TIRASPOL STATE UNIVERSITY

By way of manuscript
CZU 515.14:515.1(043.3)

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TOPOLOGICAL ALGEBRAIC SYSTEMS AND ITS APPLICATIONS

01.01.04 – Geometry and topology

Doctor Habilitat Thesis in Physical Mathematical Sciences

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CHIŞINĂU, 2011

UNIVERSITATEA DE STAT TIRASPOL

Cu titlu de manuscris
CZU 515.14:515.1(043.3)

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SISTEME TOPOLOGICO-ALGEBRICE ŞI APLICAŢIILE LOR

01.01.04 – Geometrie şi topologie

Teza de doctor habilitat în ştiinţe fizico-matematice

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CHIŞINĂU, 2011

CONTENT

ADNOTARE

la teza de doctor habilitat **"Sisteme topologico-algebrice şi aplicaţiile lor"**, prezentată de către Liubomir Chiriac pentru obţinerea titlului de doctor habilitat în ştiinţe fizico-matematice la specialitatea 01.01.04 - geometrie şi topologie.

Teza a fost perfectată la Chişinău, Universitatea de Stat Tiraspol, în anul 2011, este scrisă în limba engleză şi constă din întroducere, 6 capitole, concluzii, 210 titluri bibliografice, 200 paginini de text de bază. Rezultatele obţinute sunt publicate în 45 lucrări ştiinţifice.

Cuvinte cheie: algebră topologică universală, varietate, quasigrup topologic, unităţi multiple, izotopi omogeni, spaţiu rezolubil, grupoid medial şi paramedial, algebre fuzzy.

Teza este dedicată cercetării următoarei direcţii: *Influenţa structurilor algebrice asupra proprietăţilor topologice ale algebrelor topologice universale.* În particular, teza este consacrată studierii sistemelor topologico-algebrice şi aplicaţiile lor în diverse domenii.

Scopul şi obiectivele lucrării rezidă în: desăvârşirea metodelor de studiere a topologiilor pe algebre libere generate de spaţii pseudocompacte şi numărabil compacte; descrierea submulţimilor compacte ale algebrelor topologice libere şi ale k-algebrelor; elaborarea metodelor de cercetare a quasigrupurilor topologice cu unităţi multiple; construirea teoriei generale a descompunerilor grupoizilor topologici cu proprietăţi de invertibilitate; soluţionarea problemei omomorfismelor pentru algebre fuzzy.

Metodologia cercetărilor ştiinţifice: construcţiile şi metodele de demostraţie se bazează pe aplicarea noţiunilor de algebră topologică, algebră liberă, varietate, quasigrup cu unităţi multiple, spaţiu rezolubil, algebră fuzzy.

Noutatea şi originalitatea: Rezultatele principale sunt noi. Evidenţiem următoarele: au fost elaborate metode de studiere a topologiilor pe algebre libere generate de spaţii pseudocompacte şi numărabil compacte; au fost determinate condiţiile pentru ca omomorfismele continue a grupoizilor topologici cu diviziune continuă să fie deschise; au fost descrise submulţimile compacte ale k-algebrelor libere; au fost stabilite unele proprietăţi topologice care se păstrează la relaţia de M_K-echivalenţă; au fost introduse şi cercetate quasigrupurilor cu unităţi multiple; a fost elaborată metoda de construcţie a măsurii Haar pe quasigrupuri mediale; a fost elaborată metoda de descompunere a grupoizilor topologici cu proprietăţi de invertibilitate; a fost construită o acoperire universală pe E-algebre topologice cu signatura continuă; a fost dată soluţia generală a problemei omomorfismelor pentru algebre fuzzy.

Semnificaţia teoretică: Au fost dezvoltate teorii generale, elaborate concepţii, metode şi construcţii noi care au contribuit la realizarea obiectivelor propuse.

Valoarea aplicativă a lucrării: Metodologia aplicată, concepţiile şi metodele elaborate în lucrare au permis soluţionarea unor probleme concrete ori unele aspecte ale lor formulate de A.I. Malţev, L.S. Pontrjagin, M.M.Cioban. Aparatul matematic aplicat a condus la rezolvarea unor probleme din diverse domenii ale matematicii moderne care au conexiune cu algebra topologică.

Implementarea: Rezultatele lucrării pot fi implementate în teoria algebrelor topologice, teoria quasigrupurilor topologice, teoria automatelor, teoria algebrelor fuzzy, la elaborarea cursurilor speciale pentru masteranzi şi doctoranzi.

АННОТАЦИЯ

диссертации "**Топологических алгебраических систем и их приложение**" представлено Любомиром Кирияком на соискание степени доктора хабилитат физико-математических наук, специальность 01.01.04 - Геометрия и топология. Диссертация была разработано в Кишиневе, в Тираспольском Государственном Университете, в 2011 году. Диссертация написана на английском языке, содержит введение, 6 глав, заключение, 210 упоминаний, 200 страниц основного текста. Полученные результаты опубликованы в 45 научных работ.

Ключевые слова: топологические универсальная алгебра, многообразие, топологические квазигруппы, многократные единицы, однородный изотоп, разложение пространства, медиальный и парамедиальный группоид, нечеткая алгебра.

Диссертация посвящена исследованию: Влияния алгебраических структур на топологических свойствах топологических универсальных алгебр и применение топологических алгебраических структур в исследование свойств топологических пространств. В частности, исследовано топологические и алгебраические системы и их применение в различных областях.

Цель работы: совершенствование методами изучения топологии на свободные алгебры, порожденные псевдокомпактных и счетно-компактных пространств; описание компактных подмножеств свободных топологических алгебр и к-алгебр; разработка методов исследования топологические квазигрупп с многократными единицами; создание общей теории разложении топологических группоидов со свойством обратимости; решении проблемы гомоморфизма нечетких алгебр.

Методология исследования: конструкции и методы доказательства основаны на понятиях топологической алгебры, свободной алгебры, многообразие, квазигруппы с многократными единицами, разложение пространств, нечеткой алгебры.

Научные инноваций определяются по решению следующих проблем: разработаны методы изучения топологии на свободных алгебр, порожденными псевдокомпактных и счетно-компактных пространств; определены условия открытости для непрерывного гомоморфизмах топологических группоидов с непрерывным делением; описаны компактные подмножества свободных к-алгебр; были найдены некоторые топологические свойства, которые сохраняются при М-эквивалентности; введены и изучены квазигрупп с многократными единицами; разработан метод построения меру Хаара на медиальных квазигруппах; разработан метод разложения специальных топологических группоидов; построено универсальные покрытия на топологических Е-алгебр с непрерывными сигнатурой; приведены общие решения проблемы гомоморфизма для нечеткой алгебры.

Теоретическая ценность работы: развиты новые общие теорий, разработаны новые концепций, и методы которые способствуют достижению целей и задач исследования. Основные результаты работы являются новыми.

Практическое значение: применяемой методологии, концепций и методов, разработанных в работе позволили найти решение конкретных проблем или некоторые аспекты проблем, сформулированных А. И. Мальцевым, Л. С. Понтрягиным, М.М Чобаном. Математические средства, разработанные и применяемые привели к решению проблем из различных областях современной математики, связанных с топологической алгеброй.

Применение: результаты этой работы могут быть использованы в теории топологических универсальных алгебр, топологических квазигрупп, теории автоматов, теории нечеткой алгебры, и в разработке факультативных курсов.

SUMMARY

of the thesis **"Topological algebraic systems and its applications"** presented by Liubomir Chiriac for the competition of Ph. Habilitat Doctor degree in Physical and Mathematical Sciences, speciality 01.01.04 - geometry and topology. The thesis was elabored in Chişinău, Tiraspol State University, in 2011. The thesis is written in English, contains introduction, 6 chapters, conclusions, 210 references, 200 pages of the basic text. The main results of the thesis are published in 45 scientific works.

Keywords: topological universal algebra, variety, topological quasigroup, multiple identities, omogen isotope, resoluble space, medial and paramedial grupoid, fuzzy algebra.

The thesis is dedicated to the study: *The influence of the algebraic structures on the topological properties of the topological universal algebras.* In particular, the topological algebraic systems and its applications in diverse fields is investigated.

The purpose of the work resides in: mastering the studying methods of the topologies on free algebras generated by pseudocompact and countable compact spaces; describing the compact subsets of the free topological algebras and that of k-algebras; elaborating research methods regarding the topological quasigroups with multiple identities; establishing a general theory on the decomposition of the topological groupoids with invertibility properties; solving the homomorphism problem for fuzzy algebras.

Methodology of the research: the constructions and the methods of proofs are based on the notions of topological algebra, free algebra, variety, quasigroup with multiple identities, solvable space, fuzzy algebra.

The scientific innovation is determined by the solving of the following problems: there have been elaborated studying methods of topologies on free algebras generated by pseudocompact and countable compact spaces; there have been determined the conditions for the continuous homomorphisms of the topological groupoids with continuous division to be open; there have been described the compact subsets of the free k-algebras; there have been established some topological properties, which are preserved under the M_K-equivalence relation; there have been introduced and analyzed the quasigroups with multiple identities; there has been elaborated a method of construction of the Haar measure on medial quasigroups; there has been elaborated a method of decomposition of special topological groupoids; there has been constructed a universal covering on topological E-algebras with continuous signature; are given a general solution of the homomorphism problem for fuzzy algebras.

The theoretical value of the work: there have been development of the general theories, elaborated the new concepts, methods and constructions which contributed to achieving goals and objectives of the research. The basic results of the work are new.

The practical value: the methodology applied, the concepts and methods developed in work allowed to find the solution of concrete problems or some aspects of the problems formulated by A.I Mal'cev, L.S. Pontrjagin, M.M Choban. Mathematical tools developed and applied led to solving problems in various areas of modern mathematics.

Implementation: The results from this work can be used in the theory of topological universal algebras, of topological quasigroups, of automata, of fuzzy algebras, and in elaborating optional courses.

INTRODUCTION

The actuality of the investigated topic and its degree of research

Topological algebraic systems, as a branch of topological algebra, represent an important field of research in modern mathematics. The methods elaborated within this subject are successfully implemented not only in theoretical mathematics but also in applied mathematics. Topological algebra, being at the border of abstract algebra and general topology, started to gain more attention in the second half if the 19^{th} century, in the works of celebrated mathematicians such as H. Poincare, S. Lie, F. Klein, E. Cartan, D. Hilbert, G. Boole, A. Whitehead, etc.

The notion of universal algebra was first introduced, in 1898, in Alfred Whitehead's book "A Treatise on Universal Algebra". According to Whitehead, in order to obtain the notion of universal algebra it is necessary to introduce the notion of operation. Thus, the application $f : A^n \to A$ is called an *n-ary* algebraic operation on A, for $n \geq 0$. In this way, Whitehead defines the universal algebra as a system (A, S), where A is a non-empty set and S is a family of operations. In 1935, Garret Birkhoff formulates in his famous work "On the structure of abstract algebras" [25], the basic concepts of the universal algebra: variety, identity, cartesian product, congruence, free object, etc. From 1935 till 1950, most of the scientific research in the field of universal algebras was influenced by the ideas exposed G. Birkhoff. Therefore, those investigations focus on the methods of constructions of free algebras, the examination of the congruences and lattice subalgebras, the studying of homomorphisms between universal algebras., etc. In 1950, at the International Mathematical Congress at Cambridge, A. Tarski announces in his communication the beginning of a new period in the development of universal algebras. In that context, we mention that the results obtained by A. Tarski, C.C. Chang, A.I. Mal'cev, L. Henkin, B. Jonsson, R.C. Lyndon, A. Robinson, K. Godel, etc. have contributed to the establishment of the contemporary theory of universal algebras, with applications in topology, logistics, model theory, etc.

The introduction of the modern notions of topological group, in the works of L.E.I. Brauer, published within 1909-1910 and O. Schreier published in 1926 and also the notion of free topological group during 1944-1945 by A. A. Marcov [155] have enormously stimulated the edification of the topological group theory.

The spectacular development of the topological group theory, especially thanks to the works by A. Weil, C. Chevalley, L.S. Pontrjagin, A.A. Markov, A.D. Alexandrov, J.von Neumann, van Kampess, M.I. Graev, T. Nacayama, Sh. Kakutani, etc. positively influenced the directions of research in universal algebras and laid the foundation of universal topological algebras. It is important to mention that, in the 19^{th} century, the topological group theory has basically started to develop simultaneously with the abstract infinite group theory.

During the 1950's, A.I. Mal'cev formulated the central problems concerning the theory of universal topological algebras, therefore directing the efforts of many mathematicians to the development of the respective theory. Some of A.I. Mal'cev's problems had been solved by S. Swierczkowski.

In the relatively recent works of the famous mathematicians P. Smith, L. Zippin, K. Iwasawa, V. M. Glushkov, V. P. Platonov, A. Arhangelskii, M. Cioban, W.W. Comfort, V. Arnautov, S. Glavatsky, A. Mikhalev, K.H. Hofmann, D. Remus, M. Ursul, V. Protasov a series of cardinal problems, which influenced the evolution of the topological algebra theory, have been solved.

The purpose of the work resides in research of topological algebraic systems and its applications. In particular:

1. Studying the free topological algebras.
2. Elaborating the relevant studying methods of the topologies on free algebras generated by pseudocompact and countable, compact spaces.
3. Describing the compact subsets of the free topological algebras and that of k-algebras.
4. Elaborating research methods regarding the topological universal algebras with invariant measures. In particular, studying the concept of multiple identities.
5. Elaborating research methods regarding the topological quasigroups with multiple identities.
6. Studying the uniform structure on the topological spaces in the light of free objects.
7. Studying of the equivalences in the class of topological spaces generated by varieties of universal topological algebras.

8. Establishing a general theory on the decomposition of the topological groupoids with invertibility properties.
9. Studying the fuzzy structure on universal algebras. In particular, solving the homomorphism problem for fuzzy algebras.

Methodology of the research. Topologization of abstract algebras and sets theory are key components of research methods. The constructions and the methods of proofs are based on the notions of topological algebra, variety, quasigroup with multiple identities, solvable space, fuzzy algebra.

The scientific innovation of the work. The main problem solved in accordance with the objectives of the thesis, consist to determine the influence of the algebraic structures on the topological properties of the universal topological algebras and application of topological algebraic structures in the study of the properties of topological spaces. The scientific innovation is determined by the solving of the following particular problems:
- *there have been elaborated studying methods of topologies on free algebras generated by pseudocompact and countable, compact spaces.*
- *there have been determined the conditions for the continuous homomorphisms of the topological groupoids with continuous division to be open.*
- *there have been described the compact subsets of the free k-algebras.*
- *there have been established some topological properties, which are preserved under the M_K-equivalence relation.*
- *there have been introduced and analyzed the quasigroups with multiple identities.*
- *there has been elaborated a method of construction of the Haar measure on medial quasigroups.*
- *there has been elaborated a method of decomposition of topological groupoids with invertibility properties.*
- *there has been constructed a universal covering on topological E-algebras with continuous signature.*
- *there have been studied the paramedial topological groupoids and established the correlation between paramediality and associativity.*
- *there has been given a general solution of the homomorphism problem for fuzzy algebras.*
- *there has been studied the category of the fuzzy groupoids with division.*

The theoretical and practical value of the work. The development of the general theory of topological universal algebras prove that the free topological algebras represent an efficient instrument of topological algebras. In this context, there has been developed a theory that describes the compact subsets of the free topological algebras and of the k-algebras. There have been elaborated efficient methods, which allow the studying of topologies on free algebras generated by pseudocompact and countable, compact spaces. There has been elaborated a new concept that allows the studying of the topological quasigroups with multiple identities. There has been constructed a general theory of the decomposition of the topological groupoids with invertibility properties. There has been solved the homomorphism problem for fuzzy algebras. There has been studied the category of the fuzzy groupoids with division.

The results from this work can be used in the theory of topological universal algebras, of topological quasigroups, of automata, of fuzzy algebras, and in elaborating optional courses.

The approvement of the work. The results of the work were exposed at:
- Simpozionul al VI-lea Tiraspolean de Topologie Generală şi Aplicaţiile ei, Chişinău, 1991;
- International Conference on Group Theory, Timisoara, 17-20 September, 1992;
- Conferinţa pregatitoare pentru Congresul Matematicienilor Români de Pretutindeni, Bucureşti, 1993;
- Congresul XVIII al Academiei Româno-Americane de Ştiinte si Arte, Chişinău, 1993;
- The 4-th Conference on Applied and Industrial Mathematics, Oradea- CAIM, 1995;
- The 7-th Conference on Applied and Industrial Mathematics, Piteşti, October, 1993;
- The 8-th Conference on Applied and Industrial Mathematics, Oradea and Chişinău, October, 1994;
- II International Conferences of the Balcanic Union For Fuzzy Systems and Artificial Intelegence. Trabzon, Turkey, 1996;
- International Conference on Mathematics and Informatics (State University of Moldova). 19-21 September 1996;

- Invăţământul universitar din Moldova la 70 ani. Conferinţa ştiintifico-metodica, Chişinău, 9-10 octombrie 2000;
- First Conference of the Mathematical Society of the Republic of Moldova, Chişinău, August 16-18, 2001;
- International Seminar on Discrete Geometry dedicated to the 75-th birthday of Professor A. M. Zamorzaev, Chişinău, August 28-29, 2002;
- International Conferences on Radicals (ICOR-2003), dedicated to the memory of Prof. V. Adrunakievich, August 11-16, 2003, Chişinău, Moldova
- Second Conference of the Mathematical Society of the Republic of Moldova, Chişinău, August 17-19, 2004;
- The 13-th Conference on Applied and Industrial Mathematics, Piteşti, October 14-16, 2005;
- Materialele seminarului Ştiinţifico-metodic "Profesorul Petre Osmatescu-80", 19 noiembrie 2005, Chişinău;
- The 5-th Edition of the anual Symposion "Mathematical Applied in Biology and Biophysics", Iaşi, June 16-17, 2006;
- The XIV-th Conference on Applied and Industrial Mathematics, Satelite Conference of ICM 2006, Chişinău, August 17-19, 2006;
- 6th Congress of Romaninan Mathematicians, June 28 - July 4, 2007, Bucharest, Romania;
- Algebraic Systems and their Applications in Differential Equations and other domains of mathematics, Chişinău, August 21-23, 2007;
- Mathematics and Informatics. MITRE 1-4 October, State University of Moldova, 2008;
- The 16th Conference on Applied and Industrial Mathematics. CAIM 2008 Oradea, October 9-11, 2008;
- Conferinţa ştiinţifică republicană "Matematica-probleme actuale cu aplicaţii", 8 aprilie, 2009, ASEM, Chişinău;
-The 17th Conference on Applied and Industrial Mathematics. CAIM 2009 Constanza, September 17-29, 2009;
-Conference. Mathematics and Information Technologies. MITRE-2009, October 8-9, Mathematical Society of Moldova, State University of Moldova, 2009.
-Scientific Conference dedicated to the 80th anniversary of the foundation of the

Tiraspol State University. Actual Problems of Mathematics and Informatics. September 24-25, 2010;
-The 18th Conference on Applied and Industrial Mathematics. CAIM 2010 Iasi, October 14-18, 2010.

Publications. The main results of the thesis are published in 45 scientific works.

The content and the structure of the thesis. The work consists of six chapters (divided in paragraphs) and the bibliography. The theorems, propositions, lemmas, corollaries, remarks are numbered by three numbers, the first one indicating the number of the chapter.

Chapter 1 contains some basic topological constructions and a brief summary of the concepts related to the theory of topological universal algebras.

In the Chapter 2 we study topologies on free algebras of pseudocompact and countably compact spaces. In the presentation an important role is played by uniform structures. In this way, expanding on the ideas of A. Arhangelskii, E. Nummela, V. Pestov, T.H.Fay, B.V. Smith-Thomas and A. Tkacenko, which were successfully used in the research of free topological groups, made it possible to elaborate and implement new methods of studying topologies on free topological algebras with continuous signature generated by pseudocompact and countable compact spaces. An analog of the Nummela-Pestov theorem [126, 176] for varieties of uniform algebras is proved.

Chapter 3, Sections 3.1 - 3.11, describe the topological quasigroups with (n, m)-identities, which are obtained by using isotopies of topological groups. Such quasigroups are called (n, m)-homogeneous quasigroups. We extend some affirmations of the theory of topological groups on the class of topological (n, m)-homogeneous quasigroups. It also examines the relationship between paramediality and associativity. The concept of multiple identities and homogeneous isotopies facilitates the study of topological groupoids with (n, m)-identities and homogeneous quasigroups, (see[68, 69, 70, 71, 79]).

In Chapter 3, Sections 3.12 - 3.15, we show that every topological n-groupoid A can be embedding into a topological n-groupoid with division B. We give the conditions when continuous homomorphisms of topological groupoids with continuous homomorphisms of topological groupoids with a continuous division are open. We study one special class of topological groupoids with division, namely the class of medial topological quasigroups.

