G$, if we put $\alpha = \bigwedge_i \bar{A}(t_i(a, \ldots, b))$ then we have $\alpha \in [0,1]$ and $U(\bar{A} : \alpha) \neq \emptyset$ because $t_i(a, \ldots, b) \in U(\bar{A} : \alpha)$. On the other hand, in this case we have $t(a, \ldots, b) \notin U(\bar{A} : \alpha)$. This means that $U(\bar{A} : \alpha)$ does not satisfy $\mathcal{P}$. $\square$

The above implies that if we define a fuzzy subset $\bar{A}$ to have the property $\bar{\mathcal{P}}$ whenever a crisp subset $A$ satisfies $\mathcal{P}$ then the transfer principle holds generally.

Conversely we can show that we have to define a fuzzy subset $\bar{A}$ as in the form $\bar{\mathcal{P}}$ if the transfer principle is too hold.

**Theorem 2.** *If the transfer principle holds for a subset $A$ with the property $\mathcal{P}$, then the fuzzy subset $\bar{A}$ has the property $\bar{\mathcal{P}}$.*

*Proof.* Suppose that the transfer principle holds for $A$ but the fuzzy subset $\bar{A}$ does not have the property $\bar{\mathcal{P}}$. Thus there exist $a, \ldots, b \in G$ such that

$$\bar{A}(t(a, \ldots, b)) \ngeqslant \bar{A}(t_1(a, \ldots, b)) \wedge \ldots \wedge \bar{A}(t_n(a, \ldots, b)).$$

We take $\alpha = \bigwedge_i \bar{A}(t_i(a,\dots,b))$. It is clear that $\alpha \in [0,1]$ and $U(\bar{A} : \alpha) \neq \emptyset$ because of

$$\bar{A}(t_i(a,\dots,b)) \geqslant \alpha.$$

Since $\bar{A}(t(a,\dots,b)) \ngeqslant \alpha$, we have $t(a,\dots,b) \notin U(\bar{A} : \alpha)$ but $t_i(a,\dots,b) \in U(\bar{A} : \alpha)$. This means that $U(\bar{A} : \alpha)$ does not satisfy the property $\mathcal{P}$. This is a contradiction. □

Since any concept defined for groups so far has the forms of $\mathcal{P}$ and the corresponding fuzzy subsets are defined by the form of $\bar{\mathcal{P}}$, it is easy to get the relation between a crisp set $A$ with $\mathcal{P}$ and its fuzzy set $\bar{A}$ with $\bar{\mathcal{P}}$. For example, a non-empty subset $A$ of $G$ is called a subgroup if $x \in A$ and $y \in A$ then $xy^{-1} \in A$. So if we define a fuzzy subgroup $\bar{A}$ as $\bar{A}(xy^{-1}) \geqslant \bar{A}(x) \wedge \bar{A}(y)$ then the transfer principle holds generally. Thus from the above we can verify the next theorem immediately. The theorem is proved already Theorem 3.9 in [3] under the restriction of finiteness of the group $G$. We note that this restriction is not needed to get the theorem.

**Theorem 3.** *$\bar{A}$ is a fuzzy* (*normal*) *subgroup of $G$ if and only if for all $\alpha$ $U(\bar{A} : \alpha) \neq \emptyset \to U(\bar{A} : \alpha)$ is a* (*normal*) *subgroup of $G$.*

Let $G$, $G'$ be groups and $f : G \to G'$ be a homomorphism, that is, $f(xy) = f(x)f(y)$. For any fuzzy subgroup $\bar{A}$ of $G'$, we define a map $f^{-1}(\bar{A})$ from $G$ to $[0,1]$ by

$$f^{-1}(\bar{A})(x) = \bar{A}(f(x))$$

for all $x \in G$. We call $f^{-1}(\bar{A})$ a *preimage* of fuzzy subgroup $\bar{A}$ under $f$. For any fuzzy subgroup $\bar{A}$ of $G$ we define an *image* $f[\bar{A}]$ of $\bar{A}$ under $f$ by

$$f[\bar{A}](y) = \bigvee_{u \in f^{-1}(y)} \bar{A}(u)$$

for all $y \in G'$. It follows from definition that

**Proposition 2.** *If $f : G \to G'$ is a homomorphism from $G$ to $G'$ and $\bar{A}$ is a fuzzy* (*normal*) *subgroup of $G'$, then the preimage $f^{-1}(\bar{A})$ is a fuzzy* (*normal*) *subgroup of $G$.*

