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A COMMON FORM FOR AUTOTOPIES OF n-ARY GROUP WITH THE INVERSE PROPERTY

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Abstract

In this article it is proved that every component of an autotopy of n-IP-group is its quasiautomorphism and a common form of quasiautomorphisms and autotopies of such groups is also established.

A quasigroup $Q(A)$ of arity n is called a n-group [1] if the following identities

$$A(x_1^{i-1}, A(x_i^{i+n-1}), x_{i+1}^{2n-1}) = A(x_1^{j-1}, A(x_j^{j+n-1}), x_{j+1}^{2n-1})$$

hold in it for all $x_1^{2n-1} \in Q^{2n-1}$ and all $i, j \in \overline{1,n}$, $i \neq j$.

There exist n-ary groups without an identity element and n-ary groups with more than one identity elements [1].

A group $Q(A)$ of arity n is called *symmetric* [1] if

$$A(x_{\alpha 1}^{\alpha n}) = A(x_1^n)$$

for every $x_1^n \in Q^n$ and every $\alpha \in S_n$, where S_n is the symmetric group of the degree n.

According to the **Gluskin-Hosszu theorem** [1] each n-group $Q(A)$ is reduced to a binary group $Q(\cdot)$:

$$A(x_1^n) = x_1 \cdot \varphi x_2 \cdot \varphi^2 x_3 \cdot \ldots \cdot \varphi^{n-1} x_n \cdot k,$$

where φ is an automorphism of $Q(\cdot)$, k is a fixed element of Q and

$$\varphi k = k, \quad \varphi^{n-1} x = k \cdot x \cdot k^{-1}.$$

If $Q(A)$ is a symmetric n-group without an identity element, then

$$A(x_1^n) = x_1 \cdot x_2 \cdot x_3 \cdot \ldots \cdot x_n \cdot k. \tag{1}$$

If $Q(A)$ is a symmetric n-group with an identity element, then

$$A(x_1^n) = x_1 \cdot x_2 \cdot x_3 \cdot \ldots \cdot x_n, \tag{2}$$

where $Q(\cdot)$ is an abelian group.

A quasigroup $Q(A)$ of arity n is called a *quasigroup with the inverse property* (briefly a *n-IP-quasigroup*) [1] if there exist such substitutions ν_{ij} of Q, $i,j \in \overline{1,n}$, $\nu_{ii} = \varepsilon$ (ε is the identity substitution) that the identities

$$A(\{\nu_{ij}x_j\}_{j=1}^{i-1}, A(x_1^n), \{\nu_{ij}x_j\}_{j=i+1}^{n}) = x_i$$

hold for every $x_1^n \in Q^n$, $i \in \overline{1,n}$.

The matrix

$$(\nu_{ij}) = \begin{pmatrix} \varepsilon & \nu_{12} & \nu_{13} & \dots & \nu_{1n} & \varepsilon \\ \nu_{21} & \varepsilon & \nu_{23} & \dots & \nu_{2n} & \varepsilon \\ \nu_{31} & \nu_{32} & \varepsilon & \dots & \nu_{3n} & \varepsilon \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \nu_{n1} & \nu_{n2} & \nu_{n3} & \dots & \varepsilon & \varepsilon \end{pmatrix}$$

is called an *inverse matrix* of $Q(A)$.

It is known [2], that the inverse property holds only in the following n-groups:

a) all symmetric n-groups with an identity element. For such a group the inverse matrix is

$$(O) = \begin{pmatrix} \varepsilon & I & I & \dots & I & \varepsilon \\ I & \varepsilon & I & \dots & I & \varepsilon \\ I & I & \varepsilon & \dots & I & \varepsilon \\ \dots & \dots & \dots & \dots & \dots & \dots \\ I & I & I & \dots & \varepsilon & \varepsilon \end{pmatrix}; \qquad (3)$$

b) all symmetric n-groups without an identity element. For them

$$(O) = \begin{pmatrix} \varepsilon & IL_k & IL_k & I & I & \dots & I & \varepsilon \\ IL_k & \varepsilon & IL_k & I & I & \dots & I & \varepsilon \\ IL_k & IL_k & \varepsilon & I & I & \dots & I & \varepsilon \\ IL_k & IL_k & I & \varepsilon & I & \dots & I & \varepsilon \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ IL_k & IL_k & I & I & I & \dots & \varepsilon & \varepsilon \end{pmatrix}; \qquad (4)$$

c) all nonsymmetric n-groups without an identity element which are reduced to binary abelian groups. For such n-groups

$$\varphi^2 = \varepsilon, \quad k^2 = e,$$

where e is the identity element of $Q(\cdot)$, i.e.