In Chapter 3, Sections 3.16 - 3.18, we study universal covering algebras. L.S.Pontrjagin [178] showed that a linearly connected space, which covers topological group, has itself a structure of topological group. We obtain the analogical result for the universal algebras with continuous signature. This result, for the case of discrete signature, was also obtained by A.I. Mal'cev [154]. Result from this Chapter is stronger then Mal'cev's Theorem. In particular, the result holds for the topological R-modules, where R is a topological ring.

In Chapter 4 we investigate universal algebras with topologies. On algebras we consider topologies relatively to which operations are continuous on compact subsets. These algebras are called k-algebras. The implementation of the methods of free algebras and k-algebras contributed to the identification of some significant properties regarding the relation of M_K-equivalence. Some properties of compact subsets of free k-algebras and some facts about M_K-equivalence of spaces are established. For instance, we obtained that the homological groups obey a relation of M_K-equivalence. Some similar results for varieties of topological groups and compact spaces were proved by L. S. Pontrjagin and B. A. Pasynkov.

In Chapter 5 we study the resolvability of some special algebras with topologies. A space X is called resolvable if in X there exist two disjoint dense subsets. Improving and developing the ideas of V.I. Malykhin, W.W. Comfort, S. Van Mill, I.V.Protasov and M.M. Choban which successfully was used in research the problem of resolvability of totally bounded topological groups, in this work we elaborate a general theory of decomposition of topological groupoids with invertibility properties.

The developed theory is based on the following concepts: I_nP_k-n-groupoid, λ-k-φ-bounded topology, k-φ-total bounded topology, bounded topology of Choban.

The problem of the homomorphism for fuzzy algebras was formulated and solved for some homomorphisms of the fuzzy groupoids, groups and rings.

We introduced the concepts of fuzzy universal algebras and fuzzy homomorphism for these algebras. The problem of homomorphism for fuzzy algebras was formulated and solved by S. Nanda, A. Rozenfeld, A.Wang-Jin Liu for some homomorphisms of the fuzzy groupoids, groups and rings.

In this context, Chapter 6 gives a general solution of the homomorphism problem for fuzzy universal algebras and conditions for which there exists a homogeneous basis for the category of fuzzy groupoids with division. The results from this work are stronger than the results obtained by the authors mentioned above. In particular, we prove that for an arbitrary L-fuzzy finitely generated L-fuzzy module, L-fuzzy quasigroup, L-fuzzy group, there exists a homogeneous basis.

I would like to express my sincere thanks to Academician M. Choban for his constant interest in this work.

1. BASIC CONSTRUCTIONS AND ANALYSIS OF THE CURRENT SITUATION IN TOPOLOGICAL ALGEBRA

Chapter 1 contains:

– some basic topological and algebraic constructions;

– brief summary of the concepts related to the theory of topological universal algebras;

– an analysis of scientific materials that relate to the topic of thesis.

1.1 Basic Topological Constructions

We accept the usual basic constructions and concept of a topological space as given in the definition below.

1.1.1 Topological Spaces. Basis for a Topology

Definition 1.1.1. *Let X be a non-empty set and let τ be a collection of subsets of X. Then the pair (X, τ) is called a **topological space** and τ is called its **topology** if the following conditions are satisfied:*

(T1). $\emptyset \in \tau$, $X \in \tau$.

(T2). If $U, V \in \tau$, *then* $U \cap V \in \tau$.

(T3). If $U_\alpha \in \tau$ *for each* $\alpha \in A$, *then* $\bigcup\limits_{\alpha \in A} U_\alpha \in \tau$.

Often the topological space (X, τ) is briefly denoted by X if no confusion could arise. Here, an element $x \in X$ is called a **point** from X. Sets belonging to τ are said to be **open**, whereas their complements in X are said to be **closed**.

A set which is open and closed at the same time is said to be **open-and-closed**.

The intersection of all closed sets containing given $A \subset X$ is called the **closure** of A and is denoted by $[A]_X$ or $[A]$.

The **neighborhood** of a point is an open set containing this point.

A point $x \in X$ is said to be **isolated** in (X, τ) if the set $\{x\}$ is a neighborhood of x.

A point x is called a **cluster** point of a set $A \subset X$ if any neighborhood of x contains points of A different from x.

If X is any set, the collection of all subsets of X is a topology on X;it is called the discrete topology. The collection consisting of X and $\emptyset$ is also topology on X; we shall call it the indiscrete topology, or the trivial topology.

Given a collection of open sets B in (X, τ). If every open set can be represented as a union of some elements of B, then B is called a **basis** of the topology (X, τ).

A collection of neighborhoods $B(x)$ of a point $x \in X$ is said to be a **basis** of the topology (X, τ) in x, if for any neighborhood V of point x there is $U \in B(x)$ such that $x \in V \subset U$. $B(x)$ is said to be a **neighborhood basis** in x.

Consider a non-empty set X and for every $x \in X$ a collection $B(x)$ of subsets of X. Suppose that the following statements are true:

(B1). For every x we have $B(x) \neq \emptyset$ and $x \in U$ for any $U \in B(x)$.

(B2). If $x \in U \in B(y)$, then there is $V \in B(x)$ such that $V \subset U$.

(B3). If $U_1, U_2 \in B(x)$, then there is $U \in B(x)$ such that $U \subset U_1 \cap U_2$.

Then the family $\cup\{B(x) : x \in X\}$ constitutes a basis for a certain topology τ on X.

If B is the collection of all open intervals in the real line,

$$(a, b) = \{x : a < x < b\},$$

the topology generated by B is called the **standard topology** on the real line.

The minimal cardinality of the bases of a given topological space X is said to be its **weight** and is denoted by $w(X)$. The minimal cardinality of neighborhood bases in $x \in X$ is called the **character** of X in $x$and and is denoted by $\chi(X), x$. $\chi(X)$ denotes the lowest upper bound of the characters of X in its points and is called the character of the space X.

A set $A \subset X$ is called **dense** in the space X if $[A] = X$. The minimal cardinality of dense sets in X is called the **density** of X and is denoted by $d(X)$.

$w(X)$, $\chi(X)$, $d(X)$ are cardinal invariants of the space X.

X is called **first countable** if $\chi(X) \leqslant \aleph_0$, and **second countable** if $w(X) \leqslant \aleph_0$, where $\aleph_0$ denotes the first infinite cardinal.

1.1.2 Continuous Mappings. Homeomorphisms

Recall that a mapping $f : X \to Y$ of a topological space X in a topological space Y is called **continuous** if for every open set $U \subset Y$ the inverse image $f^{-1}(U)$ is open in X. Under these assumptions the following statements are equivalent:

(C1). f is continuous.

(C2). For every closed set B of Y, the set $f^{-1}(B)$ is closed in X.

(C3). There exist neighborhood systems $\{B(x) : x \in X\}$ and $\{D(y) : y \in Y)\}$ in Y such that for every $V \in D(y)$, where $y = f(x)$, there is $U \in B(x)$ for which $f(u) \subset V$.

(C4). For every subset $A \subseteq X$ we have $f(\bar{A}) \subseteq \overline{f(A)}$.

Theorem 1.1.2. (Rules for constructing continuous mappings). *Let X, Y, and Z be topological spaces:*

(R1). (Constant mapping). If $f : X \to Y$ mappings all of X into the single y_0 of Y, then f is continuous.

(R2). (Inclusion). If A is a subspace of X, the inclusion mapping $f : A \to X$ is continuous.

(R3). (Composites). If $f : X \to Y$ and $g : Y \to Z$ are continuous, then the mapping $g \circ f : X \to Z$ is continuous.

(R4). (Restricting the domain). If $f : X \to Y$ is continuous, and if A is a subspace of X, then the restricted mapping $f \mid A :\to Y$ is continuous.

(R5). (Restricting or expanding the range). Let $f : X \to Y$ be continuous. If Z is a subspace of Y containing the image set $f(X)$, then the mapping $g : X \to Z$ obtained by restricting the range of f is continuous. If Z is a space having Y as a subspace, then the mapping $h : X \to Z$ obtained by expanding the range of f is continuous.

A continuous surjective mapping $f : X \to Y$ is called **open** (**closed**) if for any open (closed) $A \subset X$ the image $f(A)$ is open (closed) in Y.

Let X and Y be topological spaces; let $f : X \to Y$ be a bijection. If both the function f and the inverse function $f^{-1} : Y \to X$ are continuous, then f is a called a **homeomorphism**.

So another way to define a homeomorphism is to say that it is an open bijection mapping $f : X \to Y$. In this case the topological spaces X, Y are called **homeomorphic spaces**.

If the restriction $f : X \to f(X)$ is a homeomorphism, then $f : X \to Y$ is said to be a **topological embedding** of X into Y.

Given a topological space X and a subspace $Y \subset X$. A continuous map $f : X \to Y$ is called a **retraction** if $f(y) = y$ for every $y \in Y$. Y is called **retract** of X. Every open-and-closed subspace $Y \subset X$ is a retract of X.

1.1.3 The Quotient Topology

Let X, Y be topological spaces and let $f : X \to Y$ be a continuous surjective mapping. Then f is called a **quotient mapping** if every set $U \subset Y$ is open in Y, if and only if $f^{-1}(U)$ is open in X.

An equivalent condition is to require that a subset B of Y be closed in Y if and only if $f^{-1}(B)$ is closed in X. Equivalence of the two conditions follows from equation

$$f^{-1}(Y - B) = X - f^{-1}(B).$$

Every open (closed) mapping is a quotient mapping. Every quotient bijection is a homeomorphism. There are quotient mappings that are neither open nor closed.

Let X be a topological space and E an equivalence relation on X. Then E provides a quotient mapping as follows. Let X/E denote the quotient set (i.e. the set of all equivalence classes) and let $q : X \to X/E$ be the natural surjection. Consider on X/E the maximal topology with respect to which q is continuous. Thus, X/E becomes a topological space called the **quotient space**, and q becomes a quotient map. Each quotient mapping can be obtained by a similar procedure, with a suitable equivalence relation.

Theorem 1.1.3. *Let $p : X \to Y$ be a quotient mapping. Let Z be a space and let $g : X \to Z$ be a mapping that is constant on each set $p^{-1}(y)$, for $y \in Y$. Then g induces a mapping $f : Y \to Z$ such that $f \circ p = g$. The induced mapping f is continuous if and only if g is continuous; f is a quotient mapping if and only if g is a quotient mapping.*

1.1.4 The Separation Axioms

A topological space X is said to be a T_0**-space** if for each distinct $x, y \in X$, there exists an open subset U such that $|U \cap \{x, y\}| = 1$.

A topological space X is said to be a T_1**-space** if for each distinct $x, y \in X$ there exist two open subsets U_x, U_y, $x \in U_x$, $y \in U_y$ such that $x \notin U_y$, $y \notin U_x$. In a T_1-space every single point set is closed.

A topological space X is called a T_2**-space** (or a Hausdorff space) if each distinct $x, y \in X$ have disjoint neighborhoods in X.

A T_1-space X is called a T_3**-space** (or a regular space) if for any point $x \in X$ and any closed subset $F \subseteq X$ such that $x \notin F$ there exist two open subsets U and V containing x and F, respectively, such that $U \cap V = \emptyset$.

A T_1-space is called a $T_{3\frac{1}{2}}$**-space** (or a Tychonoff space) if for any point $x \in X$ and any neighborhood U of x there exists a continuous function $f : X \to [0, 1]$ such that $f(x) = 0$ and $f(y) = 1$ for any $y \notin U$.

A T_1-space X is called a T_4**-space** (or a normal space) if for every disjoint closed sets $A, B \subseteq X$ there are disjoint open sets $U \supset A$, $V \supset B$.

A topological space X is called a T_5**-space** if each of its subspaces is normal.

A topological space is said to be a T_6**-space** if it is normal and for each disjoint closed subsets F and Φ, there exists a continuous function $f : X \to [0, 1]$ such that $F = f^{-1}(0)$, $\Phi = f^{-1}(1)$.

It is known the validity of the following implications:

$$T_6 \Rightarrow T_5 \Rightarrow T_4 \Rightarrow T_{3\frac{1}{2}} \Rightarrow T_3 \Rightarrow T_2 \Rightarrow T_1 \Rightarrow T_0.$$

Nevertheless, the inverse implications are not true.

Let us consider any topological space as being a T_{-1}-space.

1.1.5 The Subspaces, Sum and Product Topology

Let X be a topological space and Y be a non-empty subset of X. Then Y inherits the topological structure of X in the following way:U **is regarded as open in** Y **iff** $U = Y \bigcap U'$ **for a set** U' **open in** X.

Thus, Y becomes itself a topological space - a **subspace** of X. The property of being a T_i-space is hereditary for $-1 \leqslant i \leqslant 3.5$. This means that every subspace of a T_i-space is a T_i-space itself.

Let $\{X_\alpha : \alpha \in A\}$ be a collection of pairwise disjoint topological spaces. Let us define a topological structure on $X = \bigcup_{\alpha\in A} X_\alpha$ as follows: $U \subset X$ **is regarded as open in** X **if** $U \cap X_\alpha$ **is open in** X_α **for any** $\alpha \in A$**.**

The topological space X obtained in this way is denoted by $\oplus_{\alpha\in A} X_\alpha$ and is called the **topological sum** of the spaces $\{X_\alpha : \alpha \in A\}$. Every X_α is an open-and-closed set in $\oplus_{\alpha\in A} X_\alpha$.

A topological property $\Im$ is said to be **additive** if $X = \oplus X_\alpha$ has $\Im$ as soon as X_α has $\Im$ for any $\alpha \in A$. The property of being a T_i-space is additive for $i \leqslant 6$. A map $f : \oplus_{\alpha\in A} X_\alpha \to Y$ is continuous iff $f \mid X_\alpha$ is continuous for any $\alpha \in A$.

Let $\{X_\alpha : \alpha \in A\}$ be a collection of topological spaces. Consider on the Cartesian product $\Pi_{\alpha\in A} X_\alpha$ a topology generated by the base consisting of sets $\Pi_{\alpha\in A} U_\alpha$, where U_α is open in X_α for every $\alpha \in A$ and $U_\alpha = X_\alpha$ for all α from a finite number of them.

Thus, $\Pi_{\alpha\in A} X_\alpha$ becomes a topological space which is called a **topological Cartesian product** of the spaces $\{X_\alpha : \alpha \in A\}$. The topology on $\Pi_{\alpha\in A} X_\alpha$ is called the **Tychonoff topology.** If $X_\alpha = X$ for every $\alpha \in A$, then one gets a degree of X which is denoted by X^m, where m is the cardinality of A. I^m is called a Tychonoff cube, where $I = [0; 1]$.

We mention that, if $\mid A \mid = m$, then we consider that $X^m = X^A$. The property of being a T_i-space is **multiplicative** for $-1 \leqslant i \leqslant 3\frac{1}{2}$. If $\Pi_{\alpha\in A} X_\alpha$ is a non-empty T_i-space, then every X_α is a T_i-space for $i \leqslant 6$.

1.1.6 Nets

Recall that a relation $\preceq$ on a set A is called a **partial ordered** if the following conditions hold:

(PO1). $\alpha \preceq \alpha$ for all α.

(PO2). If $\alpha \preceq \beta$ and $\beta \preceq \alpha$, then $\alpha = \beta$.

(PO3). $\alpha \preceq \beta$ and $\beta \preceq \gamma$, then $\alpha \preceq \gamma$.

Now we make the following definition:

A **directed set** A is a set with a partial order $\preceq$ such that for each pair α, β of elements of A, there exist an element γ of A having the property that $\alpha \preceq \gamma$ and $\beta \preceq \gamma$.

Let A be a directed set, X be a Hausdorff topological space, S be a mapping of A into X. A mapping S is called a **net** in the space X.

Let $S = \{x_\alpha : \alpha \in A\}$, where $x_\alpha = S(\alpha)$. The point $x \in X$ is a **cluster point** of a net S if for any neighborhood U of x and any $\alpha \in A$ there is $\alpha' \succeq \alpha$ such that $x_{\alpha'} \in U$. Moreover, x is a **limit** of S if for every neighborhood U of x there is such $\alpha_0 \in A$ that $x_\alpha \in U$ as soon as $\alpha \succeq \alpha_0$. One also says that the net S **converges** to x and writes $x = \lim_{\alpha \in A} S$. Let $x \in X$, $L \subset X$, $x \in [L]$ iff there is a net consisting of points of M, which converges to x. Each net has at least one limit. One may add the following item to the conditions of continuity listed above:

If $x = \lim_{\alpha \in A} x_\alpha$, then $f(x) = \lim_{\alpha \in A} f(x_\alpha)$ for every net $\{x_\alpha : \alpha \in A\}$.

Sequences are nets directed by the set of naturals N.

1.1.7 Connectedness

A topological space X is **connected** if, given two open sets U and V with $X = U \cup V$, $U \cap V = \emptyset$, either $X = U$ or $X = V$.

Proposition 1.1.4. *The following are equivalent:*

(Co1). X is connected.

(Co2). The only subsets of X that are both open and closed are the empty set and X itself.

(Co3). Every continuous mapping $f : X \to \{0, 1\}$ is constant.

A subset A of a topological space X is a **connected subspace** if A is a connected space in the subspace topology. The closed interval $[a, b]$ is conected, for all real numbers a and b satisfying $a \leq b$

Intermediate Value Theorem. *If $f : [a, b] \to R$ is continuous and if $c \in [f(a), f(b)]$, then there is a $r \in X$ such that $f(r) = c$.*

Proposition 1.1.5. *Let X be a connected space, and $f : X \to Y$ a continuous mapping.*

(P1). $f(X)$ is a connected subspace.

(P2). If $Y = R$, then f satisfies the Intermediate Value Theorem.

A space X is **totally disconnected** if it has no connected subset with more than one point.

A concept closely to that of connectedness is path-connectedness. Let a_0 and a_1 be elements of a topological space X. A **path** in X from a_0 to a_1 is defined to be a continuous mapping $f : [0,1] \to X$ such that $f(0) = a_0$ and $f(1) = a_1$. A topological space X is said to be **path-connected** if and only, given any two points a_0 and a_1 of X, there exists a path in X from a_0 to a_1.

Every path-connected topological space is connected. There is a connected space that is not path-connected.

A space X is **locally path-connected** if every point has a local basis consisting of path-connected sets. A locally path-connected space is connected if and only if it is path-connected.

A **connected component** in point x of topological space X is called the union of all connected subsets of X that contain x.

The components of X can also be described as follows:

Proposition 1.1.6. *The components of X are connected disjoint subspace of X whose union is X, such that each non-empty connected subspace of X interesects only one of them.*

In topology, two continuous functions from one topological space to another are called **homotopic** if one can be "continuously deformed" into the other, such a deformation being called a **homotopy** between the two functions.

Formally, **homotopy** between two continuous mappings f and g from a topological space X to a topological space Y is defined to be a continuous mapping $H : X \times [0,1] \to Y$ from the product of the space X with the unit interval $[0,1]$ to Y such that, for all points $x \in X$, $H(x,0) = f(x)$ and $H(x,1) = g(x)$.

If the mappings f and g are homotopic then we denote this fact by writing $f \simeq g$. The mapping H with the properties stated above is referred to as a homotopy between f and g.

A space X is **simply connected** if and only if it is path-connected, and whenever $p : [0,1] \to X$ and $q : [0,1] \to X$ are two path with the same start and endpoint ($p(0) = q(1)$ and $p(1) = q(1)$, then p and q are homotopic relative $0, 1$.

An equivalent formulation is this: A space X is **simply connected** if it is path-connected and every continuous mapping $F : S^1 \to X$ is homotopic to a constant mapping.

A space X is **locally simply connected** if every point $x \in X$ has a locally basis of neighborhoods U that is simply connected.

1.1.8 Universal covering space

Let X be a topological space. A **covering space** of X is a space C together with a continuous surjective mapping $p : C \to X$ such that for every $x \in X$, there exists an open neighborhood $U \ni x$, such that $p^{-1}(U)$ is a disjoint union of open sets in C each of which is mapped homeomorphically onto U by p.

The mapping p is called the **covering mapping**. The space X is often called the **base space** of the covering and the space C is called the **total space** of the covering. For any point $x \in X$ the inverse image $p^{-1}(x) \in C$ is necessarily a discrete space called the **fiber** over x.

The special open neighborhoods $U \ni x$ given in the definition are called **evenly-covered neighborhoods**. The evenly-covered neighborhoods form an open-cover of the space X. The homeomorphic copies in C of an evenly-covered neighborhood U are called the **sheets** over U.

A connected covering space is a **universal cover** if it is simply connected. The names universal cover comes from the following important property: if the mapping $q : D \to X$ is a universal cover of the space X and the mapping $p : C \to X$ is any cover of the space X where the covering space C is connected, then there exists a covering mapping $f : D \to C$ such that $p \circ f = q$. This can be phrased as:

The universal cover of the space X covers all connected covers of the space X.

