Quasigroups and Related Systems **11** (2004), 1 – 8

SQS–3–Groupoids with $\mathbf{q(x, x, y) = x}$

Magdi H. Armanious

Abstract

A new algebraic structure $(P; q)$ of a Steiner quadruple systems SQS $(P; B)$ called an SQS-3-groupoid with $q(x, x, y) = x$ (briefly: an SQS-3-quasigroup) is defined and some of its properties are described. Sloops are considered as derived algebras of SQS-skeins. Squags and also commutative loops of exponent 3 with $x(xy)^2 = y^2$ given in [7] are derived algebras of SQS-3-groupoids. The role of SQS-3-groupoids in the clarification of the connections between squags and commutative loops of exponent 3 is described.

1. Introduction

A *Steiner quadruple* (*triple*) *system* is a pair $(T; B)$, where T is a finite set and B is a collection of 4-subsets (3-subsets) called *blocks* of T such that every 3-subset (2-subset) of T is contained in exactly one block of B [4, 8]. Let SQS(m) denote a Steiner quadruple system (briefly: quadruple system) of cardinality m and STS(n) denote Steiner triple system (briefly: triple system) of cardinality n. It is will known that SQS(m) exists iff $m \equiv 2$ or $4 \pmod 6$ and STS(n) exists iff $m \equiv 1$ or $3 \pmod 6$ (cf. [4, 8]).

If we consider $T_x = T - \{x\}$ for any point $x \in T$ and delete this point from all blocks which contain it, then the obtaining system $(T_x; B(x))$ is a triple system, where $B(x) = \{\, b' = b - \{x\} : b \in B \text{ and } x \in b\}$. The system $(T_x; B(x))$ is called a *derived triple system* of $(P; B)$ [4, 7]. There are one-to-one correspondences between STSs and each of sloops and squags and also between SQSs and SQS-skeins [4, 11].

An SQS-*skein* is an algebra $\mathbf{T} = (T; q')$ with one fundamental ternary operation q' satisfying the identities:

2000 Mathematics Subject Classification: 05B30, 08A99, 05B07, 20N05
Keywords: Steiner quadruple system, distributive and medial quadruple system, SQS-3-groupoid, distributive and medial SQS-3-groupoid

$$q'(x,y,z) = q'(y,x,z) = q'(z,x,y),$$
$$q'(x,x,y) = y,$$
$$q'(x,y,q'(x,y,z)) = z.$$

A *sloop* (or a *Steiner loop*) is a commutative loop $(T;\cdot,1)$ satisfying the *Steiner identity* $x\cdot(x\cdot y) = y$. A *squag* (or a *Steiner quasigroup*) is an idempotent commutative quasigroup $(Q;\cdot)$ satisfying the Steiner identity. Note that sloops are derived algebras of SQS-skeins, while squags can't be considered as derived algebras of SQS-skeins.

Let q be a ternary operation of a nonempty finite set T, then the algebra $(T;q)$ will be called an SQS-3-*groupoid with* $q(x,x,y) = x$ (briefly: an SQS-3-*groupoid*), if the following identities are satisfied:

$$q(x,y,z) = q(x,z,y) = q(z,y,x),$$
$$q(x,x,y) = x,$$
$$q(x,y,q(x,y,z)) = z \quad if \quad x \neq y.$$

It is clear that q is a totally commutative idempotent ternary operation and satisfies the Steiner equation (the third equation). Moreover, this algebra is a 3-groupoid but not a 3-quasigroup, because the equation $q(a,a,x) = a$ has no unique solution [10]. Also, the operation q is commutative but not associative [10]. Similarly, it is idempotent, but it doesn't satisfy the generalized idempotent law, i.e. $q(x,x,y) \neq y$ for $y \neq x$.

This algebra does not seem to be a nice algebra because many algebraic constructions can't be made within this class. For example, if ρ is a congruence on an SQS-3-groupoid $(T;q)$, and if $(x,y) \in \rho$, then $(x,x) \in \rho$ and $(z,z) \in \rho$ for all $z \in T$. Hence $(q(x,x,z), q(x,y,z)) = (x, q(x,y,z)) \in \rho$ for all $z \in T$, i.e. $(x,w) \in \rho$ for all $w \in T$. This means that such algebra has no proper congruences.