$$A(x_1^n) = x_1 \cdot \varphi x_2 \cdot x_3 \cdot \varphi x_4 \cdot \ldots \cdot \varphi x_{n-1} \cdot x_n \cdot k. \tag{5}$$

In this case $Q(A)$ has an odd arity and

$$(O) = \begin{pmatrix} \varepsilon & I & I & I & \ldots & I & \varepsilon \\ I\varphi & \varepsilon & I\varphi & I\varphi & \ldots & I\varphi & \varepsilon \\ I & I & \varepsilon & I & \ldots & I & \varepsilon \\ I\varphi & I\varphi & I\varphi & \varepsilon & \ldots & I\varphi & \varepsilon \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ I & I & I & I & \ldots & \varepsilon & \varepsilon \end{pmatrix}; \tag{6}$$

An ordered $(n+1)$-tuple $T = (\alpha_1^{n+1})$ of subtitutions of Q is called an *autotopy of a n-group $Q(A)$* if

$$\alpha_{n+1}^{-1} A(\{\alpha_i x_i\}_{i=1}^n) = A(x_i^n).$$

In particular, $(\overset{n+1}{\alpha}) = \alpha$ is called an *automorphism* of $Q(A)$.

The set of all autotopies of $Q(A)$ forms a group with respect to the multiplication of substitutions. This group is denoted by $\mathfrak{I}_A$.

The chief component α_{n+1} of an autotopy $T = (\alpha_1^{n+1})$ of a n-group is called a *quasiautomorphism* of this group [1].

All quasiautomorphisms of a n-group form the group [1].

In this article it is proved that every component of an autotopy of n-IP-group is its quasiautomorphism and a common form of quasiautomorphisms and autotopies of such groups is also established.

Let $T = (\alpha_1^n, \delta)$ be an autotopy of a nonsymmetric n-IP-group $Q(A)$:

$$\delta A(x_1^n) = A(\{\alpha_i x_i\}_{i=1}^n).$$

Denote

$$(\bar{a}) = A(\{\alpha_i e\}_{i=1}^n).$$

Then, according to (5),

$$\delta A(x_1^n) = \delta(x_1 \cdot \varphi x_2 \cdot x_3 \cdot \varphi x_4 \cdot \ldots \cdot \varphi x_{n-1} \cdot x_n \cdot k) = A(\{\alpha_i x_i\}_{i=1}^n).$$

By

$$x_1 = x_2 = \ldots = x_{i-1} = x_{i+1} = \ldots = x_n = e$$

we obtain

$$\delta R_k x_i = L_i(\bar{a})\alpha_i x_i,$$

whence

$$\alpha_i = L_i^{-1}(\bar{a})\delta R_k \tag{7}$$

for each odd i, and

$$\alpha_i = L_i^{-1}(\bar{a})\delta R_k \varphi \tag{8}$$

for each even i, $i \in \overline{1,n}$, where $R_k x = x \cdot k$.

Thus,

$$T = (L_1^{-1}(\bar{a}), L_2^{-1}(\bar{a})\delta\varphi\delta^{-1}, L_3^{-1}(\bar{a}), L_4^{-1}(\bar{a})\delta\varphi\delta^{-1}, \ldots, \delta R_k^{-1}\delta^{-1})\delta R_k, \tag{9}$$

since

$$R_k\varphi = \varphi R_k: \quad R_k\varphi x = \varphi x \cdot k = \varphi(x \cdot \varphi k) = \varphi(x \cdot k) = \varphi R_k x.$$

If $Q(A)$ is a symmetric n-group without an identity element, according to (1) we put $\varphi = \varepsilon$ in (9). In the case when $Q(A)$ is a symmetric n-group with an identity element we put $\varphi = R_k = \varepsilon$ in (9) according to (2).