The mapping f is unique in the following sense: if we fix a point x in the space X and a point d in the space D with $q(d) = x$ and a point c in the space C with $p(c) = x$, then there exists a unique covering mapping $f : D \to C$ such that $p \circ f = q$ and $f(d) = c$.

If the space X has a universal cover then that universal cover is essentially unique: if the mappings $q_1 : D_1 \to X$ and $q_2 : D_2 \to X$ are two universal covers of the space X then there exists a homeomorphism $f : D_1 \to D_2$ such that $q_2 \circ f = q_1$

1.1.9 Compactness

A collection Θ of subsets of a space X is said to **cover** X, or to be a **covering** of X, if the union of the elements of Θ is egual to X. It is called an **open covering** of X if its elements are open subsets of X.

A Hausdorff topological space X is **compact** if every open covering of X contains a finite subcovering.

Every closed subspace of a compact space is also compact. If F is a compact subspace of X, then F is closed in X.

If $f : X \to Y$ is a continuous surjection and X is compact, then Y is compact too.

Let $f : X \to Y$ be a bijective continous mapping. If X is compact and Y is Hausdorff, then f is a homeomorphsim.

Every continuous function $f : X \to R$ of a compact space X is bounded.

The space X is compact iff each net in X has a cluster point. If X is a dense proper subset of a compact Y, then Y is called a **compactification** of X. If $Y - X$ equals a single point, then Y is called the **one-point compactification** of X.

The collection of all compactifications of a given Tychonoff space is non-empty and has a maximal element βX, called the **Stone-Čech compactification**.

A Hausdorff space X is **locally compact** if each point $x \in X$ has a neighborhood U such that $[U]$ is a compact subspace of X. Each locally compact space is Tychonoff.

Let Y be a compact Hausdorff space and X be a proper subspace of Y whose closure equals to Y. A space X has a one-point compactification Y if and only if X is a locally compact Hausdorff space that is not itself compact.

Let X be locally compact Hausdorff; let A be a space of X. If A is closed in X or open in X, then A is locally compact.

Local compactness is additive and is preserved by open continuous mappings.

A space X is homeomorphic to an open subspace of a compact Hausdorff space if and only if X is locally compact Hausdorff.

A space is **sequentially compact** if every sequence has a convergent subsequence. A space is **countably compact** if every countable open cover has a finite subcover. A space is **pseudocompact** if every real-valued function on the space is bounded. A space is σ-compact if it is the union of countably many compact subsets. A space is **metrizable** if it is homeomorphic to a metric space. Metrizable spaces are always Hausdorff and paracompact (and hence normal and Tychonov), and first-countable. A space X is a **Baire** space if the intersection of countably many dense open sets is dense.

1.1.10 Paracompact space

A **refinement** of a cover of a space X is a new cover of the same space such that every set in the new cover is a subset of some set in the old cover. In symbols, the cover $\mathbf{V} = \{V_\beta : \beta \in B\}$ is a refinement of the cover $\mathbf{U} = \{U_\alpha : \alpha \in A\}$ if and only if, for any $V_\beta \in \mathbf{V}$, there exists some $U_\alpha \in \mathbf{U}$ such that V_β is contained in U_α.

An open cover of a space X is **locally finite** if every point of the space has a neighborhood which intersects only finitely many sets in the cover. In symbols, $\mathbf{U} = \{U_\alpha : \alpha \in A\}$ is locally finite if and only if, for any $x \in X$, there exists some neighborhood $V(x)$ of x such that the set $\{\alpha \in A : U_\alpha \cap V(x) \neq \varnothing\}$ is finite.

Note the similarity between the definitions of compact and paracompact: for paracompact, we replace "subcover" by "open refinement" and "finite" by "locally finite". Both of these changes are significant: if we take the above definition of paracompact and change "open refinement" back to "subcover", or "locally finite" back to "finite", we end up with the compact spaces in both cases.

In this sense, a space is **paracompact** if every open cover has an open locally finite refinement. A hereditarily paracompact space is a space such that every subspace of it is paracompact. This is equivalent to requiring that every open subspace be paracompact.

Comparison with compactness

Paracompactness is similar to compactness in the following respects:

(S1). Every closed subset of a paracompact space is paracompact.

(S2). Every paracompact Hausdorff space is normal.

It is different in these respects:

(D1). A paracompact subset of a Hausdorff space need not be closed. In fact, for metric spaces, all subsets are paracompact.

(D2). A product of paracompact spaces need not be paracompact. The square of the real line R in the lower limit topology is a classical example for this.

A cover of a space X is **pointwise finite** if every point of the space belongs to only finitely many sets in the cover. In symbols, U is pointwise finite if and only if, for any $x \in X$, the set $\{\alpha \in A : x \in U_\alpha\}$ is finite.

A topological space is **metacompact** if every open cover has an open pointwise finite refinement. Every paracompact space is metacompact.

Product related properties

Although a product of paracompact spaces need not be paracompact, the following are true:

(R1). The product of a paracompact space and a compact space is paracompact.

(R2). The product of a metacompact space and a compact space is metacompact.

Both these results can be proved by the tube lemma which is used in the proof that a product of finitely many compact spaces is compact.

Properties and examples

(Pr1). Every compact space is paracompact.

(Pr2). (Theorem of A. H. Stone). Every metric space is paracompact.

(Pr3). (Smirnov metrization theorem). A topological space is metrizable if and only if it is paracompact, Hausdorff, and locally metrizable.

(Pr4). Every regular Lindelöf space is paracompact. In particular, every locally compact Hausdorff second-countable space is paracompact.

(Pr5). The Sorgenfrey line is paracompact, even though it is neither compact, locally compact, second countable, nor metrizable.

(Pr6). (Theorem of Jean Dieudonne'). Paracompact Hausdorff spaces are normal.

(Pr7). If every open subset of a space is paracompact, then it is hereditarily paracompact.

(Pr8). A regular space is paracompact if every open cover admits a locally finite refinement.

1.1.11 k_ω-spaces

A T_2-space X is called a **k-space** if the subset $F \subset X$ is closed in X iff $F \cap \Phi$ is compact for each compact $\Phi \subset X$.

A mapping f of a k-space X to a topological space Y is continuous iff for every compact $\Phi \subset C$ the restriction $f \mid \Phi$ is continuous.

The property of being k-space is additive and is preserved by quotient maps onto Hausdorff spaces.

The sequence $\{\Phi_n : n \in N\}$ of the compact subspaces of a space X is called a k_ω-sequence if:

1. $X = \bigcup\{\Phi_n : n \in N\}$ and $\Phi_n \subseteq \Phi_{n+1}$ for every $n \in N$.
2. A subset $F \subset X$ is closed in X iff $F \cap \Phi_n$ is compact in X, for any $n \in N$.

A T_2-space with a k_ω-sequence is called a **k_ω-space**.

Several properties of k_ω-spaces are of special interest to us. The reader can find their proofs in [5].

Proposition 1.1.7. *Every k_ω-space is a normal Lindelöf space.*

Proposition 1.1.8. *Every k_ω-space is a normal k-space.*

Proposition 1.1.9. *The product of a finite number of k_ω-spaces is a k_ω-space.*

Proposition 1.1.10. *Every locally compact Lindelöf space is a k_ω-space.*

Proposition 1.1.11. *k_ω-property is invariant under quotient maps.*

The property of X of being a space of point-countable type can be generalized to that of being a q-space.

A topological space X is said to be a **q-space** if every point $x \in X$ has a surrounding countably compact set of countable character.

Other properties of topological spaces can be find in N. Bourbaki [29], R. Engelking [89], J. Kelley [129], K. Kuratowski [146, 147], J. Munkres [166], L. Calmutchi [34], D. Ipate [120].

1.2 Basic Algebraical Constructions

Abstract algebra, sometimes also called modern algebra, in which algebraic structures such as groups, rings and fields are axiomatically defined and investigated. Let us review some basic constructions and terminology from algebra.

1.2.1 Groups

A group is a set, G, together with an operation $(\cdot)$ that combines any two elements a and b to form another element denoted $a\cdot b$. The symbol $(\cdot)$ is a general placeholder for a concretely given operation, such as the addition above. To qualify as a **group**, the set and operation, $(G, \cdot)$, must satisfy four requirements known as the group axioms:

(A1). **Closure**. For all a, b in G, the result of the operation $a * b$ is also in G.

(A2). **Associativity**. For all a, b and $c \in G$, the equation $(ab)c = a(bc)$ holds.

(A3). **Identity element.** There exists an element $e \in G$, such that for all elements $a \in G$, the equation $ea = ae = a$ holds.

(A4). **Inverse element.** For each $a \in G$, there exists an element $b \in G$ such that $ab = ba = e$, where e is the identity element.

Suppose G and $G^{'}$ are groups, written multiplicatively. A **homomorphism** $f : G \to G^{'}$ is a mapping such that $f(x\cdot y) = f(x)\cdot f(y)$ for all $x, y \in G$; it automatically satisfies the equations $f(e) = e^{'}$ and $f(x^{-1}) = f(x)^{-1}$, where e and

e' are the identities of G and G', respectively, and the exponent -1 denotes the inverse. The **kernel** of f is the set $f(e')^{-1}$; it is a subgroup of G. The image of f, similarly, is a subgroup of G'. The **monomorphism** if it is injective (or equivalently, if the kernel of f consists of e alone). It is called an **epimorphism** if it is surjective; and it is called an **isomorphism** if is bijective.

Suppose G is a group and H is a subgroup of G. Let xH denote the set of all products xh, for $h \in H$; it is called a **left coset** of H in G. The collection of all such cosets forms a partition of G. Similarly, the collection of all right cosets Hx of H in G forms a partition of G. We call H a **normal subgroup** of G if $x{\cdot}h{\cdot}x^{-1} \in H$ for each $x \in G$ and each $h \in H$. In this case, we have $xH = Hx$ for each x, so that our two partitions of G are the some. We denote this partition by G/H; if one defines

$$(xH){\cdot}(yH) = (x{\cdot}y)H,$$

one obtain a well-defined operation on G/H that makes it a group. This group is called the **quotient** of G by H. The mapping $f : G \to G/H$ carrying x to xH is an epimorphism with kernel H. Conversely, if $f : G \to G'$ is an epimorphism, then its kernel N is a normal subgroup of G, and induces an isomorphism $G/N \to G'$ that carries xN to $f(x)$ for each $x \in G$.

1.2.2 Rings

A **ring** is a set R equipped with two binary operations $+ : R \times R \to R$ and $\cdot : R \times R \to R$ (where $\times$ denotes the Cartesian product), called addition and multiplication. To qualify as a ring, the set and two operations, $(R, +, \cdot)$, must satisfy the following requirements known as the ring axioms.

A. $(R, +, \cdot)$ **is required to be an Abelian group under addition:**

(A1). **Closure under addition.** For all a, b in R, the result of the operation $a + b$ is also in R.

(A2). **Associativity of addition.** For all a, b and c in R, the equation $(a+b)+c = a + (b + c)$ holds.

(A3). **Existence of additive identity.** There exists an element $0 \in R$, such that for all elements $a \in R$, the equation $0 + a = a + 0 = a$ holds.

(A4). **Existence of additive inverse.** For each $a \in R$, there exists an element $b \in R$ such that $a + b = b + a = 0$

(A5). **Commutativity of addition.**For all $a, b \in R$, the equation $a + b = b + a$ holds.

B. $(R, +, \cdot)$ **is required to be a monoid under multiplication:**

(M1). **Closure under multiplication.** For all $a, b \in R$, the result of the operation $a \cdot b$ is also in R.

(M2). **Associativity of multiplication.** For all a, b, and c in R, the equation $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ holds.

(M3). **Existence of multiplicative identity.** There exists an element $1 \in R$, such that for all elements $a \in R$, the equation $1 \cdot a = a \cdot 1 = a$ holds.

C. The distributive laws:

(D1). **For all** a, b **and** c **in** R**, the equation** $a \cdot (b + c) = (a \cdot b) + (a \cdot c)$ **holds**.

(D2). **For all** a, b **and** c **in** R**, the equation** $(a + b) \cdot c = (a \cdot c) + (b \cdot c)$ **holds**.

Rings that satisfy the ring axioms as given above but do not contain a multiplicative identity are called **pseudo-rings**. Rings which do have multiplicative identities (and also satisfy the above axioms) are sometimes referred to **unitary rings, or simply rings with unity or rings with identity**. The term **ring** (jocular; ring without the multiplicative identity) is also used for such rings.

A homomorphism of rings is a function $f : R_1 \to R_2$ between two rings R_1 and R_2 preserving the ring operations, in the sense that for all $a, b \in R_1$ the following identities are required to hold:

(H1). $f(a + b) = f(a) + f(b)$;

(H2). $f(a \cdot b) = f(a) \cdot f(b)$;

(H3). $f(1_{R_1}) = 1_{R_2}$.

As in any category a map f possessing an inverse $g : R_2 \to R_2$, i.e. a mapping in the opposite direction such that the two compositions $f \circ g$ and $g \circ f$ equal the identity mapping of R_2 and R_1, respectively, is called an **isomorphism**. Equivalently, f is bijective.

Informally, a **subring** is a ring, $(S, +, \cdot)$, contained in a bigger one, $(R, +, \cdot)$. More formally, let $(R, +, \cdot)$ be a ring. A subset S of R is said to be a subring of R if:

(Su1). For every $a, b \in S$, $a + b$ is in S.

(Su2). For every $a, b \in S$, $a \cdot b$ is in S.

(Su3). For every $a \in S$, $-a$ (the additive inverse of $a \in R$) is in S.

(Su4). The multiplicative identity of R is in S, i.e 1 is in S.

If S is a subring of R, then S is a ring in its own right with $+$ and $\cdot$ restricted to the cartesian product $S \times S$. Suppose $f : R_1 \to R_2$, is a ring morphism. Then the image, $f(R_1)$ is a subring of R_2.

The purpose of an ideal in a ring is to somehow allow one to define the quotient ring of a ring. An **ideal** in a ring can therefore be thought of as a generalization of a normal subgroup in a group. More formally, let $(R, +, \cdot)$ be a ring. A subset I of R is said to be a **right ideal** in R if:

(RI1). $(I, +)$ is a subgroup of the underlying additive group in $(R, +, \cdot)$ (i.e $(I, +)$ is a subgroup of $(R, +)$).

(RI2). For every $x \in I$ and $r \in R$, $x \cdot r$ is in I.

A **left ideal** is similarly defined with the second condition being replaced. More specifically, a subset I of R is a left ideal in R if:

(LI1). $(I, +)$ is a subgroup of the underlying additive group in $(R, +, \cdot)$ (i.e $(I, +)$ is a subgroup of $(R, +)$).

(LI2). For every $x \in I$ and $r \in R$, $r \cdot x$ is in I.

Notes

(N1). If k is in R, then $k \cdot R$ is a right ideal in R, and $R \cdot k$ is a left ideal in R. These ideals (for any $k \in R$) are called the principal right and left ideals generated by k.

(N2). If every ideal in a ring $(R, +, \cdot)$ is a principal ideal in $(R, +, \cdot)$, $(R, +, \cdot)$ is said to be a principal ideal ring.

(N3). An ideal in a ring, $(R, +, \cdot)$, is said to be a two-sided ideal if it is both a left ideal and right ideal in $(R, +, \cdot)$. It is preferred to call a two-sided ideal, simply an ideal.

(N4). If $I = 0$ (where 0 is the additive identity of the ring $(R, +, \cdot)$), then I is an ideal known as the trivial ideal. Similarly, R is also an ideal in $(R, +, \cdot)$ called the unit ideal.

Examples

(E1). Any additive subgroup of the integers is an ideal in the integers with its natural ring structure.

(E2). There are no non-trivial ideals in R (the ring of all real numbers) (i.e., the only ideals in R are 0 and R itself). More generally, a field cannot contain any non-trivial ideals.

(E3). From the previous example, every field must be a principal ideal ring.

(E4). A subset, I, of a commutative ring $(R, +, \cdot)$ is a left ideal if and only if it is a right ideal. So for simplicity's sake, we refer to any ideal in a commutative ring as just an ideal.

Quotient ring

The quotient ring of a ring, is a generalization of the notion of a quotient group of a group. More formally, given a ring $(R, +, \cdot)$ and an two-sided ideal I of $(R, +, \cdot)$, the quotient ring (or factor ring) R/I is the set of cosets of I (with respect to the underlying additive group of $(R, +, \cdot)$; i.e cosets with respect to $(R, +)$) together with the operations:

$(a+I)+(b+I)=(a+b)+I$ and $(a+I)(b+I)=(a\cdot b)+I$,
for every $a, b \in R$.

Direct product of rings

Let R and S be rings. Then the product $R \times S$ can be equipped with the following natural ring structure:

(P1). $(r1, s1)+(r2, s2)=(r1+r2, s1+s2)$

(P2). $(r1, s1)\cdot(r2, s2)=(r1\cdot r2, s1\cdot s2)$,

for every $r1, r2 \in R$ and $s1, s2 \in S$.

The ring $R \times S$ with the above operations of addition and operation (derived from R and S) is called the direct product of R with S.

A ring R is called **Boolean ring** if every ring element a is an idempotent, i.e. $a \cdot a = a$. For example, the (unique) ring with two elements has this property. There is a one-to-one correspondence between Boolean algebras and Boolean rings, by expressing set difference and set intersection in terms of addition and multiplication and vice versa. Any Boolean ring is commutative and of characteristic two, i.e. $a+a=0$ for all $a \in R$.

Let (X, τ) is a topological space and $(X, +, \cdot)$ be a ring. Then $(X, \tau, +, \cdot)$ is said to be a topological ring, if its ring structure and topological structure are both compatible (i.e work together) over each other. That is, the addition mapping $+ : X \times X \to X$ and the multiplication mapping $\cdot : X \times X \to X$ have to be both continuous as maps between topological spaces where $X \times X$ inherits the product topology. So clearly, any topological ring is a topological group (under addition).

Examples

(E1). The set of all real numbers, R, with its natural ring structure and the standard topology forms a topological ring.

(E2). The direct product of two topological rings is also a topological ring.

1.2.3 Topological Groups

A **topological group** is a group G together with a topology on G such that the group's binary operation and the group's inverse mapping are continuous.

Formally, in this sense, a topological group G is a topological space and group such that the group operations $G \times G \to G : (x, y) \mapsto xy$ and $G \to G : x \mapsto x^{-1}$ are continuous mapping.

Here, $G \times G$ is viewed as a topological space by using the product topology.

Suppose G and $G^{'}$ are topological groups, written multiplicatively. A **homomorphism** $f : G \to G^{'}$ is just continuous group homomorphism f. An **isomorphism** of topological groups is a group isomorphism which is also a homeomorphism of the underlying topological spaces. This is stronger than simply requiring a continuous group isomorphism-the inverse must also be continuous. There are examples of topological group which are isomorphic as ordinary groups but not as topological groups. Indeed, any indiscrete topological group is also a topological group when considered with the discrete topology. The underlying groups are the same, but as topological groups there is not an isomorphism.

Every **subgroup** of a topological group is itself a topological group when given the subspace topology. If H is a subgroup of G the set of left or right cosets G/H is a topological space when given the **quotient topology** (the finest topology on G/H which makes the natural projection $p : G \to G/H$ continous). One can show that the quotient mapping $p : G \to G/H$ is always open.

If H is a normal subgroup of G, then the factor-group, G/H becomes a topological group when given the quotient topology. However, if H is not closed in the topology of G, then G/H will not be T_0 even if G is.

A topological group is said to be **connected, totally disconnected, compact, locally compact**, etc., if the corresponding property holds for its underlying topological space. The connected component of the identity G^0 is the largest connected closed subgroup of G. The quotient group G/G^0 is totally disconnected. A locally compact totally-disconnected group has an open compact subgroup. If G is a compact totally-disconnected group, then every neighborhood of the identity contains an open normal subgroup of G.

Every topological group is a uniform space in a natural way. Specifically, a left uniform group structure on a topological group G is defined by the collection of sets

$$L(U) = \{(x, y) \in G \times G : x^{-1}y \in U\},$$

where U runs over a system of neighborhoods of the identity in G; a right structure is defined by symmetry. The topology arising from the uniform structure is the same as the original topology on the group. The existence of a uniform structure on a topological group allows one to introduce and apply the notions of uniform continuity, Cauchy sequences, completeness, and completion. A locally compact topological group is complete in its uniform structure.

