Groupoids which satisfy certain associative laws

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Definition

A binary groupoid (Q, A) is understood to be a non-empty set Q together with a binary operation A.

Let $(Q, \cdot)$ be a groupoid. The associative law stats that

$$x \cdot (y \cdot z) = (x \cdot y) \cdot z \tag{1}$$

holds for arbitrary elements $x, y, z \in Q$. By interchanging the order of the neighboring "factors" in some of the "multiplications" figuring in (1) it is possible to get 16 equations [7].

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Here we shall study groupoids with two modifications of associative law, namely cyclic associative law (identity):

$$x \cdot (y \cdot z) = (z \cdot x) \cdot y \tag{2}$$

and Tarki (in Hosszu terminology) (in fact, Tarskii) associative law:

$$x \cdot (z \cdot y) = (x \cdot y) \cdot z \tag{3}$$

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Definition

A groupoid $(Q, \cdot)$ is called a left cancelation groupoid, if the following implication fulfilled: $a \cdot x = a \cdot y \Rightarrow x = y$ for all $a, x, y \in Q$, i.e. translation L_a is an injective map for any $a \in Q$.

Definition

A groupoid $(Q, \cdot)$ is called right cancelation, if the following implication fulfilled: $x \cdot a = y \cdot a \Rightarrow x = y$ for all $a, x, y \in G$, i.e. translation R_a is an injective map for any $a \in Q$.

Definition

A groupoid $(Q, \cdot)$ is called a cancelation groupoid, if it is a left and a right cancelation groupoid.

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A groupoid $(Q, \cdot)$ is said to be a left (right) division groupoid if the mapping L_x (R_x) is surjective for every $x \in Q$.

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Definition

An element f of a groupoid $(Q, \cdot)$ is called a *left identity element*, if $f \cdot x = x$ for all $x \in Q$. An element e of a groupoid $(Q, \cdot)$ is called a *right identity element*, if $x \cdot e = x$ for all $x \in Q$. An element e of a groupoid $(Q, \cdot)$ is called a *identity element*, if $x \cdot e = x = e \cdot x$ for all $x \in Q$.

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A groupoid $(Q, \circ)$ is called a *right quasigroup* (a *left quasigroup*) if, for all $a, b \in Q$, there exists a unique solution $x \in Q$ to the equation $x \circ a = b$ $(a \circ x = b)$, i.e. in this case any right (left) translation of the groupoid $(Q, \circ)$ is a bijective map of the set Q.

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A left and right quasigroup is called a *quasigroup*.

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Definition

A left and right quasigroup is called a *quasigroup*.

Definition

A quasigroup with identity element is called a *loop*.

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In this paper an algebra (or algebraic structure) is a set A together with a collection of operations on A. T. Evans [6] defined a binary quasigroup as an algebra $(Q, \cdot, /, \backslash)$ with three binary operations. He has defined the following identities:

$$x \cdot (x \backslash y) = y \tag{4}$$

$$(y/x) \cdot x = y \tag{5}$$

$$x \backslash (x \cdot y) = y \tag{6}$$

$$(y \cdot x)/x = y \tag{7}$$

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Theorem

[11, 12]

1. *A groupoid* $(Q, \cdot)$ *is a left division groupoid if and only if there exists a left cancelation groupoid* $(Q, \backslash)$ *such that in algebra* $(Q, \cdot, \backslash)$ *identity (4) is fulfilled.*
2. *A groupoid* $(Q, \cdot)$ *is a right division groupoid if and only if there exists a right cancelation groupoid* $(Q, /)$ *such that in algebra* $(Q, \cdot, /)$ *identity (5) is fulfilled.*

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Theorem

[11, 12]

1. *A groupoid* $(Q, \cdot)$ *is a left cancelation groupoid if and only if there exists a left division groupoid* $(Q, \backslash)$ *such that in algebra* $(Q, \cdot, \backslash)$ *identity (6) is fulfilled.*
2. *A groupoid* $(Q, \cdot)$ *is a right cancelation groupoid if and only if there exists a right division groupoid* $(Q, /)$ *such that in algebra* $(Q, \cdot, /)$ *identity (7) is fulfilled.*

Results
Cyclic associative law

Lemma

If a right division groupoid $(Q, \cdot, /)$ satisfies the cyclic associative law (2), then then it satisfies associative identity (1).

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Cyclic associative law

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If a right division groupoid $(Q, \cdot, /)$ satisfies the cyclic associative law (2), then then it satisfies associative identity (1).