Lemma 1. *All components of an autotopy $T = (\alpha_1^n, \delta)$ of a n-IP-group $Q(A)$ are quasiautomorphisms if $L_i(\bar{a})$, φ, R_k, $(i \in \overline{1,n})$, are quasiautomorphisms of $Q(A)$.*

The **proof** follows from (7), (8) and the fact that the set of all quasiautomorphisms of $Q(A)$ forms a group.

Proposition. *All components of any autotopy of a n-group $Q(A)$ of the form*

$$A(x_1^n) = x_1 \cdot \varphi x_2 \cdot x_3 \cdot \varphi x_4 \cdot \ldots \cdot \varphi x_{n-1} \cdot x_n \cdot k,$$

where $Q(\cdot)$ is an abelian group, are its quasiautomorphisms.

Proof. We have for each odd i:

$$L_i(\bar{a})A(x_1^n) = A(\{\alpha_j e\}_{j=1}^{i-1}, A(x_1^n), \{\alpha_j e\}_{j=i+1}^n) =$$
$$= \alpha_1 e \cdot \varphi\alpha_2 e \cdot \alpha_3 e \cdot \varphi\alpha_4 e \cdot \ldots \cdot \varphi\alpha_{i-1} e \cdot x_1 \cdot \varphi x_2 \cdot x_3 \cdot \varphi x_4 \cdot \ldots \cdot \varphi x_{i-1} \cdot$$
$$\cdot x_i \cdot \varphi x_{i+1} \cdot x_{i+2} \cdot \ldots \cdot \varphi x_{n-1} \cdot x_n \cdot k \cdot \varphi\alpha_{i+1} e \cdot \alpha_{i+2} e \cdot \ldots \cdot \varphi\alpha_{n-1} e \cdot \alpha_n e \cdot k =$$
$$= x_1 \cdot \varphi x_2 \cdot x_3 \cdot \varphi x_4 \cdot \ldots \cdot \varphi x_{i-1} \cdot (\alpha_1 e \cdot \varphi\alpha_2 e \cdot \alpha_3 e \cdot \varphi\alpha_4 e \cdot \ldots \cdot \varphi\alpha_{i-1} e \cdot x_i \cdot$$
$$\cdot \varphi\alpha_{i+1} e \cdot \alpha_{i+2} e \cdot \ldots \cdot \varphi\alpha_{n-1} e \cdot \alpha_n e \cdot k) \cdot \varphi x_{i+1} \cdot x_{i+2} \cdot \ldots \cdot \varphi x_{n-1} \cdot x_n \cdot k =$$
$$= A(x_1^{i-1}, A(\{\alpha_j e\}_{j=1}^{i-1}, x_i, \{\alpha_j e\}_{i+1}^n), x_{i+1}^n) = A(x_1^{i-1}, L_i(\bar{a})x_i, x_{i+1}^n),$$

i.e. $L_i(\bar{a})$ is a quasiautomorphism of $Q(A)$.

For each even i

$$L_i(\bar{a})A(x_1^n) = A(\{\alpha_j e\}_{j=1}^{i-1}, A(x_1^n), \{\alpha_j e\}_{j=i+1}^n) =$$
$$= \alpha_1 e \cdot \varphi\alpha_2 e \cdot \alpha_3 e \cdot \varphi\alpha_4 e \cdot \ldots \cdot \varphi\alpha_{i-1} e \cdot \varphi x_1 \cdot x_2 \cdot \varphi x_3 \cdot x_4 \cdot \ldots \cdot \varphi x_{i-1} \cdot$$
$$\cdot x_i \cdot \varphi x_{i+1} \cdot \ldots \cdot x_{n-1} \cdot \varphi x_n \cdot k \cdot \alpha_{i+1} e \cdot \varphi\alpha_{i+2} e \cdot \ldots \cdot \varphi\alpha_{n-1} e \cdot \alpha_n e \cdot k =$$
$$= \varphi x_1 \cdot x_2 \cdot \varphi x_3 \cdot x_4 \cdot \ldots \cdot x_{i-2} \cdot (\alpha_1 e \cdot \varphi\alpha_2 e \cdot \alpha_3 e \cdot \varphi\alpha_4 e \cdot \ldots \cdot \alpha_{i-1} e \cdot \varphi x_{i-1} \cdot$$
$$\cdot \alpha_{i+1} e \cdot \varphi\alpha_{i+2} e \cdot \ldots \cdot \varphi\alpha_{n-1} e \cdot \alpha_n e \cdot k) \cdot x_i \cdot \varphi x_{i+1} \cdot x_{i+2} \cdot \ldots \cdot x_{n-1} \cdot \varphi x_n \cdot k =$$
$$= A(\{\varphi x_j\}_{j=1}^{i-2}, L_i(\bar{a}) x_{i-1}, \{\varphi x_j\}_{j=i}^n),$$

i.e. $L_i(\bar{a})$ is a quasiautomorphism of $Q(A)$.