1.2.4 The Fundamental Groups

Now we define a certain operation on path-homotopy classes as follows:

Definition. *If f is a path in X from x_0 to x_1, and if g is a path in X from x_1 to x_2, we define the product $f * g$ of f and g to be the path given by the equations: $h(s) = f(2s)$ for $s \in [0, 2^{-1}]$ and $h(s) = g(2s - 1)$ for $s \in [2^{-1}, 1]$.*

The function h is well-defined and continuous. We think of h as the path whose first half is the path f and whose second half is the path g.

The product operation on path induces a well-defined operation on path-homotopy classes, defined by the equation

$$[f] * [g] = [f * g].$$

The operation $(*)$ on path-homotopy classes turns out to satisfy properties that look very math like axioms for a group. They are called the **groupoid properties of** $(*)$. One difference from the properties of a group is that $[f] * [g]$ is not defined for every pair of classes, but only for those pairs $[f]$,$[g]$ for which $f(1) = g(0)$.

Theorem. *The operation $(*)$ has the following properties:*

1. *(Associativity). If $[f] * ([g] * [h])$ is defined, so is $([f] * [g]) * [h]$, and they are equal.*

2. *(Right and left identities). Given $x \in X$, let e_x denote the constant path $e_x : I \to X$ carrying all of I to the point x. If f is a path in X from x_0 to x_1, then $[f] * [e_{x_1}] = [f]$ and $[e_{x_0}] * [f] = [f]$.*

3. *(Inverse). Given the path f in X from x_0 to x_1, let $f^{'}$ be the path defined by $f^{'} = f(1-s)$. It is called the* ***reverse*** *of f. Then* $[f] * [f^{'}] = [e_{x_0}]$ *and* $[f^{'}] * [f] = [e_{x_1}]$.

Definition. *Let X be a space; let x_0 be a point of X. A path in x begins and ends in at x_0 is called* ***a loop*** *based'in x_0. The set of path homotopy classes of loops based at x_0 with the operation $(*)$ is called the* ***fundamental group*** *of X relative to the base point x_0. It is denoted by $\pi_1(X, x_0)$ or simply $\pi(X, x_0)$.*

It follows from Theorem that the operation $(*)$, when restricted to this set, satisfies the axioms for a group. Given two loops f and g based at x_0, the product $f * g$ is always defined and is a loop based in x_0. Associativity, the existence of an identity element e_{x_0}, and the existence of an inverse $f^{'}$ for f are immediate.

Sometimes this group is called the **first homotopy group** of X, which term implies that there is a second homotopy group. Indeed, there are groups $\pi_n(X, x_0)$ for all $n \in Z_+$.

A path-connected topological space X is simply-connected if and only if $\pi_1(X, x)$ is trivial for all $x \in X$.

Let S^1 be an unit circle in R^2. In this case $\pi_1(S^1, b) \cong Z$ for any $b \in S^1$. So the fundamental group of the circle is isomorphic to $(Z, +)$, the additive group of integers. This fact can be used to give proofs of the Brouwer fixed point Theorem and the Borsuk-Ulam Theorem in dimension 2.

Since the fundamental group is a homotopy invariant, the theory of the winding number for the complex plane minus one point is the same for the circle.

1.2.5 Universal Algebras

The discrete sum $E = \oplus\{E_n : n \in N = 0, 1, \ldots\}$ of the pairwise disjoint topological spaces $\{E_n : v \in N\}$ is called the continuous signature. If E is a discrete space then the signature E is said to be discrete.

Definition 1.2.1. *An E-algebra or an universal algebra of the signature E is a family $\{G, e_{nG} : n \in N\}$ for which:*

1. *G is a nonempty set.*

2. *$e_{nG} : E_n \times G^n \longrightarrow G$ is a mapping for every $n \in N$.*

The set G is called the support of the E-algebra and the mappings e_{nG} are called an algebraical structure on G.

The signature E is the set of symbols of operations.

Definition 1.2.2. *An E-algebra G together with a given topology on it is called a topological E-algebra if all the mappings $e_{nG} : E_n \times G^n \to G$ are continuous.*

Consider an E-algebra A and $\{e_{nA} : E_n \times A^n \to A : n \in N\}$ is the algebraical structure on A. If B is a nonempty subset of A and $e_{nA}(E_n \times B^n) \subseteq B$ for every $n \in N$, then B is called a subalgebra of A and $\{e_{nB} = e_{nA} \mid E_n \times A^n : n \in N\}$ is the algebraical structure on B.

Example 1.2.3. *Consider a nonempty set A. We fix a sequence $\{a_n \in A : n \in N\}$. We construct the mapping $e_{nA} : E_n \times A^n \to A$ such that $e_{nA}(E_n \times A^n) \subseteq \{a_n\}$. Let $N' = \{n \in N : E_n \neq 0\}$. Then $e_{nA}(E_n \times A^n) = \{a_n\}$ for every $n \in N'$ and $e_{nA}(E_m \times A^m) = \emptyset$ for every $n \in N'$. Let $B \subset A$. Then B is subalgebra of A iff $B \neq 0$ and $\{a_n : n \in N'\} \subset B$.*

Definition 1.2.4. *A mapping $f : A \to B$ of an E-algebra A into an E-algebra B is said to be a homomorphism if*

$$f(e_{0A}(\{a\} \times A^0)) = e_{0B}(\{a\} \times B^o)$$

for any $a \in E_0$ and

$$f(e_{nA}(\omega, x_1, \ldots, x_n)) = e_{nB}(\omega, f(x_1), \ldots, f(x_n))$$

for any $n > 0$, $\omega \in E_n$ and $x_1, \ldots, x_n \in A$.

If f is one-to-one and f are homomorphism, then f is called an isomorphism. In this case f^{-1} is an isomorphism, too.

A mapping $f : A \to B$ of a topological E-algebra A into a topological E-algebra B is said to be a continuous homomorphism if:

1. f is a homomorphism;

2. f is a continuous mapping of a topological space A into topological space B.

If the mapping f is an isomorphism and a homeomorphism then f is said to be a topological isomorphism.

Consider a nonempty family $\{A_\alpha : \alpha \in L\}$ of E-algebras. Let $A = \Pi\{A_\alpha : \alpha \in L\}$ and $A^n = \Pi\{A^n_\alpha : \alpha \in L\}$. We consider the mappings $e_{nA} : E_n \times A^n \to A$ with $e_{0A}(\{\omega\} \times A^0) = (e_{0A_\alpha}(\{\omega_\alpha\}) \times A^0_\alpha : \alpha \in L)$ for all $\omega \in E_0$ and $e_{nA}(\omega, x) = (e_{nA_\alpha}(\omega, x_\alpha) : \alpha \in A)$ for every $n > 0$, $\omega \in E_n$ and $x = \{x_\alpha \in A^n_\alpha : \alpha \in L\} \in A^n$. The set A with the mappings e_{nA} is called a Cartesian product of the algebras A_α. The natural projections $\Pi_\alpha : A \to A_\alpha$ are homomorphisms. If $L = \emptyset$, then $\mid \Pi\{A_\alpha : \alpha \in L\} \mid = 1$.

The Tychonoff product of topological E-algebras is a topological E-algebra. Any topological space is said to be a T_{-1}-space.

If $-1 \leqslant i < j \leqslant 3.5$, then every T_j-space is also a T_i space.

If a topological E-algebra G is a T_i-space, then G is called a $T_i - E$-algebra, for $i \in \{-1, 0, 1, 2, 3, 3\frac{1}{2}\}$.

Definition 1.2.5. *A class K of E-algebras is said to be a variety if the following conditions are fulfield:*

1. *K is closed with respect to Cartesian products.*
2. *K is closed with respect to subalgebras.*
3. *K is closed with respect to homomorphic images.*

However, if the last condition is not satisfied then K is said to be a quasivariety.

Definition 1.2.6. *A class K of topological E-algebras which are also T_i-spaces is called a complete T_i-variety if the following conditions are fulfilled:*

(M1). K is closed with respect to Tychonoff products.

(M2). K is closed with respect to subalgebras.

(M3). If $(G, \tau) \in K$ and (G, τ') is a topological E-algebra and also a T_i-space, then $(G, \tau') \in K$.

(M4). If $(G, \tau) \in K$ and (G', τ') is a topological E-algebra, T_i-space and there exists a continuous homomorphism $f : G \overset{onto}{\to} G'$, then $(G', \tau') \in K$.

A class K of topological E-algebras is called:

- T_i-quasivariety if conditions M1, M2 hold;

- T_i-variety if conditions M1, M2, M4 hold;
- complete T_i-quasivariety if conditions M1 - M3 hold;

Fix a continuous signature E. Let G be an E-algebra, $L \subseteq E$, and $H \subseteq G$. We set:

$$
\begin{aligned}
d_0(L, H) &= H; \\
d_1(L, H) &= H \cup (\cup\{e_{nG}((L \cap E_n) \times H^n) : n \in N\}); \\
d_{n+1}(L, H) &= d_1(L, d_n(L, H)); \\
d(L, H) &= \cup\{d_n(L, H) : n \in N\}.
\end{aligned}
$$

If $H \neq \emptyset$ and $L = E$, then $d(E, H)$ is an E-subalgebra of G generated by H. If $d(E, H) = G$, then the set H algebraically generates the E-algebra G.

Definition 1.2.7. *We fix a T_i-quasivariety K of topological E-algebras and a nonempty space X. A couple $(F(X, K), i_X)$ is called a topological free algebra of a space X in the class K if the following conditions hold:*

(T1). $F(X, K) \in K$.

(T2). $i_X : X \to F(X, K)$ *is a continuous mapping;*

(T3). The set $i_X(X)$ *algebraically generates* $F(X, K)$.

(T4). For each continuous mapping $f : X \to G \in K$ *there exists a continuous homomorphism* $\hat{f} : F(X, K) \to G$ *such that* $f(x) = \hat{f}(i_X(x))$ *for every* $x \in X$. *The homomorphism* $\hat{f}$ *is called the homomorphism generated by* f.

Definition 1.2.8. *We fix a T_i-quasivariety K of topological E-algebras and a nonempty space X. A couple $(F^a(X, K), a_X)$ is called an algebraically free algebra of a space X in the class K if the following conditions hold:*

(A1). $F^a(X, K) \in K$.

(A2). $a_X : X \to F^a(X, K)$ *is a mapping;*

(A3). The set $a_X(X)$ *algebraically generates* $F^a(X, K)$.

(A4). For each mapping $f : X \to G \in K$ *there exists a homomorphism* $\tilde{f} : F^a(X, K) \to G$ *such that* $f(x) = \tilde{f}(a_X(x))$ *for every* $x \in X$.

Theorem 1.2.9. (see [56, 57, 103].) *Let K be a T_i-quasivariety of topological E-algebras. Then for every nonempty space X there exist:*
– a unique algebraically free E-algebra $(F^a(X,K), a_X)$;
– a unique topologically free E-algebra $(F(X,K), i_X)$;
– a unique continuous homomorphism $k_X : F^a(X,K) \to F(X,K)$ with $i_X = k_X \cdot a_X$.

We fix $E = \oplus\{E_n : n \in N = \{0,1,2,\dots\}\}$. If $\omega \in E_0$ and G is an E-algebra, then $e_{0G}(\{\omega\} \times G^0) = 1_\omega$. If $n \in N$ and $\omega \in E_n$, then $\omega : G^n \to G$, where $\omega(x_1,\dots,x_n) = e_{nG}(\omega, x_1,\dots,x_n)$ is an operation of type n on G.

The set of terms $T(E)$ is the smallest class of the operations on the E-algebras such that:

1. $E \subseteq T(E)$ and $e_G \in T(E)$, where $e_G(x) = x$ for every $x \in G$.
2. If $n > 0$, $e \in E_n$ and $u_1, \dots, u_n \in T(E)$, then $e(u_1,\dots,u_n) \in T(E)$. The type of $e(u_1,\dots,u_n)$ is equal to the sum of types of terms $u_1, \dots, u_n$.

Let $1 \leqslant m \leqslant n$, $N_m = \{1,2,\dots,m\}$ and $h : N_m \overset{\text{onto}}{\to} N_n$ be a mapping. The operation $\omega : G^n \to G$ of type n and h generate the operation $\Psi : G^m \to G$, where $\Psi(x_1,\dots,x_m) = \omega(x_{h(1)},\dots,x_{h(n)})$, and Ψ is called an h-permutation of the operation ω. The set of the polynomials $P(E)$ or of the derived operations is the smallest class of operations on E-algebras such that:

1. $T(E) \subseteq P(E)$.
2. If $f \in P(E)$ and g is an h-permutation of f, then $g \in P(E)$.

If G is a topological E-algebra and ω is a polynomial of type n, then the mapping $\omega : G^n \to G$ is continuous.

Denote by $P_n(E)$ the polynomials of the type n, where $n \in N$. If ω_1 and ω_2 are polynomials of types n and m, then the form
$\omega_1(x_1,\dots,x_n) = \omega_2(y_1,\dots,y_m)$ is called an identity on the class of E-algebras.

Let us consider the identity $\omega_1(x_1,\dots,x_n) = \omega_2(y_1,\dots,y_m)$.

If $y_i \notin \{x_1,\dots,x_n\}$, then y_i is called a free variable of the identity $\omega_1 = \omega_2$.

If $1 \leqslant i \leqslant n$ and

$$p(x_1, \ldots, x_{i-1}, x, x_{i+1}, \ldots, x_n) = p(x_1, \ldots, x_{i-1}, y, x_{i+1}, \ldots, x_n)$$

for every x, y, x_1, $\ldots$, x_{i-1}, x_{i+1}, $\ldots$, $x_n \in G$, then x_i is a free variable of the algebra G.

Example 1.2.10. *Let $E_0 = \{1\}$, $E_1 = \{^{-1}\}$, $E_2 = \{\cdot\}$ and $E = E_0 \cup E_1 \cup E_2$. The operation $1 : G^0 \to G$ fixes the unit 1_G in G. The operation $^{-1} : G \to G$ determines the inverse element x^{-1}. The operation $\cdot : G^2 \to G$ determines the product $x \cdot y$ of $x, y \in G$. The operation $(xy) \cdot (zt)$ is a polynomial of type 4. The operation $(xy) \cdot (zt) \cdot (yt)$ is a polynomial of type 6. An E-algebra G with the identities*:

a. $1_G \cdot x = x \cdot 1_G = x$, $x \cdot x^{-1} = x^{-1} \cdot x = 1_G$ *is called a IP-loop.*

b. $(x \cdot y) \cdot t = x \cdot (y \cdot t)$, $1_G \cdot x = x \cdot 1_G = x$, $x \cdot x^{-1} = x^{-1} \cdot x = 1_G$ *is called a group.*

c. $(x \cdot y) \cdot t = x \cdot (y \cdot t)$, $x \cdot y = y \cdot x$, $1_G \cdot x = x \cdot 1_G = x$, $x \cdot x^{-1} = x^{-1} \cdot x = 1_G$ *is called an Abelian group.*

Example 1.2.11. *Let $E_0 = \{0, 1\}$, $E_1 = \{-\}$, $E_2 = \{+, \cdot\}$ and $E = E_0 \cup E_1 \cup E_2$. The E is a signature of rings with unit.*

Example 1.2.12. *Let $E_2 = \{A, B, C\}$ and $E = E_2$. An E-algebra G with the identities $A(x, B(x, y)) = A(C(y, x), x) = B(x, A(x, y)) = C(A(y, x), x) = C(x, B(y, x)) = B(C(x, y), x) = y$ is called a quasigroup.*

In [91] T.H. Fay, E.T. Ordman, B.V.S. Thomas formulated the following problem: is the free topological group over a locally compact space necessarily a k-space? The authors established that the free topological group over the space of rationales Q is not a k-space and made a general assumption that the free topological group cannot be a k-space once the space of generators is not locally compact. In [84] S. Dumitrascu shows that for metrizable spaces the assumption is true. In [130] we give the necessary and sufficient conditions for a free topological algebra with Mal'cev conditions to be a k-space.

In [132] we explore the problem on the quasicomponents of the topological algebra in a given variety of topological algebras. In the case of the variety of topological groups or of topological Abelian groups the problem of the quasicomponents was studied by M.I. Craev [103] and V.V. Tkaciuk [200]. We mention that the structure of the components of topological groups and semigroups was studied in [206, 209] by M.I. Ursul and A.S. Iunusov.

Other properties of algebraical structures can be find in A. Arhangel'ski, M. Tkacenko [13], N. Bourbaki [30], M. Choban [48], G. Gratzer [106, 107], G. Grothendieck [108], M. Hall [111], B.A. Hausmann, O. Ore [112], T. Husain [116], M. Kargapalov, Yu. Merzleakov [127], A. G. Kurosh [148, 149], R. Miron, I. Pop [168], H. Neumann [172], L. Skorneakov [187], J.D.H. Smith [188, 189], M. Stefanescu [195].

1.3 On methods of constructing some free universal algebras and solving of equations over them

In this section we examine the solvability of equations over universal and free universal algebras. We also present methods of constructing some free objects. The construction of a free object in the class $V(\varphi E)$ and the results from section 1.3.5 were also obtained by M.M. Choban using other techniques [54]. The main result establishes that each of the equations $ax = b$ and $ya = b$ over a free primitive gruppoid has no more than two solutions. Here, we use the terminology from [16, 17, 20, 42]. The results of this section were published in [133].

1.3.1 General definitions

Fix a signature $E = \oplus\{E_n : n = \{0, 1, 2, \dots\}\}$. We refer to a set A with a sequence of mappings $\{e_{nA} : E_n \times A^n \to A : n \in N\}$ as to a universal E-algebra (or simply - an E-algebra). If $\omega \in E_n$ and $1 \leqslant i \leqslant n$, then the expression $\omega(a_1, \dots, a_{i-1}, x, a_{i+1}, \dots, a_n) = a_i$ is called an equation in the class of all E-algebras. Here $\omega(a_1, \dots, a_n) = e_{nA}(\omega, a_1, \dots, a_n)$. Now we consider a system of equations

$$\varphi = \{\omega_\alpha(a_1, \dots, a_{i_\alpha - 1}, x_\alpha, a_{i_\alpha + 1}, \dots, a_{n_\alpha}) = a_{i_\alpha} : \alpha \in L\}, \qquad (\varphi)$$

where $\{\omega_\alpha : \alpha \in L\} \subset \cup\{E_n : n = 1, 2, \dots\}$.

We say that the equations:

$$\omega_\alpha(a_1, \dots, a_{i_\alpha - 1}, x_\alpha, a_{i_\alpha + 1}, \dots, a_{n_\alpha}) = a_{i_\alpha} \qquad (1.1)$$

$$\omega_\beta(b_1, \ldots, b_{i_\beta - 1}, x_\beta, b_{i_\beta+1}, \ldots, b_{n_\beta}) = b_{i_\beta} \quad (1.2)$$

are identical if $\omega_\alpha = \omega_\beta$, $n_\alpha = n_\beta$, $i_\alpha = i_\beta$, and $a_i = b_i$ for any $i \leq n_\alpha$.

The equations (1.1) and (1.2) are therefore different if at least one of the following conditions is true:

1. $\omega_\alpha \neq \omega_\beta$;

2. $\omega_\alpha = \omega_\beta,\ i_\alpha \neq i_\beta$;

3. $\omega_\alpha = \omega_\beta,\ i_\alpha\ = i_\beta$ and $a_i \neq b_i$ for some $i \leq n_\alpha$.

The equations (1.1)and (1.2) are contradictory if:

1. $\omega_\alpha = \omega_\beta$;

2. $i_\alpha\ = i_\beta$;

3. $a_{i_\alpha} \neq b_{i_\alpha}$

4. $a_i = b_i$ for some $i \leq n_\alpha$ and $i = i_\alpha$.

For commutative $n-$ary operations appear other type of contradictory equations. We assume that the system of equations (φ) does not contain contradictory equations.

We denote by $V(E)$ the class of all E-algebras and by $V(E, \varphi)$ the class of all E-algebras in which equations (φ) can be solved.

Let $V(E, u\varphi)$ be the class of all algebras in which the equations from (φ) have exactly one solution. Let $\varphi E_n = E_n \cup L_n$, where $L_n = \{\alpha \in L : n_\alpha = n\}$, and $\varphi E = E \cup L$. Additionally, we define for each $\alpha \in L$ an n_α-ary operation $\alpha : A^{n_\alpha} \to A$ for which

$$\omega_\alpha(x_1, \ldots, x_{i_\alpha - 1}, \alpha(x_1, \ldots, x_{i_\alpha}, \ldots, x_{n_\alpha}), x_{i_\alpha+1}, \ldots, x_{n_\alpha}) = x_{i_\alpha}$$

We say that the operation α as the i_α-inverse operation for ω_α. Denote by x_k^n a sequence $\{x_k, \ldots, x_n\}$, where $n > k$.