An SQS is called *distributive* or *medial*, if the associated SQS-3-groupoid satisfies the distributive or the medial law for SQSs, respectively (or more precisely, if all derived squags of the associated SQS-3-groupoid are distributive or medial, respectively).

Let $(T;B)$ be an SQS, we define the ternary operation q_B on T putting

$$q_B(x,y,z) = \begin{cases} w & if \quad \{x,y,z,w\} \in B \\ x & if \quad x = y \;\; or \;\; x = z \\ y & if \quad y = z \end{cases}$$

Obviously such defined $(T;q_B)$ is an SQS-3-groupoid.

Conversely, let $(T;q)$ be an SQS-3-groupoid. Consider the set:

$$B_q := \{\, \{x,y,z,q(x,y,z)\} \;:\; \textit{for all} \;\; \{x,y,z\} \subseteq T \;\; \textit{with} \;\; |\{x,y,z\}| = 3\}.$$

It is clear that $q(x,y,z) \notin \{x,y,z\}$, otherwise if $q(x,y,z) = x$, then $q(x,y,q(x,y,z)) = q(x,y,x) = x$ and from the Steiner equation we have $q(x,y,q(x,y,z)) = z$, which is a contradiction. Since $|\{x,y,z\}| = 3$, hence $|\{x,y,z,q(x,y,z)\}| = 4$. Since q is commutative, then $(T;B_q)$ is an SQS. Moreover one can deduce that $q_{B_q} = q$ and $B_{q_B} = B$.

This proves that there is a one-to-one correspondence between SQS's and SQS-3-groupoids.

Carmichael [3] and Lüneburg [9] constructed an SQS(3^n+1). In section 2, we prove that the associated SQS-3-groupoids with this construction of SQS(3^n+1) satisfies the medial law for SQSs. Next, in section 3, we prove that a commutative loop of exponent 3 satisfying the identity $x(xy)^2 = y^2$ (i.e. an interior Steiner loop [2]) is also a derived algebra of the SQS-3-groupoid.

2. Medial SQS-3-groupoids

For any element $a \in T$, we define the *derived algebra* $(T_a;\circ)$ of the SQS-3-groupoid $(T;q)$ putting: $T_a = T - \{a\}$ and $x \circ y = q(a,x,y)$ for all $x,y \in T_a$. Since

$$x \circ x = x$$

$$x \circ y = y \circ x$$

$$x \circ (x \circ y) = y,$$

the derived algebra $(T_a;\circ)$ is the well-known squag.

The class of SQS-3-groupoids is not variety, but the class of all derived algebras forms the well-known variety of squags.

The interesting subclass of squags forms medial squags which are squags satisfying the medial identity:

$$(x \circ y) \circ (z \circ w) = (x \circ z) \circ (y \circ w).$$

The finite medial squags correspond to the class of affine geometries over GF(3) (see Klossek [7] and Guelzow [5]). Medial squags are derived algebras of the subclass of so-called *medial* SQS-3-groupoids, i.e. SQS-3-groupoids satisfying the following medial law for SQSs:

$$q(a, q(a,x,y), q(a,z,w)) = q(a, q(a,x,z), q(a,y,w)).$$

Associated SQSs are called *medial*.

The smallest nontrivial medial SQS-3-groupoid is the associated SQS-3-groupoid of the quadruple system SQS(10).