We also have

$$R_k A(x_1^n) = R_k(x_1 \cdot \varphi x_2 \cdot x_3 \cdot \varphi x_4 \cdot \ldots \cdot \varphi x_{n-1} \cdot x_n \cdot k) =$$
$$= x_1 \cdot \varphi x_2 \cdot x_3 \cdot \varphi x_4 \cdot \ldots \cdot \varphi x_{n-1} \cdot R_k x_n \cdot k = A(x_1^{n-1}, R_k x_n),$$

i.e. $(\overset{n-1}{\varepsilon}, R_k, R_k) \in \Im_A$.

Note that φ is an automorphism of $Q(A)$. Indeed,

$$\varphi A(x_1^n) = \varphi(x_1 \cdot \varphi x_2 \cdot x_3 \cdot \varphi x_4 \cdot \ldots \cdot \varphi x_{n-1} \cdot x_n \cdot k) =$$
$$= \varphi x_1 \cdot x_2 \cdot \varphi x_3 \cdot x_4 \cdot \ldots \cdot x_{n-1} \cdot \varphi x_n \cdot k = A(\{\varphi x_i\}_{i=1}^n).$$

If $Q(A)$ is a symmetric n-group without an identity element, then in the proof we put $\varphi = \varepsilon$ according (1). If $Q(A)$ is a symmetric n-group with an identity element, then put $\varphi = R_k = \varepsilon$. Using **Lemma 1** we complete the proof.

Lemma 2. *Each quasiautomorphism δ of a n-IP-group $Q(A)$*

$$A(x_1^n) = x_1 \cdot \varphi x_2 \cdot x_3 \cdot \varphi x_4 \cdot \ldots \cdot \varphi x_{n-1} \cdot x_n \cdot k$$

has the form

$$\delta = R_{\delta k} \theta_0 R_k^{-1}, \tag{10}$$

where θ_0 is some automorphism of $Q(\cdot)$.

Proof. Let $T = (\alpha_1^n, \delta)$ be an autotopy of a nonsymmetric n-IP-group $Q(A)$ without an identity element, which is reduced to a binary abelian group $Q(\cdot)$:

$$A(x_1^n) = x_1 \cdot \varphi x_2 \cdot x_3 \cdot \varphi x_4 \cdot \ldots \cdot \varphi x_{n-1} \cdot x_n \cdot k.$$

Transforming (7) and (8) we receive

$$\alpha_i = L_i^{-1}(\bar{a}) \delta R_k = L_i(\overline{Ia}) \delta R_k$$

for each odd i and

$$\alpha_i = L_i^{-1}(\bar{a}) \delta R_k \varphi = L_i(\overline{I\varphi a}) \delta R_k \varphi$$

for each even i, where

$$(\overline{Ia}) = A(\{I\alpha_i e\}_{i=1}^n),$$

$$(\overline{I\varphi a}) = A(\{I\varphi\alpha_i e\}_{i=1}^n).$$

Indeed, according to (6), by odd i it follows from

$$A(\{I\alpha_j e\}_{j=1}^{i-1}, A(\{\alpha_j e\}_{j=1}^{i-1}, x, \{\alpha_j e\}_{j=i+1}^n), \{I\alpha_j e\}_{j=i+1}^n) = x$$

that

$$L_i(\overline{Ia})L_i(\bar{a})x = x, \quad L_i^{-1}(\bar{a})x = L_i(\overline{Ia})x.$$

By even i from

$$A(\{I\varphi\alpha_j e\}_{j=1}^{i-1}, A(\{\alpha_j e\}_{j=1}^{i-1}, x, \{\alpha_j e\}_{j=i+1}^n), \{I\varphi\alpha_j e\}_{j=i+1}^n) = x$$

it follows that

$$L_i(\overline{I\varphi a})L_i(\bar{a})x = x, \quad L_i^{-1}(\bar{a})x = L_i(\overline{I\varphi a})x.$$