Considering the image $f[\bar{A}]$ of the fuzzy (normal) subgroup $\bar{A}$ of $G$ under $f$, we have the following theorem by use of the transfer principle. The following result is proved in Proposition 4.2 [4] under the restriction of the sup property:

> *A homomorphic image of a fuzzy subgroupoid which has the sup property is a fuzzy subgroupoid.*

A fuzzy set $\bar{A}$ of $G$ has the *sup property* if for any subset $S \subseteq G$ there exists $s_0 \in S$ such that

$$\bar{A}(s_0) = \bigvee_{s \in S} \bar{A}(s).$$

We note here that the restriction is not essential, that is, we can prove the theorem without such a restriction. To prove the theorem we need a lemma. Let $G$, $G'$ be groups and $f : G \to G'$ be a homomorphism.

**Lemma 1.** *For all* $\alpha \in [0,1]$, $U(f[\bar{A}] : \alpha) = \bigcap_{\epsilon>0} f(U(\bar{A} : \alpha - \epsilon))$.

*Proof.*

$$\begin{aligned}
y \in U(f[\bar{A}] : \alpha)) &\Longleftrightarrow f[\bar{A}](y) \geqslant \alpha \\
&\Longleftrightarrow \bigvee_{u \in f^{-1}(y)} \bar{A}(u) \geqslant \alpha \\
&\Longleftrightarrow \forall \epsilon > 0 \ \exists u \in f^{-1}(y) \text{ s.t. } \bar{A}(u) \geqslant \alpha - \epsilon \\
&\Longleftrightarrow \forall \epsilon > 0 \ \exists u \in f^{-1}(y) \text{ s.t. } u \in U(\bar{A} : \alpha - \epsilon) \\
&\Longleftrightarrow \forall \epsilon > 0 \ y = f(u) \in f(U(\bar{A} : \alpha - \epsilon)) \\
&\Longleftrightarrow y \in \bigcap_{\epsilon>0} f(U(\bar{A} : \alpha - \epsilon))
\end{aligned}$$ □

Using this lemma we can prove the next theorem.

**Theorem 4.** *Let* $f : G \to G'$ *be a surjective homomorphism and* $\bar{A}$ *be a fuzzy* (*normal*) *subgroup of* $G$*, then the image* $f[\bar{A}]$ *is a* (*normal*) *subgroup of* $G'$.

*Proof.* It follows from the transfer principle that

> $f[\bar{A}]$ is a fuzzy (normal) subgroup of $G'$ if and only if $\forall \alpha$
> $U(f[\bar{A}] : \alpha) \neq \emptyset \to U(f[\bar{A}] : \alpha)$ is a (normal) subgroup of $G'$.

It is sufficient to show that $U(f[\bar{A}] : \alpha)$ is a (normal) subgroup of $G'$ if $U(f[\bar{A}] : \alpha) \neq \emptyset$. Since $\bar{A}$ is a fuzzy (normal) subgroup, it follows from the transfer principle that $U(\bar{A} : \alpha - \epsilon)$ is also a (normal) subgroup of $G$ for all $\epsilon > 0$ if $U(\bar{A} : \alpha - \epsilon) \neq \emptyset$. From $f$ being surjective, we have $f(U(\bar{A} : \alpha - \epsilon))$ is a (normal) subgroup of $G'$ for all $\epsilon > 0$ by well-known results about crisp group theory. Hence $\bigcap_{\epsilon>0} f(U(\bar{A} : \alpha - \epsilon)) = U(f[\bar{A}] : \alpha)$ is the (normal) subgroup of $G'$. □

It is well known that the class of all normal subgroups of $G$ forms a modular lattice. Similarly we may expect that the class $FNS(G)$ of all fuzzy normal subgroups of $G$ is a modular lattice. In the following, we shall show that $FNS(G)_\alpha$ forms a modular lattice for every $\alpha \in [0,1]$. Before doing, we define an order $\leqslant$ on $FNS(G)$ by

$$\bar{A} \leqslant \bar{B} \iff \bar{A}(x) \leqslant \bar{B}(x)$$

for all $x \in G$.