All E-algebras from $V(E, \varphi)$ turn into φE-algebras. Let $V(\varphi E)$ be the class of all φE-algebras for which the following equalities are true:

$$\{\omega_\alpha(x_1^{i_\alpha - 1}, \alpha(x_1^{n_\alpha}), x_{i_\alpha+1}^{n_\alpha}) = x_{i_\alpha} : \alpha \in L\}.$$

The classes $V(\varphi E)$ and $V(E, \varphi)$ are the same as E-algebras. Let

$$V(\varphi E, u) = \{A \in (V(\varphi E) \cap V(E, u\varphi))\}.$$

Let us refer to algebras from $V(\varphi E, u)$ as to $Q - \varphi E$-algebras. The classes $V(E, u\varphi)$ and $V(\varphi E, u)$ are the same as E-algebras. Notice that it is not difficult to prove that the classes $V(\varphi E, u)$ and $V(\varphi E)$ are varieties of φE-algebras. The homomorphisms in these classes are called the φ-homomorphisms. The class $V(E)$ is a variety of E-algebras.

1.3.2 Free E-algebras in the class $V(E)$

Let E be a signature and X be a non-empty set. Let $\Gamma_0(X, E) = X \oplus E_0$. Words from $\Gamma_0(X, E)$ are words of rank zero. Denote the rank $r(x) = 0$ for all $x \in \Gamma_0(X, E)$. Consider the set $\Gamma_k(X, E)$ of all words with the rank not greater than k. Let us define:

$$\begin{aligned}\Gamma_{k+1}(X, E) =&\{\omega(x_1, \ldots, x_n) : \omega \in E_n : n \geqslant 1 : k \in \{r(x_1), \ldots, r(x_n)\}\}\cup \\ &\cup \Gamma_k(X, E).\end{aligned}$$

Then the words from $\Gamma_{k+1}(X, E) \backslash \Gamma_k(X, E)$ have the rank $k+1$. In this way, by induction the words of rank k are defined. The symbol of the word of the rank k are the form $\omega(u_1, \ldots, u_n)$, where $n \geqslant 1$, $\omega \in E$, and $(k-1) = sup\{r(u_1), \ldots, r(u_n)\}$. We say that the operation $\omega \in E$ and the words $u_1, u_2, \ldots, u_n$ generated the word $\omega(u_1, \ldots, u_n)$.

Each word consists from letters. If $x \in \Gamma_0(X, E)$, then the word x consists from letter x. If $n \geqslant 1$, $\omega \in E_n$, and $x_1, \ldots, x_n \in \Gamma_0(X, E)$, then the word $\omega(x_1, \ldots, x_n)$ consists from letters $x_1, \ldots, x_n$ in this exact order. A letter can be repeated but we will formally say that letters with different indexes in the sequence are distinct. If $u_1, \ldots, u_n$ are words of the ranks not greater than $m-1$, then $\omega(u_1, \ldots, u_n)$ consists from all letters from the words $u_1, u_2, \ldots, u_n$. In this word the letters from u_1 are taken first, then the letters from u_2 and so on. The length of the word $l(u)$ is the total number of the letters in it. We mention, that the identical letters from the different words or by the distinct position are considered distinct.

Each word has subwords and each subword is of a concrete rank. Words of rank 0 have no proper subwords. Subwords of rank 0 are letters following a definite order. Subwords of the word $u = \omega(u_1, \dots, u_n)$ are the words $u_1, \dots, u_n$ in this exact order. They define the word u. Subwords of the rank k of the word u are all subwords of the rank k in words $u_1, \dots, u_n$. The order of these subwords is defined as follows:in the first we take the subwords of the rank k of the word u_1 in the order from u_1, then the subwords of the rank k of the word u_2 and so on. The identical words with different numbers in the sequence are considered different.

It can be easily shown that the set $\Gamma(X, E) = \cup\{\Gamma_n(X, E) : n = 0, 1, \dots\}$ is the free E-algebra over the set X in the class $V(E)$. The following conditions are true:

1. The set X generated algebraically the algebra $\Gamma(X, E)$;

2. For each algebra $A \in V(E)$ and a mapping $f : X \to A$ there exists a homomorphism $\hat{f} : \Gamma(X, E) \to A$ such that $\hat{f} \mid X = f$.

1.3.3 Free φE-algebras in the class $V(\varphi E)$

Let E be the fixed signature and

$$\varphi = \{\omega_\alpha(a_1, \dots, a_{i_\alpha - 1}, x_\alpha, a_{i_\alpha + 1}, \dots, a_{n_\alpha}) = a_{i_\alpha} : \alpha \in L\}$$

be a fixed family of equation. Let us also fix a nonempty set X. Using letters from $E_0 \oplus X$ and the operations φE as in section 1.3.2. we construct the words $F(X, \varphi E)$. For these words their rank, length, and subwords are well defined. The word of the form

$$\omega_\alpha(u_1, \dots, u_{i_\alpha - 1}, \alpha(u_1, \dots, u_{n_\alpha}), u_{i_\alpha + 1}, \dots, u_{n_\alpha}),$$

where $u_1, \dots, u_{n_\alpha}$ are words, is called a φ-marked word or, simply, a marked. Denote by S the set of all marked words of any rank. If a word is not marked, then any of its subword is not marked too. We define:

$$\begin{aligned}
F_0(X,\varphi E) &= X \oplus E_0;\\
F_1(X,\varphi E) &= \{\omega(x_1,\dots,x_n),\alpha(x_1,\dots,x_n) : x_1,\dots,x_n \in F_0(X,\varphi E),\\
&\qquad \omega \in E_n, \alpha \in L_n, n \geqslant 1\} \cup F_0(X,\varphi E);\\
F_2(X,\varphi E) &= \big(\{\omega(x_1,\dots,x_n),\alpha(x_1,\dots,x_n) : x_1,\dots,x_n \in F_1(X,\varphi E),\\
&\qquad \omega \in E_n, \alpha \in L_n, n \geqslant 1 : 1 \in \{r(x_1),\dots,r(x_n)\}\} \setminus S\big)\cup\\
&\qquad \cup F_1(X,\varphi E);\\
&\quad \dots
\end{aligned}$$

For each $n \geq 3$ we have:

$$\begin{aligned}
F_n(X,\varphi E) = &\big(\{\omega(x_1,\dots,x_n),\alpha(x_1,\dots,x_n) : x_1,\dots,x_n \in F_{n-1}(X,\varphi E),\\
&\omega \in E_n, \alpha \in L_n, n \geqslant 1 : n-1 \in \{r(x_1),\dots,r(x_n)\}\} \setminus S\big)\cup\\
&\cup F_{n-1}(X,\varphi E).
\end{aligned}$$

Now we put $F(X,\varphi E) = \bigcup\{F_n(X,\varphi E) : n = 0,1,2,\dots\}$.

Theorem 1.3.1. *The set $F(X,\varphi E)$ is a free E-algebra relative to signature φE in the class $V(\varphi E)$, i.e. it satisfies the following conditions:*

1. *The set X algebraically generated the algebra $F(X,\varphi E)$;*
2. *For each mapping $f : X \to A$, where $A \in V(\varphi E)$, there exists a φ-homomorphism $\hat{f} : F(X,\varphi E) \to A$ such that $\hat{f} \mid X = f$.*

Proof. It is clear that $F(X,\varphi E)$ is a φE-algebra in the class $V(\varphi E)$. From its construction it follows that X generates the algebra $F(X,\varphi E)$. Take a mapping $f : X \to A$, where $A \in V(\varphi E)$. If $\omega \in E_0$, then $f(\omega) = e_{0A}(\omega, A^0) \in A$. We put $f_0 = f : F_0(X,\varphi E) \to A$. Now let the mapping $f_n : F_n(X,\varphi E) \to A$ be constructed, where $f_{n-1} = f_n \mid F_{n-1}(X,\varphi E)$, and

$$f_n(\omega(x_1,\dots,x_n)) = \omega(f_n(x_1),\dots,f_n(x_n))$$

for each $x_1, \dots, x_n \in F_{n-1}(X, \varphi E)$ and $\omega \in E_n \cup L_n$. If $x_1, \dots, x_n \in F_n(X, \varphi E)$, $\omega \in E_n \cup L_n$, and $\omega(x_1, \dots, x_n) \in F_{n+1}(X, \varphi E)$, then we define:

$$f_{n+1}(\omega(x_1, \dots, x_n)) = \omega(f_{n+1}(x_1), \dots, f_{n+1}(x_n)).$$

If $\omega(x_1, \dots, x_n) \in S$, then $\omega(x_1, \dots, x_n) = y_i \in F_{n-1}(X, \varphi E)$, and $f_{n+1}(y_i)$ is already defined.

We have constructed the mapping $f_{n+1} : F_{n+1}(X, \varphi E) \to A$ for which $f_n = f_{n+1} \mid F_n(X, \varphi E)$.

Let us now take the mapping $\hat{f} : F(X, \varphi E) \to A$ where $\hat{f} \mid F_n(X, \varphi E) = f_n$ for each $n \in N$. It is clear that $\hat{f}$ is a φ-homomorphism and we are done. The proof is complete.

Under which conditions two words $u, v \in F(X, \varphi E)$ are equal? The answer is simple: two words u and v are equal if and only if the following conditions are satisfied:

1*. Ranks of the words u and v are equal.

2*. For each k, the sequence of subwords of the rank k in u is the same as the sequence of subwords of rank k in v.

3*. The equal subwords from u and v of the same rank and the same position in the respective sequences contains the equal subwords in the same orders.

Example. *Condition 2* is not satisfied for the words $\omega(a, \omega(b, c))$ and $\omega(\omega(a, b), c)$ because $\omega(b, c)$ and $\omega(a, b)$ are different. The words $\omega(a, \omega(a, a))$ and $\omega(\omega(a, a), a)$ do not satisfy the condition 3* because $\omega(a, \omega(a, a))$ is generated from the sequence of subwords a, $\omega(a, a)$, but $\omega(\omega(a, a), a)$ is generated by the words $\omega(a, a)$ and a.*

We mention that the word $x = \alpha(a_1, \dots, a_{i_\alpha - 1}, b, a_{i_\alpha + 1}, \dots, a_{n_\alpha})$ is a solution to the equation:

$$\omega_\alpha(a_1, \dots, a_{i_\alpha - 1}, x, a_{i_\alpha + 1}, \dots, a_{n_\alpha}) = b.$$

for each $b, a_1, \dots, a_{n_\alpha} \in F(X, \varphi E)$.

Nevertheless, a natural question might be arise: **Are there any other solutions except this one?**
For that reason we explain how the rank and the length of the word b depend on the ranks and the lengths respectively of the words $a_1, \dots, a_{n_\alpha}$.

Lemma 1.3.2. *If the word* $b = \omega_\alpha(a_1, \dots, a_{i_\alpha-1}, a_{i_\alpha}, a_{i_\alpha+1}, \dots, a_{n_\alpha})$ *is marked, then* $r(b) < \max\{r(a_1), \dots, r(a_{i_\alpha}), \dots, r(a_{n_\alpha})\}$*, and*
$l(b) < \max\{l(a_1), \dots, l(a_{i_\alpha}), \dots, l(a_{n_\alpha})\}$.

Proof. Let $b = \omega_\alpha(a_1, \dots, a_{i_\alpha-1}, \alpha(a_1, \dots, a_{n_\alpha}), a_{i_\alpha+1}, \dots, a_{n_\alpha})$ for some α and some words $a_1, \dots, a_{n_\alpha}$. Then $b = a_{i_\alpha}$, and a_{i_α} is a subword of the word $\alpha(a_1, \dots, a_{n_\alpha}) = a_{i_\alpha}$.
Therefore $r(b) = r(a_{i_\alpha}) < r(\alpha(a_1, \dots, a_{n_\alpha})) = r(a_{i_\alpha})$. It is obvious that $l(b) < l(a_{i_\alpha})$. The proof is complete.

Lemma 1.3.3. *If the word* $b = \omega_\alpha(a_1, \dots, a_{i_\alpha-1}, a_{i_\alpha}, a_{i_\alpha+1}, \dots, a_{n_\alpha})$ *is not marked, then* $r(b) = 1 + \max\{r(a_1), \dots, r(a_{i_\alpha}), \dots, r(a_{n_\alpha})\}$*, and*
$l(b) = l(a_1) + \dots + l(a_{i_\alpha}) + \dots + l(a_{n_\alpha})$.

*Proof.*The word b consists of the letters from the words $a_1, \dots, a_{n_\alpha}$, and the letters are taken first from a_1, then from $a_2, \dots, a_{i_\alpha-1}, a_{i_\alpha}, a_{i_\alpha+1}, \dots, a_{n_\alpha}$. Therefore it is clear that $r(b) = 1+\max\{r(a_1), \dots, r(a_{n_\alpha})\}$, and $l(b) = l(a_1)+\dots+l(a_{i_\alpha}) + \dots + l(a_{n_\alpha})$. The proof is complete.

Lemma 1.3.4. *If the words* $b_1 = \omega_\alpha(a_1, \dots, a_{i_\alpha-1}, c, a_{i_\alpha+1}, \dots, a_{n_\alpha})$ *and* $b_2 = \omega_\alpha(a_1, \dots, a_{i_\alpha-1}, d, a_{i_\alpha+1}, \dots, a_{n_\alpha})$ *are not marked, and* $b_1 = b_2$*, then* $c = d$.

*Proof.*From the method of construction of the words in $F(X, \varphi E)$ it follows that the words b_1 and b_2 do not contain marked subwords. Therefore it is impossible to change any letters in b_1 and b_2 without changing b_1 and b_2.

The word b_1 is generated by the sequence of subwords
$a_1, \dots, a_{i_\alpha-1}, c, a_{i_\alpha+1}, \dots, a_{n_\alpha}$, and the word b_2 is generated by the sequence of subwords $a_1, \dots, a_{i_\alpha-1}, d, a_{i_\alpha+1}, \dots, a_{n_\alpha}$. Since $b_1 = b_2$ in $F(X, \varphi E)$ by virtue of the conditions 1*–3* we obtain that the generating sequences of subwords of the words b_1 and b_2 are the same. This means that $c = d$. The proof is complete.

Definition. *A word* b *is* i_α*-factorizable over the words sequence* $a_1, \dots, a_{i_\alpha-1}, a_{i_\alpha+1}, \dots, a_{n_\alpha}$ *if there exists a word* d *such that* $b = \omega_\alpha(a_1, \dots, a_{i_\alpha-1}, d, a_{i_\alpha+1}, \dots, a_{n_\alpha})$ *and the word* $\omega_\alpha(a_1, \dots, a_{i_\alpha-1}, d, a_{i_\alpha+1}, \dots, a_{n_\alpha})$ *is not marked.*

From Lemma 1.3.4 it follows that such element d is uniquely defined.

Theorem 1.3.5. *Each equation of the system* (φ) *has no more than two solutions in a free* φE*-algebra* $F(X, \varphi E)$ *of the set* X *in the variety* $V(\varphi E)$.

Proof. Consider the equation $\omega_\alpha(a_1, \dots, a_{i_\alpha-1}, x, a_{i_\alpha+1}, \dots, a_{n_\alpha}) = b$. Obviously, that $x_1 = \alpha(a_1, \dots, a_{i_\alpha-1}, b, a_{i_\alpha+1}, \dots, a_{n_\alpha})$ is a solution of this equation.

Suppose that this equation has some other two solutions $x_2 = c$ and $x_3 = d$. Since $c \neq \alpha(a_1^{i_\alpha-1}, b, a_{i_\alpha+1}^{n_\alpha})$ and $d \neq \alpha(a_1^{i_\alpha-1}, b, a_{i_\alpha+1}^{n_\alpha})$, the words $\omega_\alpha(a_1, \ldots, a_{i_\alpha-1}, c, a_{i_\alpha+1}, \ldots, a_{n_\alpha})$ and $\omega_\alpha(a_1, \ldots, a_{i_\alpha-1}, d, a_{i_\alpha+1}, \ldots, a_{n_\alpha})$ are not marked. Consequently, $\omega_\alpha(a_1^{i_\alpha-1}, c, a_{i_\alpha+1}^{n_\alpha}) = \omega_\alpha(a_1^{i_\alpha-1}, d, a_{i_\alpha+1}^{n_\alpha})$ and by Lemma 1.3.4 we get $c = d$. Therefore each equation from the system (φ) has at most two solutions. The proof is complete.

Remark. *The equation $\omega_\alpha(a_1, \ldots, a_{i_\alpha-1}, x, a_{i_\alpha+1}, \ldots, a_{n_\alpha}) = b$ from the system (φ) has two solutions if the word b is i_α-factorizable over the words sequence $a_1, \ldots, a_{i_\alpha-1}, a_{i_\alpha+1}, \ldots, a_{n_\alpha}$.*

There exists another way to construct a free φE-algebra. We denote $\Phi_0(X, \varphi E) = X \oplus E_0$, and

$$\begin{aligned}\Phi_n(X, \varphi E) = \{\omega(x_1, \ldots, x_n), \alpha(x_1, \ldots, x_n), \omega \in E_n, \alpha \in L_n :\\ x_1, \ldots, x_n \in \Phi_{n-1}(X, \varphi E), n \geqslant 1\} \cup \Phi_{n-1}(X, \varphi E)\end{aligned}$$

for each $n = 1, 2, \ldots.$ Now we put

$$\Phi(X, \varphi E) = \bigcup \{\Phi_n(X, \varphi E) : n \in \{0, 1, 2, \ldots\}\}.$$

We say that in $\Phi(X, \varphi E)$, the word of the form

$$\omega_\alpha(a_1, \ldots, a_{i_\alpha-1}, \alpha(a_1, \ldots, a_{i_\alpha-1}, a_{i_\alpha}, a_{i_\alpha+1}, \ldots, a_{n_\alpha}), a_{i_\alpha+1}, \ldots, a_{n_\alpha}) \qquad (*)$$

is identical to the word a_{i_α}, where $a_1, \ldots, a_{n_\alpha} \in \Phi(X, \varphi E)$.

If the word $b \in \Phi(X, \varphi E)$ does not contain subwords of the kind $(*)$, then it is a called reduced word.

We say that the words a and b are adjacent if one of them is produced from the other by replacing a subword a_{i_α} by subwords of the kind $(*)$.

Two words $f, g \in \Phi(X, \varphi E)$ are equivalent (that is denoted by $f \sim g$) if there exist the words $f_1 = f, f_2, \ldots, f_{m-1}, f_m = g$ such that f_i and f_{i+1} are adjacent words for each $1 \leqslant i < m$.

The relation $f \sim g$ is an equivalence relation. All words from $\Phi(X, \varphi E)$ that are equivalent to the word f define its equivalence class $[f]$.

The following properties can be easily proved:

Property 1. *$[f] \cap F(X, \varphi E)$ is unique. The word $f_0 \in [f] \cap F(X, \varphi E)$ is called the reduced representatione of the class $[f]$.*

Property 2. *The following operations:*

$$\omega_\alpha([a_1], \ldots, [a_{n_\alpha}]) = [\omega_\alpha(a_1, \ldots, a_{n_\alpha})];$$
$$\alpha([a_1], \ldots, [a_{n_\alpha}]) = [\alpha(a_1, \ldots, a_{n_\alpha})];$$

depend only on the multiplied classes $[a_1], \dots, [a_{n_\alpha}]$ but do not depend on their representations.

From these facts it follows that in the result of identification of the equivalent words words in $\Phi(X, \varphi E)$ we obtain classes of equivalent words that coincide with algebra $F(X, \varphi E)$.

1.3.4 Free $Q - \varphi E$-algebra in the class $V(\varphi E, u)$

Let us keep all notations from the previous section. It follows from the theory of n-quasigroups [16, 17] that the uniqueness of solutions of the equations (φ) is equivalent to the following equalities:

$$\omega_\alpha(a_1, \dots, a_{i_\alpha - 1}, \alpha(a_1, \dots, a_{i_\alpha - 1}, a_{i_\alpha}, a_{i_\alpha + 1}, \dots, a_{n_\alpha}), a_{i_\alpha + 1}, \dots, a_{n_\alpha}) = a_{i_\alpha};$$
$$\alpha(a_1, \dots, a_{i_\alpha - 1}, \omega_\alpha(a_1, \dots, a_{i_\alpha - 1}, a_{i_\alpha}, a_{i_\alpha + 1}, \dots, a_{n_\alpha}), a_{i_\alpha + 1}, \dots, a_{n_\alpha}) = a_{i_\alpha},$$

where $\alpha \in L$ and $\{\omega_\alpha : \alpha \in L\} \subset \bigcup\{E_n : n = 1, 2, \dots\}$. The φE-algebras of this kind are called $Q - \varphi E$-algebras. Denote by $V(\varphi E, u)$ the class of all $Q - \varphi E$-algebras . Fix a set X. The words of the kind

$$\omega_\alpha(a_1, \dots, a_{i_\alpha - 1}, \alpha(a_1, \dots, a_{i_\alpha - 1}, a_{i_\alpha}, a_{i_\alpha + 1}, \dots, a_{n_\alpha}), a_{i_\alpha + 1}, \dots, a_{n_\alpha}), \text{ and}$$
$$\alpha(a_1, \dots, a_{i_\alpha - 1}, \omega_\alpha(a_1, \dots, a_{i_\alpha - 1}, a_{i_\alpha}, a_{i_\alpha + 1}, \dots, a_{n_\alpha}), a_{i_\alpha + 1}, \dots, a_{n_\alpha}),$$

are called the $q\varphi$-marked words.
Denote by S_1 the set of all $q\varphi$-marked words of the arbitrary rank. It is clear that $S \subset S_1$. Also, denote

$$Q_n(X, u\varphi E) = F_n(X, u\varphi E) \setminus S_1, \text{ and}$$
$$Q(X, u\varphi E) = \bigcup\{Q_n(X, u\varphi E) : n \in N\} = F(X, u\varphi E) \setminus S_1.$$

It is clear that $Q_i(X, u\varphi E) = F_i(X, u\varphi E)$ if $i = 0, 1$. We will prove that $Q(X, u\varphi E)$ is the desired free $Q - \varphi E$-algebra in the class $V(\varphi E, u)$.