Hence,

$$T = (L_1(\overline{Ia})\delta R_k, L_2(\overline{I\varphi a})\delta R_k\varphi, L_3(\overline{Ia})\delta R_k, L_4(\overline{I\varphi a})\delta R_k\varphi, \ldots, L_n(\overline{Ia})\delta R_k, \delta),$$

i.e.

$$\begin{aligned}\delta A(x_1^n) &= A(L_1(\overline{Ia})\delta R_k x_1, L_2(\overline{I\varphi a})\delta R_k\varphi x_2, L_3(\overline{Ia})\delta R_k x_3, \ldots, L_n(\overline{Ia})\delta R_k x_n) = \\ &= A(A(\delta R_k x_1, I\alpha_2 e, \ldots, I\alpha_n e), A(I\varphi\alpha_1 e, \delta R_k\varphi x_2, I\varphi\alpha_3 e, \ldots, I\varphi\alpha_n e), \\ &\ldots, A(I\alpha_1 e, \ldots, I\alpha_{n-1}e, \delta R_k x_n)) = \delta R_k x_1 \cdot I\varphi\alpha_2 e \cdot \ldots \cdot I\alpha_n e \cdot k \cdot \\ &\cdot \varphi(I\varphi\alpha_1 e \cdot \varphi\delta R_k\varphi x_2 \cdot I\varphi\alpha_3 e \cdot I\alpha_4 e \cdot \ldots \cdot I\varphi\alpha_n e \cdot k) \cdot \ldots \cdot I\alpha_1 e \cdot I\varphi\alpha_2 e \cdot \ldots \cdot \delta R_k x_n \cdot k \cdot k = \\ &= \delta R_k x_1 \cdot I\alpha_1 e \cdot I\varphi\alpha_2 e \cdot I\alpha_3 e \cdot I\varphi\alpha_4 e \cdot \ldots \cdot I\alpha_n e \cdot k \cdot \delta R_k\varphi x_2 \cdot I\alpha_1 e \cdot I\varphi\alpha_2 e \cdot I\alpha_3 e \cdot I\varphi\alpha_4 e \cdot \ldots \cdot I\alpha_n e \cdot \\ &\cdot k \cdot \ldots \cdot \delta R_k\varphi x_{n-1} \cdot I\alpha_1 e \cdot I\varphi\alpha_2 e \cdot I\alpha_3 e \cdot I\varphi\alpha_4 e \cdot \ldots \cdot I\alpha_n e \cdot k \cdot \delta R_k x_n,\end{aligned}$$

since $I\varphi x = \varphi I x$. But

$$\delta A(x_1^n) = \delta(x_1 \cdot \varphi x_2 \cdot x_3 \cdot \varphi x_4 \cdot \ldots \cdot x_n \cdot k) = \delta R_k(x_1 \cdot \varphi x_2 \cdot x_3 \cdot \varphi x_4 \cdot \ldots \cdot x_n),$$

and

$$I\alpha_1 e \cdot I\varphi\alpha_2 e \cdot I\alpha_3 e \cdot I\varphi\alpha_4 e \cdot \ldots \cdot I\alpha_n e \cdot k = I(\alpha_1 e \cdot \varphi\alpha_2 e \cdot \alpha_3 e \cdot \varphi\alpha_4 e \cdot \ldots \cdot \alpha_n e \cdot k) =$$

$$= IA(\{\alpha_i e\}_{i=1}^n) = I\delta A(\overset{n}{e}) = I\delta(e \cdot \varphi e \cdot e \cdot \varphi e \cdot \ldots \cdot e \cdot k) = I\delta k,$$

therefore

$$\delta R_k(x_1 \cdot \varphi x_2 \cdot x_3 \cdot \varphi x_4 \cdot \ldots \cdot x_n) = (\delta R_k x_1 \cdot I\delta k) \cdot (\delta R_k\varphi x_2 \cdot I\delta k) \cdot \ldots \cdot (\delta R_k\varphi x_{n-1} \cdot I\delta k) \cdot \delta R_k x_n.$$

