THE SPECIAL SESSION OF THE SCIENTIFIC SEMINAR
IN MEMORIAM OF PROF. V. BELOUSOV,
21 february 2014

DIFFERENTIAL STRUCTURES AND MONOID ALGEBRAS OVER NON-COMMUTATIVE RINGS

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We define on an arbitrary ring A a family of mappings $(\sigma_{x,y})$ subscripted with elements of a multiplicative monoid G. The assigned properties allow to call these mappings as derivations of the ring A. The notion of a monoid algebra in our context extends those of a group ring, a skew polynomial ring, Weyl algebra and other related ones.

Let A be a ring with $1 \neq 0$ and G a multiplicative monoid with the unit element e. Let $\sigma = (\sigma_{x,y})_{x,y \in G}$ be a family of mappings $\sigma_{x,y} : A \longrightarrow A$ which satisfy the following properties (cf. the assumption (A) in [1]):

(i) $\sigma_{x,y}(a+b) = \sigma_{x,y}(a) + \sigma_{x,y}(b) \quad (a, b \in A; x, y \in G)$;
(ii) $\sigma_{x,y}(ab) = \sum_{z \in G} \sigma_{x,z}(a)\sigma_{z,y}(b) \quad (a, b \in A; x, y \in G)$;
(iii) $\sigma_{xy,z} = \sum_{uv=z} \sigma_{x,u} \circ \sigma_{y,v} \quad (x, y, z \in G)$;
(iv_1) $\sigma_{x,y}(1) = 0 \ (x \neq y;\ x, y \in G)$; (iv_2) $\sigma_{x,x}(1) = 1 \ (x \in G)$;
(iv_3) $\sigma_{e,x}(a) = 0 \ (x \neq e;\ x \in G)$; (iv_4) $\sigma_{e,e}(a) = a \ (a \in A)$.

In plus we assume that for $x \in G$ and $a \in A$ the value $\sigma_{x,y}(a)$ is equal to zero for almost all $y \in G$, that is, $\sigma_{x,y}(a) \neq 0$ only for a finite set of $y \in G$. Thus, the sums in (ii) and (iii) are taken only on a finite number of non-zero terms. In the case of the property (iii) this refers to the every element $a \in A$, i.e.

$$\sigma_{xy,z}(a) = \sum_{uv=z} \sigma_{x,u}(\sigma_{y,v}(a)), \quad a \in A.$$

We call the family $\sigma = (\sigma_{x,y})_{x,y \in G}$ satisfying (i)-(iv) a *D-structure* [2] on the ring A.

The operations of differentiation defined traditionally on a ring, for instance in [3], [4], are particular examples in our case. We connect the structure of differentiation defined by means of the family σ with a monoid algebra $A\langle G\rangle$. The elements of $A\langle G\rangle$ are mappings α from G into A such that $\alpha(x) = 0$ for almost all $x \in G$. We make $A\langle G\rangle$ into an A - module by defining the (left) module operations in the natural way. But the law of multiplication is defined specifically, by involving the mappings of the family σ. Namely, we write the elements α of $A\langle G\rangle$ as sums $\alpha = \sum_{x \in G} a_x \cdot x$, where $a_x \cdot x$ denotes the function from G into A whose value is a_x at x and 0 at y different of x. For two elements $\alpha = \sum_{x \in G} a_x \cdot x$ and $\beta = \sum_{x \in G} b_x \cdot x$ we define the law of multiplication by the following formulas $\alpha\beta = \sum_{x,y \in G}(a_x \cdot x)(b_y \cdot y)$, and

$$(a \cdot x)(b \cdot y) = \sum_{z \in G} a\sigma_{x,z}(b) \cdot zy \ (a, b \in A; x, y \in G).$$

In respect with this law of composition $A\langle G\rangle$ becomes to be a ring. This ring $A\langle G\rangle$ is also called a G-algebra over A (or simply a monoid algebra over A). It turns out that $A\langle G\rangle$ represents a free G-algebra over A. In order to prove this fact we construct a suitable category $\mathcal{C}$ in which the ring $A\langle G\rangle$ together with the canonical maps $\varphi_0 : G \longrightarrow A\langle G\rangle$, $\varphi_0(x) = 1 \cdot x$ $(x \in G)$ and $f_0 : A \longrightarrow A\langle G\rangle$, $f_0(a) = a \cdot e$ $(a \in A)$ is a universal object. We note that free algebras over commutative rings (see, for instance, [4, Ch. V, p. 106]), group algebras (when G is a group) [5], Weyl algebras (for the concept see [8], for instance) are concrete realizations of monoid algebras $A\langle G\rangle$. Certain special cases of crossed products (as, for example, twisted semigroup rings or skew group rings) [9] (see also [10]) can be considered as concrete situations of our approach as well.

We study in detail the particular case of a monoid G generated by two elements. This case is important especially for the theory of skew-polynomials, in one variable. The obtained results concerning this special case extend and generalize some related results of T. H. M. Smits [6], [7] (see also in [3]).

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