A fuzzy subset $\bar{A} \wedge \bar{B}$ of $G$ is defined by

$$(\bar{A} \wedge \bar{B})(x) = \bar{A}(x) \wedge \bar{B}(x)$$

for $x \in G$. It is obvious that $\bar{A} \wedge \bar{B} \in FNS(G)$ and

$$\bar{A} \wedge \bar{B} = \inf\{\bar{A}, \bar{B}\}$$

with respect to the order.

As for $\sup\{\bar{A}, \bar{B}\}$, we consider a fuzzy subset $\bar{A}\bar{B}$ defined by

$$(\bar{A}\bar{B})(x) = \bigvee_{ab=x} (\bar{A}(a) \wedge \bar{B}(b)).$$

For this fuzzy subset $\bar{A}\bar{B}$, it follows from definition that

$$(*) \qquad \bar{A}\bar{B}(x) = \bigvee_{a} \bar{A}(a) \wedge \bar{B}(a^{-1}x) = \bigvee_{b} \bar{A}(xb^{-1}) \wedge \bar{B}(b).$$

First of all we show that $\bar{A}\bar{B} \in FNS(G)$ for $\bar{A}, \bar{B} \in FNS(G)$ by using the transfer principle.

**Lemma 2.** *For $\bar{A}, \bar{B} \in FNS(G)$, we have $\bar{A}\bar{B} \in FNS(G)$*

*Proof.* By transfer principle we have that $\bar{A}\bar{B}$ is a fuzzy normal subgroup if and only if for all $\alpha$ $U(\bar{A}\bar{B} : \alpha) \neq \emptyset \to U(\bar{A}\bar{B} : \alpha)$ is a normal subgroup.

It is sufficient to show that $U(\bar{A}\bar{B} : \alpha)$ is a normal subgroup of $G$ provided that $U(\bar{A}\bar{B} : \alpha) \neq \emptyset$. Let $\alpha$ be such that $U(\bar{A}\bar{B} : \alpha) \neq \emptyset$ and $x \in U(\bar{A}\bar{B} : \alpha)$. For every $y \in G$ we have to show that $yxy^{-1} \in U(\bar{A}\bar{B} : \alpha)$.

Since

$$\begin{aligned}
\bar{A}\bar{B}(yxy^{-1}) &= \bigvee_u \bar{A}(u) \wedge \bar{B}(u^{-1}yxy^{-1}) \\
&= \bigvee_v \bar{A}(vy^{-1}) \wedge \bar{B}(yv^{-1}yxy^{-1}) \qquad \text{(by } (*)) \\
&= \bigvee_v \bar{A}(y^{-1}v) \wedge \bar{B}(v^{-1}yxy^{-1}y) \\
&= \bigvee_v \bar{A}(y^{-1}v) \wedge \bar{B}(v^{-1}yx) \\
&= \bigvee_v \bar{A}(y^{-1}v) \wedge \bar{B}((y^{-1}v)^{-1}x) \\
&= \bar{A}\bar{B}(x) \geqslant \alpha,
\end{aligned}$$

we have $yxy^{-1} \in U(\bar{A}\bar{B} : \alpha)$, that is, $U(\bar{A}\bar{B} : \alpha)$ is the normal subgroup of $G$ if $U(\bar{A}\bar{B} : \alpha) \neq \emptyset$. So we can conclude that $\bar{A}\bar{B} \in FNS(G)$. □

As to $\bar{A}\bar{B}$ we also have

**Lemma 3.** *$\bar{A}\bar{B} = \sup\{\bar{A}, \bar{B}\}$ for $\bar{A}, \bar{B} \in FNS(G)$ such that $\bar{A}(e) = \bar{B}(e)$.*

*Proof.* It is easy to show that $\bar{A}, \bar{B} \leqslant \bar{A}\bar{B}$. Let $\bar{C}$ be any fuzzy normal subgroup such that $\bar{A}, \bar{B} \leqslant \bar{C}$. For every $x \in G$, since

$$\begin{aligned}
\bar{A}\bar{B}(x) &= \bigvee_u \bar{A}(u) \wedge \bar{B}(u^{-1}x) \\
&\leqslant \bigvee_u \bar{C}(u) \wedge \bar{C}(u^{-1}x) \\
&\leqslant \bigvee_u \bar{C}(uu^{-1}x) = \bar{C}(x)
\end{aligned}$$

it follows that $\bar{A}\bar{B} = \sup\{\bar{A}, \bar{B}\}$ when $\bar{A}(e) = \bar{B}(e)$. □

By the above we denote $\sup\{\bar{A}, \bar{B}\}$ by $\bar{A} \vee \bar{B}$.