Changing x_{2i} for φx_{2i} ($2i<n$), and multiplying both parts of the last equality by $I\delta k$ we get

$$\delta R_k^*(x_1 \cdot x_2 \cdot \ldots \cdot x_n) I\delta k = (\delta R_k x_1 \cdot I\delta k) \cdot (\delta R_k x_2 \cdot I\delta k) \cdot \ldots \cdot (\delta R_k x_n \cdot I\delta k).$$

Let

$$\delta R_k x \cdot I\delta k = \theta_0 x. \tag{11}$$

Then

$$\theta_0(x_1 \cdot x_2 \cdot \ldots \cdot x_n) = \theta_0 x_1 \cdot \theta_0 x_2 \cdot \ldots \cdot \theta_0 x_n,$$

i.e. θ_0 is an automorphism of $Q(\cdot)$. It follows from (11) that

$$\delta R_k x = R_{\delta k} \theta_0 x$$

whence (10) follows.

Note, that if $Q(A)$ is a symmetric n-IP-group without an identity element, then $\varphi = \varepsilon$, i.e.

$$A(x_1^n) = x_1 \cdot x_2 \cdot \ldots \cdot x_n \cdot k,$$

and according to (1) and (4)

$$\alpha_i = L_i^1(\bar{a})\delta R_k = L_i(\overline{IL_k a})\delta R_k, \quad i \in \overline{1,n},$$

where

$$\begin{gathered}
L_1(\overline{IL_k a})x = A(x, IL_k\alpha_2 e, IL_k\alpha_3 e, I\alpha_4 e, \ldots, I\alpha_n e), \\
L_2(\overline{IL_k a})x = A(IL_k\alpha_1 e, x, IL_k\alpha_3 e, I\alpha_4 e, \ldots, I\alpha_n e), \\
L_3(\overline{IL_k a})x = A(IL_k\alpha_1 e, IL_k\alpha_2 e, x, I\alpha_4 e, \ldots, I\alpha_n e), \\
L_4(\overline{IL_k a})x = A(IL_k\alpha_1 e, IL_k\alpha_2 e, I\alpha_3 e, x, I\alpha_5 e, \ldots, I\alpha_n e), \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
L_n(\overline{IL_k a})x = A(IL_k\alpha_1 e, IL_k\alpha_2 e, I\alpha_3 e, \ldots, I\alpha_{n-1} e, x).
\end{gathered}$$

Thus,

$$T = (\{L_i(\overline{IL_k a})\delta R_k\}_{i=1}^n, \delta),$$

i.e.

$$\begin{gathered}
\delta A(x_1^n) = A(A(\delta R_k x_1, IL_k\alpha_2 e, IL_k\alpha_3 e, I\alpha_4 e, \ldots, I\alpha_n e), \\
A(IL_k\alpha_1 e, \delta R_k x_2, IL_k\alpha_3 e, I\alpha_4 e, \ldots, I\alpha_n e), \\
A(IL_k\alpha_1 e, IL_k\alpha_2 e, \delta R_k x_3, I\alpha_4 e, \ldots, I\alpha_n e), \\
A(IL_k\alpha_1 e, IL_k\alpha_2 e, I\alpha_3 e, \delta R_k x_4, I\alpha_5 e, \ldots, I\alpha_n e), \\
\ldots, A(IL_k\alpha_1 e, IL_k\alpha_2 e, I\alpha_3 e, \ldots, I\alpha_{n-1} e, \delta R_k x_n)) = \\
= \delta R_k x_1 \cdot I\alpha_2 e \cdot Ik \cdot I\alpha_3 e \cdot Ik \cdot I\alpha_4 e \cdot \ldots \cdot I\alpha_n e \cdot k \cdot \\
\cdot I\alpha_1 e \cdot Ik \cdot \delta R_k x_2 \cdot I\alpha_3 e \cdot Ik \cdot I\alpha_4 e \cdot \ldots \cdot I\alpha_n e \cdot k \cdot \\
\cdot I\alpha_1 e \cdot Ik \cdot I\alpha_2 e \cdot Ik \cdot \delta R_k x_3 \cdot I\alpha_4 e \cdot \ldots \cdot I\alpha_n e \cdot k \cdot \\
\cdot I\alpha_1 e \cdot Ik \cdot I\alpha_2 e \cdot Ik \cdot I\alpha_3 e \cdot \delta R_k x_4 \cdot I\alpha_5 e \cdot \ldots \cdot I\alpha_n e \cdot k \cdot \\
\ldots \cdot I\alpha_1 e \cdot Ik \cdot I\alpha_2 e \cdot Ik \cdot I\alpha_3 e \cdot \ldots \cdot I\alpha_{n-1} e \cdot \delta R_k x_n \cdot k \cdot k = \\
= (\delta R_k x_1 \cdot I\alpha_1 e \cdot I\alpha_2 e \cdot \ldots \cdot I\alpha_n e \cdot Ik) \cdot (\delta R_k x_2 \cdot I\alpha_1 e \cdot I\alpha_2 e \cdot \ldots \cdot I\alpha_n e \cdot Ik) \cdot \\
\ldots \cdot \delta R_k x_{n-1} \cdot I\alpha_1 e \cdot I\alpha_2 e \cdot \ldots \cdot I\alpha_n e \cdot Ik) \cdot \delta R_k x_n.
\end{gathered}$$