Theorem 1.3.6. *$Q(X, u\varphi E)$ is a free $Q - \varphi E$-algebra in the class $V(\varphi E, u)$ relative to the signature φE, i.e. it satisfies the following conditions:*

1. $X \subset Q(X, u\varphi E)$.

2. If Y is an E-algebra, where $X \subset Y \subset Q(X, u\varphi E)$ and $Y \neq Q(X, u\varphi E)$, then is not a φE-algebra.

3. For each mapping $f : X \to A$, where $A \in V(\varphi E, u)$, there exists a φ-homomorphism $\hat{f} : Q(X, u\varphi E) \to A$ such that $\hat{f} \mid X = f$.

Proof. From its construction it follows that $Q(X, u\varphi E)$ is an E-subalgebra of the algebra $F(X, \varphi E)$. Suppose that $Q(X, u\varphi E)$ is not a $Q - \varphi E$-algebra. Then there exist such words $a_1, \ldots, a_{i_\alpha - 1}, a_{i_\alpha + 1}, \ldots, a_{n_\alpha}, b$ of the lowest rank for which one of equations from the system (φ) does not have a unique solution. Assume that the equation

$$\omega_\alpha(a_1, \ldots, a_{i_\alpha - 1}, x, a_{i_\alpha + 1}, \ldots, a_{n_\alpha}) = b \tag{1.3}$$

does not have a unique solution. Therefore it has no solution at all or it has at least two solutions. Let $a_1, \ldots, a_{i_\alpha - 1}, a_{i_\alpha + 1}, \ldots, a_{n_\alpha}, b \in Q_n(X, u\varphi E)$, and $n \in \{r(a_1), \ldots, r(a_{i_\alpha - 1}), r(a_{i_\alpha + 1}), \ldots, r(a_{n_\alpha}), r(b)\}$. Then $x = \alpha(a_1, \ldots, a_{i_\alpha - 1}, b, a_{i_\alpha + 1}, \ldots, a_{n_\alpha})$ is a solution in $F(X, \varphi E)$. Let $\alpha(a_1^{i_\alpha - 1}, b, a_{i_\alpha + 1}^{n_\alpha}) \in Q(X, u\varphi E)$. Then the word $\alpha(a_1^{i_\alpha - 1}, b, a_{i_\alpha + 1}^{n_\alpha})$ is $q\varphi$-marked, i.e. is of the kind $\alpha(c_1^{i_\alpha - 1}, \omega_\alpha(c_1^{i_\alpha - 1}, d, c_{i_\alpha + 1}^{n_\alpha}), c_{i_\alpha + 1}^{n_\alpha})$. Now

$$\alpha(a_1^{i_\alpha - 1}, b, a_{i_\alpha + 1}^{n_\alpha}) \;=\; \alpha(c_1^{i_\alpha - 1}, \omega_\alpha(c_1^{i_\alpha - 1}, d, c_{i_\alpha + 1}^{n_\alpha}),\, c_{i_\alpha + 1}^{n_\alpha}),$$

and $\omega_\alpha(c_1^{i_\alpha - 1}, d, c_{i_\alpha + 1}^{n_\alpha})$ is not $q\varphi$-marked. This is possible in $F(X, \varphi E)$ only when $a_1^{i_\alpha - 1} = c_1^{i_\alpha - 1}$, $a_{i_\alpha + 1}^{n_\alpha} = c_{i_\alpha + 1}^{n_\alpha}$, and $b = \omega_\alpha(c_1^{i_\alpha - 1}, d, c_{i_\alpha + 1}^{n_\alpha})$. Now $b = \omega_\alpha(c_1^{i_\alpha - 1}, d, c_{i_\alpha + 1}^{n_\alpha})$, and $d \in Q(X, u\varphi E)$. Therefore $x = d$ is a solution of the equation (1.3). The equation (1.3) has no other solutions by Lemma 1.3.4.

Let $\alpha(a_1^{i_\alpha - 1}, b, a_{i_\alpha + 1}^{n_\alpha}) \in Q(X, u\varphi E)$. Then $x = \alpha(a_1^{i_\alpha - 1}, b, a_{i_\alpha + 1}^{n_\alpha})$ is a solution of the equation (1.3). Let us consider another solution $x_2 = d \neq \alpha(a_1^{i_\alpha - 1}, b, a_{i_\alpha + 1}^{n_\alpha})$ of the equation (1.3). In this case,

$$b = \omega_\alpha(a_1^{i_\alpha - 1}, d, a_{i_\alpha + 1}^{n_\alpha}),$$

the word $\omega_\alpha(a_1^{i_\alpha - 1}, d, a_{i_\alpha + 1}^{n_\alpha})$ is not marked in $F(X, \varphi E)$, but the word

$$\alpha(a_1^{i_\alpha - 1}, b, a_{i_\alpha + 1}^{n_\alpha}) = \alpha(a_1^{i_\alpha - 1}, \omega_\alpha(a_1^{i_\alpha - 1}, d, a_{i_\alpha + 1}^{n_\alpha}), a_{i_\alpha + 1}^{n_\alpha})$$

is $q\varphi$-marked. Thus $Q(X, u\varphi E)$ is a $Q - \varphi E$-algebra.
Let $f : X \to A$ be a mapping on a $Q - \varphi E$-algebra A. Then there exists a

φ-homomorphism $\tilde{\tilde{f}} : F(X, \varphi E) \to A$, where $\tilde{\tilde{f}} \mid X = f$. Let $\tilde{f} = \tilde{\tilde{f}} \mid Q(X, u\varphi E)$. Then $\tilde{f} : Q(X, u\varphi E) \to A$ is a homomorphism. This homomorphism is unique. The proof is complete.

It is possible to construction the free $Q - \varphi E$-algebra with other methods by using the definitions and properties of the classes of the equivalent words.

1.3.5 A free object in the class $V(E, \varphi)$

As a set of words the object - $Q(X, u\varphi E)$ is a subset of the set $F(X, \varphi E)$. It is clear that $Q(X, u\varphi E)$ is an E-subalgebra of the E-algebra $F(X, \varphi E)$. For that reason we will have that $Q(X, u\varphi E) \subset F(X, \varphi E)$.

Theorem 1.3.7. *For each mapping $f : X \to A$, where $A \in V(E, \varphi)$, there exists a homomorphism $\tilde{f} : Q(X, u\varphi E) \to A$ such that $f = \tilde{f} \mid X$.*

Proof. Follows from Theorem 1.3.6.

However, the next natural question one can: **When the homomorphism $\tilde{f} : Q(X, u\varphi E) \to A$ is unique?**

Theorem 1.3.8. *If a mapping $f : X \xrightarrow{onto} A$ is given, then a homomorphism $\tilde{f} : Q(X, u\varphi E) \to A$ is unique if and only if $A \in V(\varphi E, u)$.*

Proof. Let $A \in V(\varphi E, u)$. Then there exist $\alpha \in L$ and elements $a_1, \ldots, a_{i_\alpha}, \ldots, a_{n_\alpha} \in A$ such that the equation

$$\omega_\alpha(a_1, \ldots, a_{i_\alpha - 1}, x, a_{i_\alpha + 1}, \ldots, a_{n_\alpha}) = a_{i_\alpha}$$

has at least two different solutions $x_1 = b$, $x_2 = c$. We convert A into φE-algebra twice. In the first time, we set $\alpha(a_1, \ldots, a_{i_\alpha}, \ldots, a_{n_\alpha}) = b$, but for the second time we set $\alpha(a_1, \ldots, a_{i_\alpha}, \ldots, a_{n_\alpha}) = c$. The word $\alpha(a_1, \ldots, a_{i_\alpha}, \ldots, a_{n_\alpha})$ is not $q\varphi$-marked in $F(X, \varphi E)$, i.e.

$$d = \alpha(a_1, \ldots, a_{i_\alpha}, \ldots, a_{n_\alpha}) \in Q(X, u\varphi E).$$

For the first conversion we got a φ-homomorphism $\tilde{f}_1 : F(X, \varphi E) \to A$ and for the second one we got a φ-homomorphism $\tilde{f}_2 : F(X, \varphi E) \to A$. By construction, we have $\tilde{f}_1(d) = b$ and $\tilde{f}_2(d) = c$. Therefore $\tilde{f}_1 \mid Q(X, u\varphi E) \neq \tilde{f}_2 \mid Q(X, u\varphi E)$. Hence the homomorphism $\tilde{f}$ is not unique.

If $A \in V(\varphi E, u)$, then A is uniquely converted into a φE-algebra. Therefore the homomorphism $\tilde{f}$ is unique too.

Let $\tilde{f}_1, \tilde{f}_2 : Q(X, u\varphi E) \to A$ be two different homomorphisms for which $\tilde{f}_1 \mid X = \tilde{f}_2 \mid X = f$. There exists $a \in Q(X, u\varphi E)$ such that $\tilde{f}_1(a) \neq \tilde{f}_2(a)$. For some $n \geqslant 1$ we have $a \in Q_n(X, u\varphi E) \setminus Q_{n-1}(X, u\varphi E)$. We will consider that a has the smallest rank of words for which $\tilde{f}_1(x) \neq \tilde{f}_2(x)$. Then $a = \omega(u_1, \dots, u_n)$, where $u_1, \dots, u_n \in Q_{n-1}(X, u\varphi E)$. The following two cases are possible:

1. $\omega \in E_m$, $m \geqslant 1$.
   In this case we have:

   $$\tilde{f}_1(a) = \omega(\tilde{f}_1(u_1), \dots, \tilde{f}_1(u_n)) = \omega(\tilde{f}_2(u_1), \dots, \tilde{f}_2(u_n)) = \tilde{f}_2(a).$$

   This case is impossible.

2. $\omega = \alpha \in L_m$.
   Then $a = \alpha(u_1, \dots, u_n)$. Therefore

   $$\omega_\alpha(u_1, \dots, u_{i_\alpha - 1}, a, u_{i_\alpha + 1}, \dots, u_{n_\alpha}) = u_{i_\alpha}.$$

   Let us denote $h_1 = \tilde{f}_1(u_1) = \tilde{f}_2(u_1)$, $\dots$, $h_n = \tilde{f}_1(u_n) = \tilde{f}_2(u_n)$. The equation $\omega_\alpha(h_1, \dots, h_{i_\alpha - 1}, x, h_{i_\alpha + 1}, \dots, h_{n_\alpha}) = h_{i_\alpha}$ has two solutions in A: $x_1 = \tilde{f}_1(a)$ and $x_2 = \tilde{f}_2(a)$. Thus $A \in V(\varphi E, u)$.

The proof is complete.

Define a free object of a set X in the class $V(E, \varphi)$ according to next definition of M. Choban.

Definition 1.3.9. *The free E-algebra of a set X in the class $V(E, \varphi)$ is an E-algebra $G(X, E, \varphi) \in V(E, \varphi)$ such that:*

1. *$X \subset G(X, E, \varphi)$ and the set X algebraically generates the E-algebra $G(X, E, \varphi)$, i.e. if $X \subset Y \subset G(X, E, \varphi)$, $Y \neq G(X, E, \varphi)$, and Y is a subalgebra of the algebra $G(X, E, \varphi)$, then $Y \notin V(E, \varphi)$.*

2. *For every mapping $f : X \to A$, where $A \in V(E, \varphi)$, there exists a homomorphism $\hat{f} : G(X, E, \varphi) \to A$ such that $\hat{f} \mid X = f$.*

Corolary 1.3.10. *$Q(X, u\varphi E)$ as an E-algebra is a free object of the set X in the class $V(E, \varphi)$.*

Theorem 1.3.11. *A free object of the set X in the class $V(E, \varphi)$ is unique up to an isomorphism.*

Proof. Let A be a free object of the set X in the class $V(E,\varphi)$. Then $X \subset A$, and for each $X \subset B \subset A$ and $B \neq A$ the subalgebra B is not an algebra from $V(E,\varphi)$. There exist two homomorphisms $\varphi : Q(X, u\varphi E) \to A$ and $\psi : A \to Q(X, u\varphi E)$ such that $\varphi(x) = \psi(x) = x$ for all $x \in X$. Then by Proposition **??**, $\varphi \circ \psi : Q(X, u\varphi E) \to Q(X, u\varphi E)$ coincide up to the identical an isomorphism. Therefore φ and ψ are isomorphisms and $\varphi = \psi^{-1}$.Then $X \subset \varphi(Q(X, u\varphi E)) = A$. The proof is complete.

1.3.6 Some remarks on free objects. Examples

Fix a signature E and a system of equations (φ). We say that an element e is the i-unit for n-ary operation ω if

$$\omega(a_1^{i-1}, x, a_{i+1}^n) = x \tag{$\star$}$$

for each $x \in A$, where $a_1 = \cdots = a_{i-1} = a_{i+1} = \cdots = a_n = e$, $i \in \{1, \ldots, n\}$, $n \geqslant 2$. If the identity $(\star)$ is true for each $x \in A$ and each $i \leq n$, then e is called the unit of the operation ω in the E-algebra A.

An element $0 \in A$ is called the i-zero of a n-ary operation ω, where $i \leq n$ and $n \geqslant 2$, if $\omega(a_1^{i-1}, 0, a_{i+1}^n) = 0$ for each $a_1, \ldots, a_n \in A$. The element $0 \in A$ is called the zero of the n-ary operation ω, where $n \geq 2$, if $\omega(a_1^{i-1}, 0, a_{i+1}^n) = 0$ for each $a_1, \ldots, a_n \in A$ and each $i \in \{1, \ldots, n\}$.

Let $e \in E_0$. For each $i \geqslant 1$ we define the sets:

1. $H(e,i) = \{\omega \in E_n : n \geqslant i, e_A = e_{0A}(e \times A^0)\}$, where e_A is an i-unit for ω;
2. $P(e,i) = \{\omega \in E_n : n \geqslant i, e_A = e_{0A}(e \times A^0)\}$, where e_A is an i-zero for ω;

such that $P(e,i) \cap \{\omega_\alpha : \alpha \in L\} = \emptyset$ and $P(e,i) \cap H(e,i) = \emptyset$. Let $T_1 = \bigcup\{H(e,i) : e \in E_0, i \geqslant 1\}$, and $T_2 = \bigcup\{P(e,i) : e \in E_0, i \geqslant 1\}$.

If the element $e_{0A}(e \times A^0)$ is i-unit in the E-algebra A for each operation $\omega \in H(e,i)$ and the same element $e_{0A}(e \times A^0)$ is i-zero for each operation $\omega \in P(e,i)$, then A is called an E-algebra with T_1-units and T_2-zeroes.

Denote by $V(E,T)$, $V(\varphi E,T)$, and $V(\varphi E,u,T)$ the classes of E-algebras, φE-algebras and $Q - \varphi E$-algebras with T_1-units and T_2-zeroes. The method of the construction of free objects in the classes $V(E,T)$, $V(\varphi E,T)$, and $V(\varphi E,u,T)$ is analogous to that from Sections $1.3.1 - 1.3.5$ with an additional condition - that it is necessary to include in the sets of marked words those marked words that define i-units, units, i-zeroes, and zeroes.

Suppose that we have the free objects $\Gamma(X, E, T)$, $F(X, \varphi E, T)$, and $Q(X, u\varphi E, T)$ in classes $V(E, T)$, $V(\varphi E, T)$, and $V(\varphi E, u, T)$ respectively. Then the following inclusions hold:

$$X \subset \Gamma(X, E) \subset \Gamma(X, E, T) \subset Q(X, u\varphi E, T) \subset F(X, \varphi E, T).$$

It is clear that

$$Q(X, u\varphi E) \subset Q(X, u\varphi E, T),$$

and

$$F(X, \varphi E) \subset F(X, \varphi E, T).$$

For the algebra $F(X, \varphi E, T)$ a theorem analogous to the Theorem 1.3.6 is true. It is clear that we can consider free objects in classes $V(E, T)$, $V(\varphi E, T)$, and $V(\varphi E, u, T)$ with properties of associativity, commutativity, etc.

Example 1.3.12. Let $E = E_2 = \{\cdot\}$. Then E-algebras from $V(E)$ are called gruppoids. If $\varphi = \{ax = b, ya = b\}$, then algebras from $V(E, \varphi)$ are called gruppoids with division, but those from $V(E, u\varphi)$ are called quasigroups. If $\varphi E = \{\cdot, ', ''\}$, where $a(ab)' = b$ and $(ab)''a = b$, then the algebras from $V(\varphi E)$ will be primitive gruppoids with divisions, or Π-gruppoids, but those from $V(\varphi E, u)$ will be Π-quasigroups. The algebra $F(X, \varphi E)$ is a free Π-gruppoid, $Q(X, u\varphi E)$ is a free Π-quasigroup, and $\Gamma(X, E)$ is a free gruppoid. Clearly, $\Gamma(X, E) \subset Q(X, u\varphi E) \subset F(X, \varphi E)$. Each of two equations $ax = b$ and $ya = b$ has no more than two solutions in a free Π-gruppoid. The gruppoid $Q(X, u\varphi E)$ is a free gruppoid with division in the class $V(E, \varphi)$.

Example 1.3.13. Let $E = E_0 \cup E_2$, $E_0 = \{e\}$, $E_2 = \{\cdot\}$, and $T = T_1 = H(e, 1) = H(e, 2) = \{\cdot\}$. Then, $\Gamma(X, E, T)$ is a free gruppoid with a unit, $Q(X, u\varphi E, T)$ is a free Π-loop, and $F(X, \varphi E, T)$ is a free Π-gruppoid with a unit.

Example 1.3.14. If $E = E_n = \{\omega\}$, $n \geqslant 2$,

$$\varphi = \{\omega(a_1, \ldots, a_{i-1}, x, a_{i+1}, \ldots, a_n) = a_i : i \leqslant n\},$$

then $\varphi E = \varphi E_n = \{\omega, \alpha_1, \dots, \alpha_n\}$, where

$$\omega(x_1, \dots, x_{i-1}, \alpha(x_1, \dots, x_i, \dots, x_n), x_{i+1}, \dots, x_n) = x_i$$

for each $i \leqslant n$. In this case $\Gamma(X, E)$ is a free n-gruppoid, $Q(X, u\varphi E)$ is a free primitive n-quasigroup (or Π-n-quasigroup) and $F(X, \varphi E)$ is a free Π-n-gruppoid for which each equation of the system (φ) has no more than two solutions in a free objects. The gruppoid $Q(X, u\varphi E)$ is a free n-gruppoid with division. All algebras from $V(E, \varphi)$ are n-gruppoids with division.

Example 1.3.15. Let $E = E_0 \cup E_2$, $E_0 = \{1, 0\}$, $E_2 = \{+, \cdot\}$, $\varphi = \{a + x = b, x + a = b\}$. Then 0 is a unit of the operation "+", and a zero of the operation "·", but 1 is a unit of the operation "·". In addition, $Q(X, u\varphi E, T)$ is a free quasiring and $F(X, u\varphi E, T)$ is a free quasiringoid.

1.4 Basic Research Topics

In the relatively recent papers of well-known mathematicians such as A. Arhangelskii [7, 8, 10, 11, 12, 13], M. Choban [38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53], V. Arnautov, S. Glavatsky, A. Mikhalev [1, 2, 3, 4], M. Ursul [205, 206, 207, 208, 209], V. Protasov [179, 180, 181], D. Botnaru [31, 32], K.H. Hofmann [117, 118, 119], S. Morris [162], S. Morris and H.B. Thompson [163, 164], T. Fay, E. Ordman, M. Rajagopolan, H.B. Smith-Thomas [91, 92, 93], J. Carruth, J. Hildebrant, R. Koch [35], A.B. Paalman de Miranda [175], W. Taylor [198], Das Phullendu [81], Iu. Muhin [165], J. van Mill [202] etc., cardinal problems that have influenced the evolution of topological algebras theory were solved.