Lemma

If a right division groupoid $(Q, \cdot, /)$ satisfies the cyclic associative law (2), then it is commutative.

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If a right division groupoid $(Q, \cdot, /)$ satisfies the cyclic associative law (2), then then it satisfies associative identity (1).

Lemma

If a right division groupoid $(Q, \cdot, /)$ satisfies the cyclic associative law (2), then it is commutative.

Theorem

If a right division right cancelation groupoid $(Q, \cdot, /)$ satisfies the cyclic associative law (2), then it is a commutative group relative to the operation $\cdot$.

Results
Cyclic associative law

Lemma

If a left division groupoid $(Q, \cdot, \backslash)$ *satisfies the cyclic associative law (2), then it satisfies and ordinary associative law* $(x \cdot y) \cdot z = x \cdot (y \cdot z)$ *(1).*

Results
Cyclic associative law

Lemma

If a left division groupoid $(Q, \cdot, \backslash)$ *satisfies the cyclic associative law (2), then it satisfies and ordinary associative law* $(x \cdot y) \cdot z = x \cdot (y \cdot z)$ *(1).*

Theorem

If a left division left cancelation groupoid $(Q, \cdot, \backslash)$ *satisfies the cyclic associative law (2), then it is a commutative group relative to the operation* $\cdot$.

Results

Tarski associative law

Lemma

If a left division groupoid $(Q, \cdot, \backslash)$ satisfies the Tarski associative law (3), then it is a commutative groupoid.

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If a left division groupoid $(Q, \cdot, \backslash)$ *satisfies the Tarski associative law (3), then it is a commutative groupoid.*

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If a left division left cancelation groupoid $(Q, \cdot, \backslash)$ satisfies the Tarski associative law (3), then it is a commutative group relative to the operation $\cdot$.

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Theorem

If a left division right cancelation groupoid $(Q, \cdot, \backslash, /)$ satisfies the Tarski associative law (3), then it is a commutative group relative to the operation $\cdot$.

Lemma

If a right division right cancelation groupoid $(Q, \cdot, /)$ satisfies Tarski law (3), then it is associative (1).

Results

Tarski associative law

As follows from the following example, if a right division groupoid $(Q, \cdot, /)$ satisfies the Tarski associative law (3), then it is not a commutative groupoid and it does not contain two-sided identity element.

Example

$\cdot$	0	1
0	0	0
1	1	1

$/$	0	1
0	0	0
1	1	1

Results

Tarski associative law

Theorem

If a right division left cancelation groupoid $(Q, \cdot, \backslash, /)$ *satisfies the Tarski associative law (3), then it is a commutative group relative to the operation* $\cdot$.

References

- **V.D. Belousov**: *Foundations of the Theory of Quasigroups and Loops*, Nauka, Moscow, (1967). (in Russian).
- **V.D. Belousov**: *Elements of Quasigroup Theory: a special course*, Kishinev State University Printing House, Kishinev, (1981) (in Russian).
- **G. Birkhoff**: *Lattice Theory*, Nauka, Moscow, (1984) (in Russian).
- **H.O. Pflugfelder**: *Quasigroups and Loops: Introduction*, Heldermann Verlag, Berlin, (1990).
- **S. Burris and H.P. Sankappanavar**: *A Course in Universal Algebra*, Springer-Verlag, (1981).

T. Evans: *On multiplicative systems defined by generators and relations*, Math. Proc. Camb. Phil. Soc., (1951) **47** 637 – 649.

M. Hosszu: *Some functional equations related with the associative law*, Publ. Math. Debrecen, (1954) **3** 205 – 214.

- **J. Ježek and T. Kepka**: *Medial groupoids*, sešit 2 of Rozpravy Československe Academie VĚD, Academia, Praha., (1983) volume 93.
- **J. Ježek, T. Kepka, and P. Nemec**: *Distributive groupoids*, sešit 3 of Rozpravy Československe Academie VĚD, Academia, Praha, (1981) volume 91.
- **A.I. Mal'tsev**: *Algebraic Systems*, Nauka, Moscow, (1976) (in Russian).

V.A. Shcherbacov: *On definitions of groupoids closely connected with quasigroups*, Bul. Acad. Stiinte Repub. Mold., Mat., (2007) no. 2 43 – 54.

V.A. Shcherbacov, A.Kh. Tabarov, and D.I. Pushkashu: *On congruences of groupoids closely connected with quasigroups*, Fundam. Prikl. Mat., (2008) **14**(1) 237 – 251.