Let $\alpha$ be an element of $[0, 1]$ and $FNS_\alpha(G)$ be the class of all fuzzy normal subgroups of $G$ satisfying $\bar{A}(e) = \alpha$ for all $\bar{A} \in FNS(G)$, that is, $FNS_\alpha(G) = \{\bar{A} \in FNS(G) | \bar{A}(e) = \alpha\}$. We can prove that

**Theorem 5.** *$FNS_\alpha(G)$ forms a modular lattice for any $\alpha \in [0, 1]$.*

*Proof.* Let $\bar{A}, \bar{B}, \bar{C} \in FNS_\alpha(G)$ and $\bar{A} \leqslant \bar{C}$. It is sufficient to prove that

$(\bar{A} \vee \bar{B}) \wedge \bar{C} \leqslant \bar{A} \vee (\bar{B} \wedge \bar{C})$. For all $x \in G$, since

$$\begin{aligned}((\bar{A} \vee \bar{B}) \wedge \bar{C})(x) &= (\bigvee_u (\bar{A}(u) \wedge \bar{B}(u^{-1}x))) \wedge \bar{C}(x) \\ &= \bigvee_u (\bar{A}(u) \wedge \bar{B}(u^{-1}x) \wedge \bar{C}(x)) \\ &= \bigvee_u (\bar{A}(u) \wedge \bar{C}(u) \wedge \bar{B}(u^{-1}x) \wedge \bar{C}(x)) \\ &= \bigvee_u (\bar{A}(u) \wedge \bar{B}(u^{-1}x) \wedge \bar{C}(u) \wedge \bar{C}(x)) \\ &\leqslant \bigvee_u (\bar{A}(u) \wedge \bar{B}(u^{-1}x) \wedge \bar{C}(u^{-1}x)) \\ &= (\bar{A} \vee (\bar{B} \wedge \bar{C}))(x),\end{aligned}$$

it follows that if $\bar{A} \leqslant \bar{C}$ then $(\bar{A} \vee \bar{B}) \wedge \bar{C} \leqslant \bar{A} \vee (\bar{B} \wedge \bar{C})$. This means that $FNS_\alpha(G)$ is a modular lattice. □

## 4. Fuzzy congruences

We define a fuzzy congruence on an arbitrary group $G$. A binary function $\theta$ from $G \times G$ to $[0, 1]$ is called a *fuzzy congruence* on $G$ if for all elements $x, y, z, u \in G$ it satisfies the conditions:

1. $\bar{\theta}(e, e) = \bar{\theta}(x, x)$,
2. $\bar{\theta}(x, y) = \bar{\theta}(y, x)$,
3. $\bar{\theta}(x, z) \geqslant \bar{\theta}(x, y) \wedge \bar{\theta}(y, z)$,
4. $\bar{\theta}(xu, yu), \bar{\theta}(ux, uy) \geqslant \bar{\theta}(x, y)$.

**Lemma 4.** *If $\bar{\theta}$ satisfies the conditions* (2), (3) *and* (4) *above, then* (1) *is equivalent to* (1)′ $\bar{\theta}(e, e) \geqslant \bar{\theta}(x, y)$ *for all* $x, y \in G$.

*Proof.* Suppose that $\bar{\theta}(e, e) = \bar{\theta}(x, x)$. Since $\bar{\theta}$ satisfies the conditions (2) and (3), we have $\bar{\theta}(e, e) = \bar{\theta}(x, x) \geqslant \bar{\theta}(x, y) \wedge \bar{\theta}(y, x) = \bar{\theta}(x, y)$.