Next, since

$$\begin{gathered}
\delta A(x_1^n) = \delta R_k(x_1 \cdot x_2 \cdot \ldots \cdot x_n), \\
I\alpha_1 e \cdot I\alpha_2 e \cdot \ldots \cdot I\alpha_n e \cdot Ik = I(\alpha_1 e \cdot \alpha_2 e \cdot \ldots \cdot \alpha_n e \cdot k) =
\end{gathered}$$

$$= IA(\{\alpha_i e\}_{i=1}^{n}) = I\delta A(\overset{n}{e}) = I\delta(e\cdot e\cdot\ldots\cdot e\cdot k) = I\delta k,$$

then, multiplying both parts of this equality by $I\delta k$, we have

$$\delta R_k(x_1\cdot x_2\cdot\ldots\cdot x_n) = (\delta R_k x_1\cdot I\delta k)\cdot(\delta R_k x_2\cdot I\delta k)\cdot\ldots\cdot(\delta R_k x_n\cdot I\delta k).$$

Let

$$\delta R_k x\cdot I\delta k = \theta_0 x.$$

Then

$$\theta_0(x_1\cdot x_2\cdot\ldots\cdot x_n) = \theta_0 x_1\cdot\theta_0 x_2\cdot\ldots\cdot\theta_0 x_n,$$

i.e. θ_0 is an automorphism of $Q(\cdot)$ and (10) is true.

If $Q(A)$ is a symmetric n-group with an identity element, then the proof is analogous to that of a nonsymmetric n-IP-group when $\varphi=\varepsilon$, $k=e$. Note that in this case the automorphism $\theta_0 = R_{\delta e}^{-1}\delta$ of $Q(\cdot)$ is also an automorphism of $Q(A)$, since

$$\theta_0 A(x_1^n) = \theta_0(x_1\cdot x_2\cdot\ldots\cdot x_n) = \theta_0 x_1\cdot\theta_0 x_2\cdot\ldots\cdot\theta_0 x_n = A(\theta_0 x_1,\theta_0 x_2,\ldots,\theta_0 x_n).$$

Now we can easy prove the following

Theorem. *Every autotopy $T=(\alpha_1^n,\delta)$ of a n-IP-group $Q(A)$:*

$$A(x_1^n) = x_1\cdot\varphi x_2\cdot x_3\cdot\varphi x_4\cdot\ldots\cdot\varphi x_{n-1}\cdot x_n\cdot k$$

has the form

$$\begin{aligned} T = (&L_1^{-1}(\bar{a})R_{\delta k}, L_2^{-1}(\bar{a})\delta\varphi\delta^{-1}R_{\delta k}, L_3^{-1}(\bar{a})R_{\delta k},\\ &L_4^{-1}(\bar{a})\delta\varphi\delta^{-1}R_{\delta k},\ldots,L_n^{-1}(\bar{a})R_{\delta k},\delta R_k^{-1}\delta^{-1}R_{\delta k})\theta_0, \end{aligned} \tag{12}$$

where $\theta_0 = R_{\delta k}^{-1}\delta R_k$ is an automorphism of the binary abelian group $Q(\cdot)$.