Example 1. Let $(T(10); q_B)$ be the SQS-3-groupoid associated with the quadruple system of order 10. Any derived squag $(T_a; \circ)$ of $(T(10); q_B)$ is of order 9 and is associated with the triple system STS(9). The STS(9) is isomorphic to the affine plane over GF(3) (cf. [5, 7]) and then the squag $(T_a; \circ)$ satisfies the medial identity. This implies $(T(10); q_B)$ satisfies the medial law for SQSs. □

Carmichael [3] and Lüneburg [9] constructed an $\mathrm{SQS}(3^n+1) = (K^*; B^*)$ having a sharp triply transitive automorphism group Γ^*. Namely, K^* and B^* are defined by:

$$K^* = GF(3^n) \cup \{\infty\}$$

and

$$B^* = \{\Psi(B) \,:\, B = \{0, 1, -1, \infty\} \text{ and } \Psi \in \Gamma^*\},$$

where

$$\Gamma^* = \{\Psi \,:\, K^* \to K^* \,:\, \Psi(x) = (ax + b)/(cx + d),\ \ ad - bc \neq 0\}.$$

The following theorem is given in [9], helps us to show that the construction $(K^*; B^*)$ supplies us with an example of a medial SQS-3-groupoid of cardinality $3^n + 1$ for each positive integer n.

Theorem 1. *If $(K; B)$ is a triple system with a sharp doubly transitive automorphism group Γ^*, then $(K; B)$ is an affine plane over $GF(3)$.*

Example 2. For any $p \in K^*$, we have the derived STS $(K_p^*; B_p^*)$ of $(K^*; B^*)$ with the automorphism group Γ_p^* defined by:

$$\Gamma_P^* = \{\Psi \,:\, K_P^* \to K_P^* \,:\, \Psi \in \Gamma^*,\ \ \Psi(p) = p\},$$

where Γ_p^* is a sharp doubly transitive automorphism group of $(K_p^*; B_p^*)$.

According to the above theorem, the triple system $(K_p^*; B_p^*)$ is an affine plane over GF(3). This means that the squag $(K_p^*; \circ)$ associated with $(K_p^*; B_p^*)$ satisfies the medial identity. The medial law for SQSs is satisfied, too.

According to the above discussion and the construction $(K^*; B^*) = SQS(3^n + 1)$ given by Carmichael [3] and Lüneburg [9], we can say that any finite medial squag is a derived algebra from a medial SQS-3-groupoid.

In other words, we can say that there are quadruple systems in which the associated squag of any its derived triple systems is medial.

Now, we consider the question about the existence of an quadruple systems in which the squag associated with each derived STS is distributive, i. e. each derived squag of an SQS-3-groupoid satisfies the distributive law:

$$x \circ (y \circ z) = (x \circ y) \circ (x \circ z).$$

In other word, *is there a non-medial SQS-3-groupoid satisfying the distributive law for SQSs*

$$q(a, x, q(a, y, z)) = q(a, q(a, x, y), q(a, x, z)) \ \ ?$$

M. Hall [6] constructed an STS (called now a *Hall triple system*) in which each three elements generate the affine plane over GF(3) . The smallest cardinality for a Hall STS is 81. The associated squags of Hall triple systems are distributive. A vector-space model of a distributive squag of cardinalty 3^m is given by Klossek [7].

As a special case, for $m = 4$, we obtain the smallest non-medial distributive squag $(\mathrm{GF}(3)^4; \circ)$, where the binary operation $\circ$ is defined by:

$$x \circ y = 2x + 2y + \left(0, 0, 0, \begin{vmatrix} x_2 & x_3 \\ y_2 & y_3 \end{vmatrix} (x_1 - y_1)\right).$$

The above question can be formulated in the following way:

Is there SQS-3-groupoid of cardinality 82 all of whose derived squags are distributive but non-medial ?

or in the combinatorial language:

Is there SQS(82) all of whose derived STS(81)s are Hall STSs but are not isomorphic to the direct power STS(3)^4 ?

3. SQS-3-Groupoids and commutative loops

A commutative loop $(L; \cdot, e)$ of exponent 3 is called *Moufang*, if it satisfies the *Moufang identity*:

$$x \cdot (x \cdot (y \cdot z)) = (x \cdot y) \cdot (x \cdot z).$$

Commutative Moufang loops of exponent 3 and distributive squags are polynomially equivalent [7]. There is a one-to-one correspondence between triple systems and commutative loops of exponent 3 with $x(xy)^2 = y^2$ (cf. [1]). Therefore, as it is proved in [1], commutative loops of exponent 3 with $x(xy)^2 = y^2$ are polynomially equivalent to squags.