The contribution of Professor Arhangelskii's school to Topological Algebras, Mathematics in general, is enormous [5, 6, 7, 8, 9, 10, 11, 12, 14, 200, 201]. Scientific achievements of this school place it at a leading position in the topological world. An exceptional representative of Arhangelskii's Math school is Professor M. Choban. Innovative ideas generated by Academician M. Choban opened new perspectives in the development for topological algebras.

We mention that the foundation of the topological algebra theory with continuous signature has been laid in the works of M. Choban, S. Dumitraşcu [48, 50, 82, 83, 84, 85].

Additional information of the history of topological algebras and mathematicians can be find in M. Choban [49], M. Choban, I. Valutsa [51], A. Georgescu, C-L Bichir, G-V Cirlig, R. Radoveanu [102], P. Kirku [145].

We can conclude from the above that the directions of research in theory of universal topological algebra are determined by the abstract theory of universal algebras, by the general topology and by the group and topological rings theory.

Consider the continuous signature E and $-1 \leq i \leq 3.5$. Let J be the set of identities. Denote by $V(E, i, J)$ the class of topological E-algebras that are T_i-spaces and which satisfy the identities from J. The academician M. Choban, in his work "Topological Algebra, Problems ", Tiraspol State University Publishing House, 2006, formulated the following problems concerning the topological algebraic systems theory:

Problem 1. *Study the relationship between the algebraic and topological properties of the topological E-algebras G from $V(E, i, J)$.*

Usually, the aforementioned problem is examined in light of the following two problems.

Problem 1T. *Let G be an E-algebra. Determine the kinds of topologies, which can be considered on the E-algebra G that makes it a topological algebra.*

Problem 1A. *Let G be a topological space. Determine the types of algebraic structures that can be considered on the space G, which makes it a topological algebra.*

Notice that if $E = \emptyset$, the the class of E-algebras coincides with the class of topological spaces. In the case when E and G discrete spaces, then the properties of the E-algebra G coincide with the properties of the abstract E-algebra G. In this way, the theory of topological E-algebras is an intermediary theory of the topological spaces theory and of the abstract E-algebras theory. The problems 1T and 1A have been a subject of interest for many mathematicians. Problem 1T was studied, for example, by A. Kertesz and Scele who proved in 1953 that any infinite commutative group can be made a topological group by considering the indiscrete topology. Other aspects of Problem 1T had been examined by V.I. Arnautov, M.I. Ursul, P.I. Chircu, etc. Problem 1A is much more difficult. Frobenius's Theorem represents of the first results in this direction. In this sense, significant results were obtained by L.S. Pontrjagin, J. Milnor, I. M. James. Of special interest is the following result due to L.S. Pontrjagin from 1932: The field R of real numbers, the field C of complex numbers and the field of quaternions are the only associative, connected and locally compact fields. Jon Milnor proved in 1958 that the dimension of a real, finite-dimensional algebra, without any zero-divisors can only take the values $n = 1, 2, 4, 8$. From this point of view we can say that the identities influence directly the topological properties. We emphasize that on any topological space G there exists structures of topological E-algebra. It is enough to fix some continuous applications $e_{nG} : E_n \times G^n \to G$. For example, if $n \geq 1$ and $E_n \neq \emptyset$, then consider $e_{nG}(\omega, x_1, ..., x_n) = x_n$ for every $(\omega, x_1, ..., x_n) \in E_n \times G^n$. The topological E-algebra structure is unique if and only if G is a single point space. However, algebraical structures with certain identities do not exist for all spaces.

Examples:

1. Let R^n, $n \geq 1$, be the n-dimensional Euclidean space and $S^{n-1} = \{x \in R^n : \|x\| = 1\}$ the unitary sphere. Then, on S^m, there exist structures of topological groupoids with unity, only for $m \in \{0, 1, 3, 7\}$. This result was obtained by I.M. James in 1957-1958.
2. Let R^n, $n \geq 0$, be the n-dimensional Euclidean space. If $n \notin \{0, 1, 2, 4, 8\}$, then on R^n there are no structures of commutative fields.
3. On the 8-dimensional Euclidean space R^8 there exists a structure of a non-associative field and there are no structures of associative fields.
4. On the 4-dimensional Euclidean space R^4 there exists a structure of an associative field and there are no structures of commutative fields.
5. If R^1 and R^2 are 1-dimensional and, respectively, 2-dimensional Euclidean spaces then on these spaces there are structures of commutative fields.
6. On any space A with respect to a binary additive operation $(+)$, with zero but without any zero divisors, defined by the equality $x + y = y$ there exists a structure of topological semigroup with right identity. In this case any element from A will be a right identity.

Hence, algebraic identities influence the topological properties. Topological algebras with certain topological-algebraic (or algebraic-topological) properties play a significant rile in various theories. For example, J.M. Boardman and R.M. Vogt in the book [22] have considered algebraic structures that determine: H-

spaces, A_∞-spaces, $\mathcal{B}$-spaces, $W\mathcal{B}$-spaces, E_∞-spaces. All these spaces generate spacial loop structures. This theory can be exposed clearly enough only due to the utilization of universal algebras. The above mentioned matter was treated successfully in J.P. May's article [157]

Therefore, Problem 1 in its broad sense is very important. The following research topic is closely tied with that problem:

The influence of the algebraic structures on the topological properties of the universal topological algebras and application of topological algebraic structures in the study of the properties of topological spaces.

This is the topic regarding the celebrated Hilbert's problem V. The research from this work is devoted to the development of above topic. The following problems are related to the research direction exposed earlier.

Problem 2. *Let P be a topologico-algebraic property. When does the topological E-algebra $A \in V(E, i, J)$ enjoy the property P?*

Problem 3. *Let P be a topologico-algebraic property. Study the structure of the topological E-algebras $A \in V(E, i, J)$, which have the property P.*

While Problem 2 is more general, Problem 3 has a particular character. The following topological properties P can be considered: to be a compact space, to be a locally compact space, to be a locally compact metric space, to be a metric space, to be a complete space, to be a paracompact space.

Problem 4. *Let $p : A \to B$ be a continuous homomorphism from the topological E-algebra A to the topological E-algebra B. Under what conditions is p an open homomorphism ?*

L.Pontrjagin [178] has proved that for a large class of topological groups the homomorphism mapping is open. In the paper [38] M.Choban has generated this assertion for topological algebras with a continuous signature. We give the conditions when continuous homomorphisms of topological groupoids with a continuous division are open.

Let K be a quasivariety of topological E-algebras. For every space X determine the free algebra $(F(X, K), i_X)$ and the abstract free algebra $(F^a(X, K), j_X)$. There exists a continuous homomorphism $a_X : F^a(X, K) \to F(X, K)$, for which $a_X(j_X(X)) = i_X(X)$. If a_X is a continuous isomorphism, then we say that the algebra $F(X, K)$ is algebraically free.

Problem 5. *Under what conditions is the algebra $F(X, K)$ algebraically free?*
Problem 6. *Under what conditions is $i_X : X \to F(X, K)$ an embedding?*
The application i_X is an embedding if i_X is a homomorphism from the space X to the space $i_X(X)$ from $F(X, K)$.
Problem 7. *Study the relationships between the properties of the space X and of the topological E-algebra $F(X, K)$.*

Problems 5-7 were formulated by A.I. Mal'cev in the case of a discrete signature. In the context of Problems 5-7, we study topologies on free algebras with continue signature of pseudocompact and countably compact spaces. In the investigation an important role is played by uniform structures. An analog of the Nummela-Pestov theorem [173, 176] for varieties of uniform algebras is proved. The results of Tkachenko M.G. [201] are generalized to the case of arbitrary varieties formed by topological algebras. It is worth noting that the method of proving is simplified by the analog of the Nummela-Pestov theorem and Proposition 2.8.1 which was proved by T.H.Fay, B.V. Smith-Thomas in [92] for the case of free topological groups.

Two spaces X and Y are called M_K-equivalent if the algebras $F(X, K)$ and $F(Y, K)$ are topologically isomorphic.
Problem 8. *Determine the topological properties that are preserved by the relation of M_K-equivalence.*

We investigate universal algebras with topologies. On algebras we consider topologies relatively to which operations are continuous on compact subsets. These algebras are called k-algebras. Some properties of compact subsets of free k-algebras and some facts about M_K-equivalence of spaces are established.

The investigation is connected with results of A.A. Markov [155], M.I. Graev [103], A.I. Mal'cev [26], J. Milnor [160, 161], P.J. Huber [115], H.-E. Porst [177], V.M. Valov and B.A. Pasynkov [203], E.T. Ordman [174] and M.M. Choban [38, 41]. We study the category of universal topological algebras and the notions of k-continuous mappings and k-algebras.

The category of k-algebras was studied in [41, 174, 177]. The existence of the free k-algebras follows from the general result from [41]. Our attention is focused on the problem of the description of compact subsets of free topological algebras and free k-algebras.

The notion of a k-group was first considered by J. Milnor [160] and P. J. Huber [115]. Let G be a commutative group. According to J. Milnor, there exists the Eilenberg-MacLane semi-simplicial complex $K(G, n)$, where $n \in N$, which admits the structure of a commutative group. J. Milnor [160] stated that the groups $K(G, n)$ are topological for every countable group G. As was observed by P. J. Huber [115], a closer inspection of Milnor's proof shows that $K(G, n)$ are k-groups for every group G. This deep fact was widely studied and applied (see [22, 115, 157, 161, 203]).

Problem 9. *Let V complete variety of topological E-algebras with continuous signature. Consider the E-algebra $G \in V$. Under what conditions does there exist a universal covering $p : G^* \to G$, such that $G^* \in V$ and $p : G^* \to G$ is a homomorphism ?*

L.S. Pontrjagin [178] proved that a linear connected space that covers a topological group admits, in a natural way, a structure of a topological group. In this work we establish a similar result for universal algebras with continuous signature. This result, for the case of a finite discrete signature, was obtained by A.I. Mal'cev [151]. Result from this work is stronger then Mal'cev's Theorem. In particular, the result holds for the topological R-modules, where R is a topological ring.

The space X is called solvable if there exist two dense and disjoint subsets.

Problem 10. *Fix the continuous signature E and J the set of identities. Let $G \in V(E, i, J)$ be a minimal topological E-algebra. Under what conditions is the space G solvable?*

In [64] M. Choban and L. Chiriac has proved the following assertion.

Theorem *Let G be an infinite group of cardinality τ. Then there exists a disjoint family $\{B_\mu : \mu \in M\}$ of subsets of G such that:*

1. $|M| = |G|$.

2. $G = \cup\{B_\mu : \mu \in M\}$.

3. $(G \setminus B\mu) \cdot K \neq G$ for all $\mu \in M$ and every finite subset K of G.

4. The sets $\{B_\mu : \mu \in M\}$ are dense in all totally bounded topologies on G.

This fact is a generalization of one Protasov's result [180]. In this work the assertions of Theorem are proved for the special algebras - I_nP_k-n-groupoids.

Problem 11. *Under what conditions on a medial groupoid do there exist right invariant (or left invariant) Haar measures?*

We describe the topological quasigroups with (n, m)-identities, which are obtained by using isotopies of topological groups. The notion of the (n, m)-**identity** was introduced by M. Choban and L. Chiriac in [57]. Such quasigroups are called the (n, m)-homogeneous quasigroups. We extend some affirmations of the theory of topological groups on the class of topological (n, m)-homogeneous quasigroups. We establish conditions for which exist right invariant (or left invariant) Haar measures on medial grupoid.

The triplet (G, e_G, μ) is a called *L-fuzzy E-algebra* if the following conditions hold:

(A) (G, e_G) is a E-algebra;

(F) $\mu : G \to L$ is a L-fuzzy subset of G;

(AF) the set $\{x \in G : \mu(x) \geqslant l\}$ is empty or is a E-subalgebra from G for any $l \in L$.

The homomorphism $f : A \to B$ of the L-fuzzy E-algebra (A, e_A, μ) in the L-fuzzy E-algebra (B, e_B, η) is called a *fuzzy homomorphism* if $\eta(f(x)) \geqslant \mu(x)$ for every $x \in A$.

Problem 12. (the problem on fuzzy homomorphisms). *Let $f : A \to B$ be a homomorphism of the L-fuzzy E-algebra (A, e_A, μ) on the E-algebra B. Under what conditions $\lambda(f, \mu) = f(\mu)$, that is $(B, e_B, f(\mu))$ is a L-fuzzy E-algebra*?

Certain problems about fuzzy universal algebras were considered in [15, 42, 52, 58, 101]. In papers [169, 182] the problem of the homomorphism for fuzzy algebras was formulated and solved for some homomorphisms of the fuzzy groupoids, groups and rings. We give a general solution of the homomorphism problem for fuzzy universal algebras.

The purpose of the work resides in research of topological algebraic systems and its applications. In particular:

1. Studying the free topological algebras.
2. Elaborating the relevant studying methods of the topologies on free algebras generated by pseudocompact and countable, compact spaces.
3. Describing the compact subsets of the free topological algebras and that of k-algebras.

4. Elaborating research methods regarding the topological universal algebras with invariant measures. In particular, studying the concept of multiple identities.
5. Elaborating research methods regarding the topological quasigroups with multiple identities.
6. Studying the uniform structure on the topological spaces in the light of free objects.
7. Studying of the equivalences in the class of topological spaces generated by varieties of universal topological algebras.
8. Establishing a general theory on the decomposition of the topological groupoids with invertibility properties.
9. Studying the fuzzy structure on universal algebras. In particular, solving the homomorphism problem for fuzzy algebras.

Methodology of the research. Topologization of abstract algebras and sets theory are key components of research methods. The constructions and the methods of proofs are based on the notions of topological algebra, variety, quasigroup with multiple identities, solvable space, fuzzy algebra.

The obtained results in the respective piece of work are directly intertwined with the solving of Problems 1-12, which were formulated above. The main results of the work are new. There have been solved concrete problems, or some aspects of the problems formulated by A.I. Mal'cev, L.S. Pontrjagin, M.M.Cioban.

1.5. Conclusions for Chapter 1

Topological algebraic systems, as a branch of topological algebra, represent an important field of research in modern mathematics. The methods elaborated and the received results within this theory are successfully implemented not only in theoretical mathematics, but also in applied mathematics, physics, computer science, fuzzy sets, etc.

The directions of research in the theory of universal topological algebra are determined by the abstract theory of universal algebras, by the general topology and by the group and topological rings theory.

The study of the relationship between the algebraic and topological properties of universal topological algebras from complete varieties or quasivarieties is a central problem of the topological algebraic systems theory. For successful research on this key issue, it is necessary:

- to elaborate methods of uniform structures to various classes of topological algebras.
- to develop a method of free algebras.
- to elaborate a method of k-algebras.
- to elaborate a general theory on the decomposition of the topological algebras.
- to introduce the concept of multiple identities.
- to develop methods of investigation of topological quasigroups with multiple identities.
- to develop methods of fuzzy algebras and fuzzy homomorphism.

We carried out studies related to the following directions of investigations:

The influence of the algebraic structures on the topological properties of the universal topological algebras and application of topological algebraic structures in the study of the properties of topological spaces.

This topic is linked to the celebrated Hilbert's problem V.

2. ON APPLYING UNIFORM STRUCTURES TO STUDY OF FREE TOPOLOGICAL ALGEBRAS

In this Chapter we study topologies on free algebras of pseudocompact and countably compact spaces. The results of this Chapter were published in [56]. In the presentation an important role is played by uniform structures. An analog of the Nummela-Pestov theorem [173, 176] for varieties of uniform algebras is proved. The results of the article [201] are generalized to the case of arbitrary varieties formed by topological algebras. It is worth noting that the method of proving is simplified by the analog of the Nummela-Pestov theorem and Proposition 6.8.1 which was proved in [92] for the case of free topological groups. The results of Sections 6.5 and 6.10 were announced in [39, 40]. The rest of results were announced in [46, 131, 136] for the case of the variety of groups with operators. Some interesting properties of topological groups and uniform groups was studied in [27, 28]. We denote by R the space of reals endowed with the natural topology. We denote by $[A]_X$ and $[A]$ the closure of a set A in a space X. A set $L \subset X$ is called null or a null-set if $L = g^{-1}(0)$ for a certain continuous function g on X. A mapping $\varphi : X \longrightarrow Y$ is said to be z-closed provided that the image of every null-set is closed. A mapping φ is referred to as R-factor whenever $Y = \varphi(X)$ and the space Y is furnished with the finest Tychonoff topology ensuring the continuity of φ. Every factor mapping of Tychonoff spaces is R-factor [9].
Let $\varphi : X \longrightarrow Y$ be a continuous mapping from a Tychonoff space X into a Tychonoff space Y. Designate as $\beta\varphi : \beta X \longrightarrow \beta Y$ the continuous extension of the mapping φ onto the Stone-Čech compactifications βX and βY of the spaces X and Y. The mapping φ is dense, provided $\varphi(X) = Y$ and $[\varphi^{-1}(y)]_{\beta X} = \beta\varphi^{-1}(y)$ for each point $y \in Y$.

2.1. Mappings of Pseudocompact Spaces

All spaces are assumed to be Tychonoff and mappings, to be continuous.

Lemma 2.1.1 *Let $\varphi : X \longrightarrow Y$ be a z-closed mapping onto a Tychonoff space Y. Then the mapping φ is R-factor.*

Proof. Let τ be the topology of Y and τ' be a Tychonoff topology on Y such that $\tau \subset \tau'$ and $\varphi^{-1}U$ is open in X for all $U \in \tau'$. If F is a null-set in

(Y,τ') then $\varphi^{-1}F$ is a null-set in X. Therefore, φF is closed in (Y,τ). Since the null-sets constitute a closed base of the Tychonoff topology, the mapping $i : (Y,\tau') \longrightarrow (Y,\tau)$, where $i(y) = y$ for all $y \in Y$, is a homomorphism. The lemma is proved.

Lemma 2.1.2 [201]. *Every dense mapping $\varphi : X \longrightarrow Y$ from a normal space X onto Y is closed.*

A mapping $\varphi : X \longrightarrow Y$ is pseudocompact if all inverse images of points, $\varphi^{-1}(y)$, are pseudocompact. The space X is pseudocompact iff all continuous functions on X are bounded. The mapping $\varphi : X \longrightarrow Y$ is relatively pseudocompact iff, for every point $y \in \varphi(X)$ and every continuous function f in X, there exists a point $x_0 \in \varphi^{-1}(y)$ such that $f(x_0) = supf(x) : x \in \varphi^{-1}(y)$.

Lemma 2.1.3 *Let $\varphi : X \longrightarrow Y$ be dense and relatively pseudocompact. Then the mapping φ is z-closed.*

Proof. Let f be a nonnegative bounded continuous function on X. The set $\beta\varphi(\beta f^{-1}(0))$ is closed in βY. Fix $y \notin \varphi(f^{-1}(0))$. Then $f(x) > 0$ for all $x \in \varphi^{-1}(y)$ and there exists a point $x_0 \in \varphi^{-1}(y)$ such that $inf\{f(x) : x \in \varphi^{-1}(y)\} = f(x_0) = b > 0$. If $y \in Y$ then $\beta f(x) \geq b$ for all $x \in \beta\varphi^{-1}(y) = [\varphi^{-1}(y)]_{\beta X}$. Consequently, $y \notin \beta\varphi(\beta f^{-1}(0))$ and $Y \cap \beta\varphi(\beta f^{-1}(0)) = \varphi(f^{-1}(0))$.

Corollary 2.1.4 *Every dense pseudocompact mapping is z-closed.*

2.2.Π_ω-Spaces

A sequence $\{X_n : n \in N = \{0, 1, 2, ...\}\}$ of subsets in a space X is monotonic if $X_n \subset X_{n+1}$ for all $n \in N$.

Definition 2.2.1 *The Tychonoff space X is the inductive limit of a sequence $\{X_n : n \in N\}$ if:*
1. $X = \cup\{X_n : n \in N\}$ and the sets X_n are closed in X;
2. a set F is closed in X whenever all intersections $F \cap X_n$ are closed in X.

As is known, the passage to the inductive limit preserves the properties of being normal, or sequential, or a k-space. Any inductive limit of a sequence of compact sets is called a k_ω-space.

Definition 2.2.2 *The space X is the functional limit or C-limit of a sequence $\{X_n : n \in N\}$ if:*
1. X is a Tychonoff space;
2. $X = \cup\{X_n : n \in N\}$ and the sets X_n are closed in X;
3. a function $f : X \longrightarrow R$ is continuous whenever the restriction $f|X_n$ is continuous on X_n for every $n \in N$.

The following three lemmas are immediate.