Conversely, from (4) we have $\bar{\theta}(e, e) \leqslant \bar{\theta}(xe, xe) = \bar{\theta}(x, x)$. □

**Proposition 3.** *If $\bar{\theta}$ is a fuzzy congruence on $G$, then $\bar{\theta}(x, y) = \bar{\theta}(xy^{-1}, e)$ for all $x, y \in G$.*

*Proof.* $\bar{\theta}(x, y) \leqslant \bar{\theta}(xy^{-1}, yy^{-1}) = \bar{\theta}(xy^{-1}, e) \leqslant \bar{\theta}(xy^{-1}y, ey) = \bar{\theta}(x, y)$. Hence $\bar{\theta}(x, y) = \bar{\theta}(xy^{-1}, e)$ □

For every element $a \in G$, we define a subset

$$a/\bar{\theta} = \{b \in G | \bar{\theta}(a, b) = \bar{\theta}(e, e)\}$$

of $G$ and $G/\bar{\theta} = \{a/\bar{\theta} | a \in G\}$. We also define an operator "$\cdot$" on the set $\{a/\bar{\theta} | a \in G\}$ by

$$a/\bar{\theta} \cdot b/\bar{\theta} = (ab)/\bar{\theta}.$$

This operator is well-defined. Because, if $a/\bar{\theta} = a'/\bar{\theta}$ and $b/\bar{\theta} = b'/\bar{\theta}$, then we have $\bar{\theta}(a, a') = \bar{\theta}(b, b') = \bar{\theta}(e, e)$. Since $\bar{\theta}(e, e) = \bar{\theta}(a, a') \leqslant \bar{\theta}(ab, a'b)$ and $\bar{\theta}(e, e) = \bar{\theta}(b, b') \leqslant \bar{\theta}(a'b, a'b')$, we have $\bar{\theta}(e, e) \leqslant \bar{\theta}(ab, a'b) \wedge \bar{\theta}(a'b, a'b') \leqslant \bar{\theta}(ab, a'b') \leqslant \bar{\theta}(e, e)$. This means that $\bar{\theta}(ab, a'b') = \bar{\theta}(e, e)$ and $ab/\bar{\theta} = a'b'/\bar{\theta}$ hence that the operator is well-defined. It is easy to show that $G/\bar{\theta}$ forms a group with respect to this operator. So we omit its proof.

**Proposition 4.** *If $\bar{\theta}$ is a fuzzy congruence on $G$, then $G/\bar{\theta}$ is a group with the unit element $e/\bar{\theta}$.*

**Proposition 5.** *If $\bar{A}$ is a fuzzy normal subgroup of $G$, then the fuzzy relation $\theta_{\bar{A}}(x, y)$ defined by $\theta_{\bar{A}}(x, y) = \bar{A}(xy^{-1})$ is a fuzzy congruence.*

*Proof.* We only show that $\theta_{\bar{A}}$ satisfies the conditions (3) and (4). For the case of (3), we have

$$\begin{aligned} \theta_{\bar{A}}(x, z) = \bar{A}(xz^{-1}) &= \bar{A}(xy^{-1}yz^{-1}) \\ &\geqslant \bar{A}(xy^{-1}) \wedge \bar{A}(yz^{-1}) = \theta_{\bar{A}}(x, y) \wedge \theta_{\bar{A}}(y, z) \end{aligned}$$

For the case of (4), it follows that

$$\begin{aligned} \bar{\theta}(xu, yu) &= \bar{A}((xu)(yu)^{-1}) = \bar{A}((xu)(u^{-1}y^{-1})) \\ &= \bar{A}(xy^{-1}) = \theta_{\bar{A}}(x, y) \end{aligned}$$

The case of $\theta_{\bar{A}}(ux, uy) \geqslant \theta_{\bar{A}}(x, y)$ is similar. □

Conversely,

**Proposition 6.** *If $\bar{\theta}$ is a fuzzy congruence, then the function $A_{\bar{\theta}}$ from $G$ to $[0, 1]$ defined by $A_{\bar{\theta}}(x) = \bar{\theta}(x, e)$ is a fuzzy normal subgroup of $G$.*

*Proof.* We have $A_{\bar{\theta}}(x) \wedge A_{\bar{\theta}}(y) = \bar{\theta}(x, e) \wedge \bar{\theta}(y, e) = \bar{\theta}(x, e) \wedge \bar{\theta}(e, y) \leqslant \bar{\theta}(x, y) = \bar{\theta}(xy^{-1}, e) = A_{\bar{\theta}}(xy^{-1})$. Thus $A_{\bar{\theta}}$ is a fuzzy subgroup of $G$.