Moreover, any commutative loop of exponent 3 with $x(xy)^2 = y^2$ is a derived algebra of the constructed SQS-3-groupoid.

Indeed, let $(T; q)$ be an SQS-3-groupoid and $a \in T$. Then $(T - \{a\}; \circ)$ is a squag, where $\circ$ is defined by the formula

$$x \circ y = q(a, x, y).$$

Fixing $e \in T - \{a\}$ and putting

$$x \cdot y = q(a, e, q(a, x, y)),$$

we can see that $(T - \{a\}; \cdot, e)$ is a commutative loop of exponent 3 with $x \cdot (x \cdot y)^2 = y^2$.

Moreover, for SQS-3-groupoids, we have

$$x \cdot y = q(a, e, q(a, x, y) = e \circ (x \circ y)$$

and

$$\begin{aligned} x \circ y &= q(a, x, y) = q(a, q(a, e, x^2), q(a, e, y^2)) \\ &= q(a, e, q(a, x^2, y^2)) = x^2 \cdot y^2 \end{aligned}$$

for distributive SQS-3-groupoids.

Thus, using results from [7] and [1], we can see that

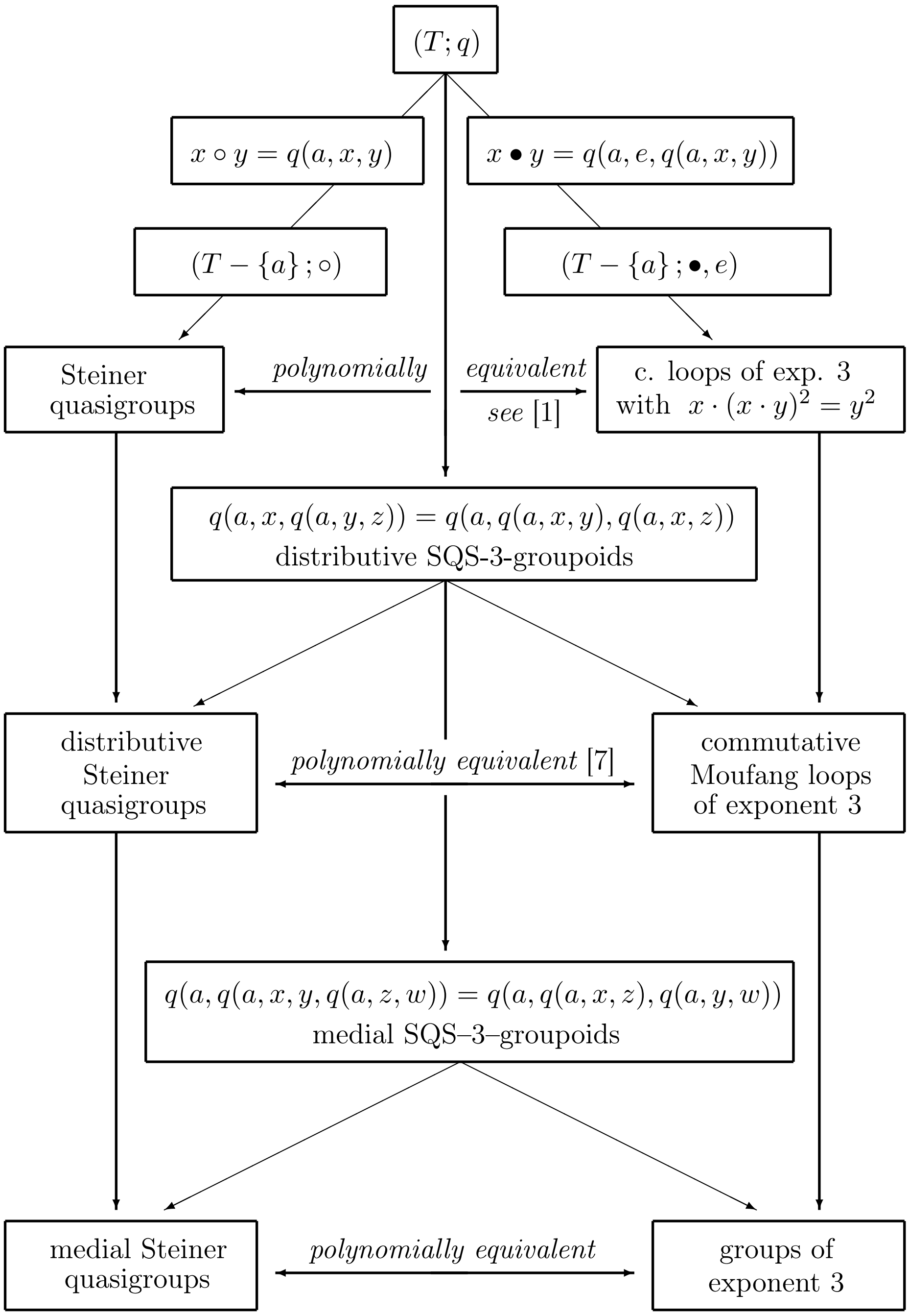
(i) if $(T; q)$ is a distributive SQS-3-groupoid, then for each $a \in T$ and each $e \in T - \{a\}$, the squag $(T - \{a\}; \circ)$ is distributive and $(T - \{a\}; \cdot, e)$ is a commutative Moufang loop of exponent 3,