Proof. Let $T=(\alpha_1^n,\delta)$ be an autotopy of a n-IP-group $Q(A)$. Then according to (9) this autotopy has the form

$$\begin{aligned} T = (&L_1^{-1}(\bar{a})R_{\delta k}, L_2^{-1}(\bar{a})\delta\varphi\delta^{-1}R_{\delta k}, L_3^{-1}(\bar{a})R_{\delta k},\\ &L_4^{-1}(\bar{a})\delta\varphi\delta^{-1}R_{\delta k},\ldots,L_n^{-1}(\bar{a})R_{\delta k},\delta R_k^{-1}\delta^{-1}R_{\delta k})R_{\delta k}^{-1}\delta R_k. \end{aligned}$$

But, by **Lemma 2**,

$$R_{\delta k}^{-1}\delta R_k = \theta_0$$

is an automorphism of $Q(\cdot)$ in all cases. The theorem is proved.

From this **theorem** it follows that

1) if $Q(A)$ is a symmetric n-IP-group without an identity element, then according (1) with $\varphi=\varepsilon$ we have

$$T=(L_1^{-1}(\bar{a})R_{\delta k}, L_2^{-1}(\bar{a})R_{\delta k},\ldots,L_n^{-1}(\bar{a})R_{\delta k},\delta R_k^{-1}\delta^{-1}R_{\delta k})\theta_0.$$

2) if $Q(A)$ is a symmetric n-IP-group with an identity element, then in accord with (2), where $\varphi=\varepsilon$ and $k=e$, (12) takes on the form

$$T=(\{L_i^{-1}(\bar{a})R_{\delta e}\}_{i=1}^{n}, R_{\delta e})\theta_0, \tag{13}$$

where $\theta_0=R_{\delta e}^{-1}\delta$ is an automorphism of $Q(\cdot)$ and $Q(A)$. In this case the form of an autotopy can be simplified. Really, since for each $i\in\overline{1,n}$

$$L_i^{-1}(\bar{a})R_{\delta e}x=L_i(\overline{Ia})R_{\delta e}x=A(\{I\alpha_j e\}_{j=1}^{i-1},x\cdot\delta e,\{I\alpha_j e\}_{j=i+1}^{n})=$$

$$=I\alpha_1 e\cdot I\alpha_2 e\cdot\ldots\cdot I\alpha_{i-1}e\cdot x\cdot\delta A(\overset{n}{e})\cdot I\alpha_{i+1}e\cdot\ldots\cdot I\alpha_n e=$$

$$=I\alpha_1 e\cdot I\alpha_2 e\cdot\ldots\cdot I\alpha_{i-1}e\cdot x\cdot\alpha_1 e\cdot\alpha_2 e\cdot\ldots\cdot\alpha_{i-1}e\cdot\alpha_i e\cdot\alpha_{i+1}e\cdot\ldots\cdot\alpha_n e\cdot I\alpha_{i+1}e\cdot\ldots\cdot I\alpha_n e=$$

$$=x\cdot\alpha_i e=R_{\alpha_i e}x,$$

$$\theta_0 x=R_{\delta e}^{-1}\delta x=R_{I\delta e}\delta x=\delta x\cdot I\delta e=\delta A(\overset{i-1}{e},x,\overset{n-i}{e})\cdot I\delta A(e)=$$

$$=\alpha_1 e\cdot\ldots\cdot\alpha_{i-1}e\cdot\alpha_i x\cdot\alpha_{i+1}e\cdot\ldots\cdot\alpha_n e\cdot I\alpha_1 e\cdot\ldots\cdot I\alpha_{i-1}e\cdot I\alpha_i e\cdot I\alpha_{i+1}e\cdot\ldots\cdot I\alpha_n e=$$

$$=\alpha_i x\cdot I\alpha_i e=R_{I\alpha_i e}\alpha_i x=R_{\alpha_i e}^{-1}\alpha_i x,$$

then (13) takes on the form

$$T=(\{R_{\alpha_i e}\}_{i=1}^{n}, R_{\delta e})\theta_0, \tag{14}$$

where $\theta_0=R_{\alpha_i e}^{-1}\alpha_i$ is an automorphism of $Q(\cdot)$ and $Q(A)$.

Note that when n=2 the known result from [3] for abelian groups follows.

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