Moreover,

$$\begin{aligned} A_{\bar{\theta}}(xy) = \bar{\theta}(xy, e) &\leqslant \bar{\theta}(xyy^{-1}, ey^{-1}) = \bar{\theta}(x, y^{-1}) \\ &\leqslant \bar{\theta}(x^{-1}x, x^{-1}y) = \bar{\theta}(e, x^{-1}y^{-1}) \\ &\leqslant \bar{\theta}(ey, x^{-1}y^{-1}y) = \bar{\theta}(y, x^{-1}) \\ &\leqslant \bar{\theta}(yx, x^{-1}x) = \bar{\theta}(yx, e) = A_{\bar{\theta}}(yx). \end{aligned}$$

Hence $A_{\bar{\theta}}$ is a fuzzy normal subgroup of $G$. □

It is natural to ask whether there is a one-to-one correspondence between the class $FNS(G)$ of all fuzzy normal subgroups of $G$ and the class $FCon(G)$ of all fuzzy congruences on $G$. We can answer the question "Yes". We see that both sets are (complete) lattices with respect to the order of set inclusion.

**Theorem 6.** *$FNS(G) \cong FCon(G)$ as lattices. In particular, $\bar{A} = A_{\theta_{\bar{A}}}$ and $\bar{\theta} = \theta_{A_{\bar{\theta}}}$.*

*Proof.* It is easy to see that the map $\xi : FNS(G) \to FCon(G)$ defined by $\xi(\bar{A}) = \theta_{\bar{A}}$ is an isomorphism. □

## 5. Homomorphism theorem

Since $\theta_{\bar{A}}$ is a fuzzy congruence when $\bar{A}$ is a fuzzy normal subgroup of $G$ , we conclude that $G/\theta_{\bar{A}}$ is a group. We denote the group simply by $G/\bar{A}$ and call it a *fuzzy quotient group* induced by a fuzzy normal subgroup $\bar{A}$.

Let $G$, $G'$ be groups and $f$ be a homomorphism from $G$ to $G'$. If $\bar{A}$ is a fuzzy normal subgroup of $G'$, then the map $f^{-1}(\bar{A})$ defined by

$$f^{-1}(\bar{A})(x) = \bar{A}(f(x))$$

for all $x \in G$ is a fuzzy normal subgroup of $G$ as proved above. Thus $G/f^{-1}(\bar{A})$ and $f(G)/\bar{A}$ are groups. In this case we can show the following result which is an extension of *homomorphism theorem*.

**Theorem 7.** *Let $G$, $G'$ be groups, $f$ a homomorphism, and $\bar{A}$ a fuzzy normal subgroup of $G'$. Then there is an isomorphism from $G/f^{-1}(\bar{A})$ onto $f(G)/\bar{A}$, that is,*

$$G/f^{-1}(\bar{A}) \cong f(G)/\bar{A}.$$

*Proof.* We define a map $h$ from $G/f^{-1}(\bar{A})$ to $f(G)/\bar{A}$ by

$$h(x/f^{-1}(\bar{A})) = f(x)/\bar{A}$$

for all $x \in G$. The map $h$ is well-defined. Because, if $x/f^{-1}(\bar{A}) = y/f^{-1}(\bar{A})$, since $f^{-1}(\bar{A})(xy^{-1}) = f^{-1}(\bar{A})(e)$ by definition, then we have $\bar{A}(f(x)(f(y))^{-1}) = \bar{A}(f(e)) = \bar{A}(e')$, where $e'$ is the unit element of $G'$, and thus $f(x)/\bar{A} = f(y)/\bar{A}$. This implies that $h$ is well-defined.

For injectiveness of $h$, we suppose that $h(x/f^{-1}(\bar{A})) = h(y/f^{-1}(\bar{A}))$, that is, $f(x)/\bar{A} = f(y)/\bar{A}$. It follows from definition that $\bar{A}(f(x)(f(y))^{-1}) = \bar{A}(e')$ and hence $f^{-1}(\bar{A})(xy^{-1}) = f^{-1}(\bar{A})(e)$. This means that $x/f^{-1}(\bar{A}) = y/f^{-1}(\bar{A})$. Therefore $h$ is injective.

It is easy to show that $h$ is a surjective homomorphism.

Thus we can conclude that $G/f^{-1}(\bar{A}) \cong f(G)/\bar{A}$. □

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