(ii) if $(T; q)$ is a medial SQS-3-groupoid, then for all $a \in T$ and $e \in T - \{a\}$, the squag $(T - \{a\}; \circ)$ is medial and the loop $(T - \{a\}; \cdot, e)$ is a commutative group of exponent 3.

This, according to results obtained in [7] and [1], means that the distributive squag $(T - \{a\}; \circ)$ is polynomially equivalent to the commutative Moufang loop $(T - \{a\}; \cdot, e)$ of exponent 3 with $x \cdot (x \cdot y)^2 = y^2$.

In the next page, we give the diagram presenting some connections between different types of algebras derived from SQS-3-groupoids.

SQS–3–groupoids with $\mathbf{q(x,x,y)=x}$



References

[1] **M. H. Armanious**: *Commutative loops of exponent* 3 *with* $x(xy)^2 = y^2$, Demonstratio Math. **35** (2002), 469 – 475.

[2] **O. Chein, H. O. Pflugfelder, J. D. H. Smith**: *Quasigroups and loops. Theory and applications*, Sigma Series in Pure Math., 8, Helderman Verlag, Berlin, 1990.

[3] **R. D. Carmichael**: *Introduction to the theory of groups of finite order*, New York, Dover Publ. 1956.

[4] **B. Ganter and H. Werner**: *Co-ordinatizing Steiner systems*, Annals Discrete Math. **7** (1980), 3 – 24.

[5] **A. J. Guelzow**: *Representation of finite nilpotent squags*, Discrete Math. **154** (1996), 63 – 76.

[6] **M. Jr. Hall**: *Automorphism of Steiner triple systems*, IBM Journal **5** (1960), 460 – 472.

[7] **S. Klossek**: *Kommutative Spiegelurgsraüme*, Mitteilungen Math. Semin. Giessen **117** (1975).

[8] **C. C. Lindner and A. Rosa**: *Steiner quadruple systems, a survey*, Discrete Math. **21** (1979), 147 – 181.

[9] **H. Lüneburg**: *Fahnenhomogene Quadrupelsysteme*, Math. Zeitschr. **89** (1965), 82 – 90.

[10] **J. Timm**: *Kommutative n-Gruppen*, Ph. D. Thises, Hamburg 1967.

[11] **R. W. Quackenbush**: *Varieties of Steiner loops and Steiner quasigroups*, Canadian J. Math. **28** (1978), 1187 – 1198.

Department of Mathematics
Faculty of Science
Mansoura University
Mansoura
Egypt
E-mail: m.armanious@excite.com

Received January 21, 2003