Lemma 2.2.3 *If X is the inductive limit of a sequence $\{X_n : n \in N\}$, then X also is the inductive limit of the monotonic sequence $\{X_n = \cup\{X_i : i \geq n\} : n \in N\}$ and if X does not coincide with the C-limit of the sequence $\{X_n : n \in N\}$ then X is not completely regular.*

Lemma 2.2.4 *If a Tychonoff space X is the C-limit of a sequence $\{X_n : n \in N\}$ then X is the C-limit of the monotonic sequence $\{X_n = \cup\{X_i : i \geq n\}\}$ as well.*

Lemma 2.2.5 *Let a Tychonoff space X be the inductive limit of a sequence $\{X_n : n \in N\}$. Then X is the C-limit of the sequence $\{X_n : n \in N\}$.*

Proposition 2.2.6 *Let a Tychonoff space X be the C-limit of a sequence $\{X_n : n \in N\}$ consisting of normal subspaces. Then X is normal and presents the inductive limit of the sequence $\{X_n : n \in N\}$.*

Proof. By Lemma 2.2.4, we can assume that $X_n \subset X_{n+1}$ for all $n \in N$. Let $F \subseteq X$ and $F \cap X_n$ be closed in X for all $n \in N$. Fix a point $X_0 \in X \setminus F$. Let $x_0 \in X_0$. Construct a continuous function $f_0 : X_0 \longrightarrow [0, 1]$ such that $f_0(x_0) = 0$ and $f_0(F \cap X_0) = 1$. Suppose that we have constructed continuous functions $f_n : X_n \longrightarrow [0, 1]$ such that $f_n(x_0) = 0$, $f_n(F \cap X_n) = 1$, and $f_i = f_n|X_i$, for all $i < n$. The sets X_n and $F \cap X_{n+1}$ are closed in X_{n+1}. On $\Phi_n = X_n \cup (F \cap X_{n+1})$, we construct a function g_n such that $g_n|X_n = f_n$ and $g_n(F \cap X_{n+1}) = 1$, i.e., $g_n(y) = 1$ for all $y \in \Phi_n \setminus X_n$. The function g_n is continuous on Φ_n and by the Urysohn lemma there exists a continuous extension $f_{n+1} : X_{n+1} \longrightarrow [0, 1]$ of the function g_n. On X, we consider a function f such that $f|X_n = f_n$ for all $n \in N$. The function f is continuous, $f(x_0) = 0$, and $f(F) = 1$. Therefore, $X_0 \notin [F]$ and the set F is closed in X. The proposition is proved.

If X is the inductive limit of a sequence of compact subspaces $\{X_n : n \in N\}$

then $\{X_n : n \in N\}$ is called a k_ω-decomposition of the space X.

Proposition 2.2.7 *Let a Tychonoff space X be the C-limit of a sequence $\{X_n : n \in N\}$, $Y_n = [X_n]_{\beta X}$, and $Y = \cup\{Y_n : n \in N\} \subset \beta X$. Then:*
1. $\{Y_n : n \in N\}$ is a k_ω-decomposition of Y;
2. if a set $\Phi \subset X$ is compact then $\Phi \subset \{X_i : i \in n\}$ for some $n \in N$;
3. if X is a k-space then X is the inductive limit of $\{X_n : n \in N\}$.

Proof. Clearly, the sets Y_n are compact. Let $F \subset Y$ and the set $F \cap Y_n$ be compact for all $n \in N$. Let us prove that the set F is closed in Y. Consider a point $y \in Y \setminus F$. As in the proof of Proposition 2.6, we construct a function $f : Y \longrightarrow [0, 1]$ such that $f(y) = 0$, $f(F) = 1$, and the restriction $f_n = f|Y_n$ is continuous for all $n \in N$. Set $g = f|X$. Since the restrictions $g|X_n = f|X_n$ are continuous on X_n, therefore g is continuous on X. By construction, $f = \beta g|Y$. Consequently, the function f is continuous and the set F is closed in Y. Item 1 is proved. Fix a compact set $\Phi \subset X$. Then $\Phi \subset \cup\{Y_i : i < n\}$ for some $n \in N$. Item 2 is proved.
Let X be a k-space. It a set $F \subset X$ is not closed in X, then there exists a compact set $\Phi \subset X$ such that $\Phi \cap X$ is not closed in X. Then, for some $m, n \in N$, we have: $m < n$, $\Phi \subset \cup\{X_i : i < n\}$, and the sets $\Phi \cap X_m \cap F$ and $X_m \cap F$ are not closed in X. Thus, X is the inductive limit of $\{X_n : n \in N\}$. The proposition is proved.

Lemma 2.2.8 *Let $\{X_n : n \in N\}$ be a sequence of closed subspaces of a Tychonoff space X, $X = \cup\{X_n : n \in N\}$, and suppose that a bounded function $f : X \longrightarrow R$ is continuous if its every restriction $f|X_n$ is continuous. Then the C-limit of $\{X_n : n \in N\}$ coincides with the space X.*

Proof. Consider a function $f : X \longrightarrow R$ such that all the restrictions $g|X_n$ are continuous. Suppose that the function g is discontinuous at a point $x_0 \in X$. Construct a function $f : X \longrightarrow R$ such that $f(x) = g(x_0) - 1$ if $g(x) < g(x_0) - 1$, $f(x) = g(x_0) + 1$ if $g(x) > g(x_0) + 1$, and $f(x) = g(x)$ for the other points $x \in g^{-1}([g(x_0) - 1, g(x_0) + 1])$. The function f is bounded, discontinuous at the point x_0 and, moreover, every restriction $f|X_n$ is continuous. The obtained contradiction completes the proof.

Proposition 2.2.9 *Let a normal space X be the inductive limit of a monotonic sequence $\{X_n : n \in N\}$, $Y \subset X$, $Y_n = Y \cap X_n$, and $X_n = [Y_n]_X \subset \beta Y_n$ for all $n \in N$. Then the C-limit of the sequence $\{Y_n : n \in N\}$ coincides with the space Y and $X \subset \beta Y$.*

Proof. Consider a function $g : Y \longrightarrow R$ such that $g_n = g|Y_n$ is continuous on Y_n for all $n \in N$. We may assume that the function g is bounded. Since $Y_n \subset X_n \subset \beta Y_n$, there exists a function $f : Y \longrightarrow R$ such that $f|X_n = \beta g_n|X_n$ for all $n \in N$. Then the function f is continuous on X and $g = f|Y$. Therefore, the function g is continuous on Y. The proposition is proved.

A space Y is C-embedded into X if every function continuous on Y has a continuous extension onto X. By vX we denote the Hewitt extension of a Tychonoff space X. If X is a Tychonoff space, $Y \subset X$, and Y is dense in X, then Y is C-embedded into X if and only if $Y \subset X \subset vY$ (cf. [10]). Set $\Pi = \{X : X^n\}$ is pseudocompact for all $n \in N$.

Definition 2.2.10 *A sequence $\{X_n : n \in N\}$ is said to be a Π_ω-decomposition of a Tychonoff space X if:*
1. X is the C-limit of the sequence $\{X_n : n \in N\}$,
2. for every $n \in N$, the subspace X_n is pseudocompact and is C-embedded into X.

Transition from Π_ω-decomposition to monotonic Π_ω-decomposition is not always possible, for the union of two C-embedded subspaces is not always C-embedded.

Example 2.2.11 Let $I = [0,1]$, $b \in X = I^\tau$, and $\tau \geq \aleph_1$. In $Y = (X \times X) \setminus (b, b)$ we consider subsets $Y_1 = (X \times \{b\}) \cap Y$ and $Y_2 = (\{b\} \times X) \cap Y$. The sets Y_1 and Y_2 are C-embedded into Y and $X \times X$, but $Y_1 \cup Y_2$ is not C-embedded into Y.

Theorem 2.2.12 *Let $\{X_n : n \in N\}$ be a Π_ω-decomposition of a Tychonoff space X. Then $\{Y_n = [X_n]_{vX} : n \in N\}$ is a Π_ω-decomposition of the space vX.*

Proof. The claim follows from Proposition 2.7.

Theorem 2.2.13 *Let $\{Z_n : n \in N\}$ be a monotonic k_ω-decomposition of a space Z, $X \subset Z$, and let the sets $X_n = X \cap Z_n$ be dense and C-embedded into*

Z_n. *Then* $Z = vX$ *and* $\{X_n : n \in N\}$ *is a* Π_ω*-decomposition of* X.

Proof. By construction. X_n is pseudocompact and C-embedded into Z and X for all $n \in N$. Thus, $Z_n = \beta X_n$. By virtue of Proposition 2.2.9, the C-limit of $\{X_n : n \in N\}$ coincides with X. Thus, $\{X_n : n \in N\}$ is a Π_ω-decomposition of X and X is C-embedded into Z. The proof is complete.

Example 2.2.14 We consider the discrete topology on N and introduce the order topology in $Z = \{\alpha : \alpha \leq \omega_1\}$, α is ordinal, here ω_1 stands for the first uncountable ordinal. After collapsing the set $\{\omega_1\} \times N$ in $Z \times N$ into a single point, we obtain the factor-space X and the natural projection $p : Z \times N \longrightarrow X$. Assign $Y = X \setminus \{b\}$, $X_n = p(Z \times \{n\})$,$Y_n = X_n \cap Y$. By construction, $\{X_n : n \in N\}$ presents a k_ω-decomposition of X; $\{Y_n : n \in N\}$ serves as a Π_ω-decomposition formed by normal and countably compact subspaces C-embedded into X; $\beta Y_n = X_n$; $vY \neq X$ and $vY = Z \times N$. For $n \neq m$, the set $Y_n \cup Y_m$ is not C-embedded into X. It follows that the requirement of monotonicity imposed on a sequence is essential in the hypotheses of Proposition 2.9 and Theorem 2.13.

Definition 2.2.15 *A Tychonoff space* X *is called a* Π_ω*-space if there exists a monotonic* Π_ω*-decomposition* $\{X_n : n \in N\}$ *such that* $X_n \in \Pi$ *for all* $n \in N$.

Theorem 2.2.16 *If* X *is a* Π_ω*-space then* X^m *is a* Π_ω*-space for all* $n \in N$.

Proof. Let $\{X_n : n \in N\}$ be a monotonic Π_ω-decomposition of X such that $X_n \in \Pi$ for all $n \in N$. Assign $Y_n = [X_n]_{\beta X}$ and $Y = \cup\{Y_n : n \in N\}$. Then $\{Y_n^m : n \in N\}$ is a k_ω-decomposition of Y^m and $\beta(X_n^m) = (\beta X_n)^m = Y_n^m$ for all $n \in N$. By Theorem 2.13, $\{X_n^m : n \in N\}$ constitutes a monotonic Π_ω-decomposition of X^m. The theorem is proved.

Corollary 2.2.17 *If* X *is a* Π_ω*-space then* $v(X^m) = (vX)^m$ *for all* $m \in N$.

Example 2.2.18 Let $\{A_n : n \in N\}$ be a disjoint family of uncountable sets, $A = \cup\{A_n : n \in N\}$, $I_\alpha = I = [0, 1]$ for all $\beta \in A, Z = I^A = \Pi\{I_\alpha : \alpha \in A\}$, $0 = \{0_\alpha = 0 : \alpha \in A\} \in Z$, and $b_n = \{0_\alpha : \alpha \in A_n\} \times \{1_\beta \beta \in A \setminus A_n\}$. Then $0 = \lim b_n$, and the space $Z \setminus \{b_n : n \in N\}$ is pseudocompact, Čech complete, and $X = \beta(Z \setminus \{b_n : n \in N\})$. Assign $Y_n = \{\{x_m \in Z : m \in N\} \in Z^N : x_m = 0$ for all $m > n\}$, $c_n = (b_n, 0, ..., 0, ...) \in Y_0$, and $X_n = Y_n \setminus \{c_m : m \in N\}$ for all $n \in N$. By construction, $\{Y_n : n \in N\}$ is a monotonic sequence of compact sets and $\beta X_n = Y_n$. We denote the inductive limit of $\{Y_n : n \in N\}$ by Y, and designate $X = \cup\{X_n : n \in N\} \subset Y$. By Theorem 2.13, X coincides with the C-limit of $\{X_n : n \in N\}$. Moreover, $X_n \in \Pi$ for all $n \in N$. Thus, X is a Π_ω-space. Let us show that X does not coincide with the inductive limit of the sequence $\{X_n : n \in N\}$. Consider the points $h = (0, 0, ..., 0, ...) \in Z^N$ and $h_{mn} = (b_n b_{m+1}, ..., b_{m+n}, 0, ..., 0, ...) \in X_n$ and sets $F_0 = \varnothing$, $F_n = \{h_{im} : i \leq n, m \in N\}$, and $F = \cup\{F_n : n \in N\}$. Since $h = \lim c_n$ and $c_n = \lim h_{nm}$, we have $0 \in [F]_X \subset [F]_Y$ and the set F is not closed in X. By construction, $F \cap X_n = F_n$ is closed in X_n for all $n \in N$. Hence, the projective limit of $\{X_n : n \in N\}$ does not coincide with X. The spaces X_n are Čech-complete spaces as well as k-spaces. Therefore, X is not a k-space. In particular, the C-limit does not preserve the property of being a k-space and the inductive limit of $\{X_n : n \in N\}$ is not completely regular.

2.3. C_ω-Spaces

Let $C = \{X : X^n\}$ is normal and countably compact for all $n \in N$.

Definition 2.3.1 *A C_ω-decomposition is a Π_ω-decomposition $\{X_n : n \in N\}$ with X_n countably compact and normal for all $n \in N$.*

Definition 2.3.2 *A space X is a C_ω-space if there exists a monotonic C_ω-decomposition $\{X_n : n \in N\}$ such that $X_n \in C$ for all $n \in N$.*

C_ω-spaces and the class C were introduced in [201]. It is clear that every C_ω-space is a Π_ω-space, but the converse is not true.

Example 2.3.3 Let $X \in C$ and assume that the space X is not compact. Then $Y = X \oplus \beta X \in \Pi \setminus C$. The space Y^n is countably compact for $n \in N$. If $n \geq 2$ then $X \times \beta X$ is closely embedded into Y^n and, by the Tamano theorem [89], the space $X \times \beta X$ is not normal. Therefore, for $n \geq 2$, the space Y^n is not normal.

Proposition 2.3.4 *If X is a C_ω-space then X^n is a normal C_ω-space for all $n \in N$.*

Proof. The claim follows from Proposition 2.2.6, Lemma 2.2.5, Theorem

2.2.16, and the definition of the class C.

Theorem 2.3.5 *Let X be a Tychonoff space, $\{Phi_n : n \in N\}$ be a monotonic k_ω-decomposition of the space vX, the set $X_n = \Phi_n \cap X$ be C-embedded into X, and $[X]_{vX} = \Phi_n$ for all $n \in N$. Assume that one of the conditions holds:*
1. the space X is normal;
2. every closed pseudocompact subspace of the space X is normal;
3. the subspace X_n is normal for all $n \in N$.
Then $\{X_n : n \in N\}$ is a C_ω-decomposition of X. If, moreover, $\{X_n : n \in N\} \subset C$ then X is a C_ω-space.

Proof. Since X_n is C-embedded into X and $[X_n]_{vX} = \Phi_n$ is compact; therefore, X_n is pseudocompact. Thus, X_n is normal and countably compact; and, moreover, $\beta X_n = \Phi_n$ for all $n \in N$. By virtue of Theorem 2.2.13, $\{X_n : n \in N$ is a $\Pi_\omega\}$-decomposition. The proof is complete.

Definition 2.3.6 *A Tychonoff space X is called strictly collectionwise normal if each open neighborhood of the diagonal $\Delta(X) = \{(x) : x \in X\}$ in X^2 belongs to the universal uniform structure of X, i.e. the finest uniformity on X comparable with the topology on X.*

Proposition 2.3.7 *Every C_ω-space is strictly collectionwise normal.*

Proof. Let $\{X_n : n \in N\}$ be a C_ω-decomposition of the space X. Consider an open set $U \supset \Delta(X) = \{(x, x) : x \in X\}$. We have $v(X \times X) = vX \times vX$ and $v(X \times X)$ is finally compact. Since $X \times X$ is normal, there exists an open set $V \supset \Delta(vX)$ in $vX \times vX$ such that $U = V \cap (X \times X)$. In vX there exists an open covering γ such that $\cup\{H \times H : H \in \gamma\} \subset V$. The covering γ is uniform. Therefore, $\omega = \{H \cap X : H \in \gamma\}$ is a uniform covering of X and $\Delta(X) \subset \cup\{H : H \in \omega\}$. The proof is complete.

2.4. Universal Algebras. Terms

Given disjoint sets $\{E_n : n \in N\}$, we refer to the set $\cup\{E_n : n \in N\}$ as a signature. We endow E with the discrete topology.
A space A is called a topological E-algebra if A is nonempty and continuous mappings $\{e_{nA} : E_n \times A^n \longrightarrow A : n \in N\}$ are given. The set E_n plays the role of symbols of n-ary operations and the mapping e_{nA} presents the joint action of all

n-ary operations. For every $n \in N$ and $\omega \in E_n$, on the E-algebra we define the action of the operation $\omega : A^n \longrightarrow A$, where $\omega(x_1, ..., x_n) = e_{nA}(\omega, x_1, ..., x_n)$. Each operation on E is referred to as a term of first degree. We also designate as terms of first degree the projections $\pi_{ni} : X^n \longrightarrow X$, where $1 \leq i \leq n$ and $\pi_{ni}(x_1, ..., x_n) = x_i$. If we have defined terms of degree $m \geq 1, n \geq 1, \omega$ is an n-ary term of first degree, and $t_1, ..., t_n$ are terms of degree at most m, where the arity of the term t_i is k_i, then $u = \omega(t_1, ..., t_n)$ will present a term of degree at most $m + 1$ and of arity $k = k_1 + k_2 + ... + k_n$ and, moreover, $u(x_1, ..., x_k) = \omega(t_1(x_1, ..., x_{k_1}), t_2(x_{k_1+1}, ..., x_{k_2}), ..., t_n(x_{k-(k_n-1)}, ..., x_{k_n}))$. If $n > m > 0$, $u(x_1, ..., x_n)$ is an n-ary term, and $h : \{1, 2, ..., n\} \longrightarrow \{1, 2, ..., m\}$ is an onto mapping, then $v(x_1, ..., x_m) = u(x_{h(1)}, x_{h(2)}, ..., x_{h(n)})$ is referred to as an m-ary term resulted in identification of separate groups of variables. If terms u and v are given, then the equality $u(x_1, ..., x_n) = v(y_1, ..., y_m)$ is called an identity. Every topological space is assumed to be a T_{-1}-space. Let us fix $i \in \{-1, 0, 1, 2, 3, 3, 5\}$.

Definition 2.4.1 *A class K of topological E-algebras presenting T_i-spaces is said to a full T_i-quasivariety, in case the following conditions are fulfilled:*
1. K is closed under Tychonoff products;
2. K is closed under taking subalgebas;
3. if $(G, \tau) \in K$ and (G, τ') is a continuous E-algebra presenting a T_i-space, then $(G, \tau') \in K$. A full T_i-quasivariety K is called a full T_i-variety, provided that the additional condition holds;
4. if (G, τ) and (G', τ') is a T_i-algebra, and there exists a continuous onto homomorphism $f : G \longrightarrow G'$, then $(G', \tau') \in K$.

Definition 2.4.2 *Fix a class K of topological E-algebras and a space X. The pair $(F(X, K), i_X)$ is called a free topological E-algebra over X in the class K if:*
1. $F(X, K) \in K$;
2. $i_X : X \longrightarrow F(X, K)$ is a continuous mapping;
3. the set $i_X(X)$ generates the algebra $F(X, K)$;
4. for every continuous mapping $f : X \longrightarrow G$, where $G \in K$, there exists a continuous homeomorphism $\hat{f} : F(X, K)$ such that $f = \hat{f} \circ i_X$.

The class K of E-algebras is nontrivial if K contains an algebra different from a singleton.

Definition 2.4.3 *Fix a class K of topological E-algebras. The pair $(F^a(X,K), j_X)$ is called an abstract free E-algebra of the space X in the class K, provided that:*
1. $F^\alpha(X,K) \in K$ and $j_X : X \longrightarrow F^\alpha(X,K)$ is a is a mapping;
2. the set $j_X(X)$ generates the algebra $F^\alpha(X,K) \in K$;
3. for every mapping $f : X \longrightarrow G$, where $G \in K$, there exists a continuous homomorphism $\bar{f} : F^\alpha(X,K) \longrightarrow G$ such that $f = \bar{f} \circ h_X$.

Let K be a full nontrivial T_i-quasivariety of topological E-algebras. The articles [82, 153] prove that a free topological algebra always exists and is unique to within a topological isomorphism. If X is completely regular then $i_X : X \longrightarrow F(X,K)$ is a topological embedding [15, 14]. If X is discrete then $F(X,K) = F^\alpha(X,K)$ and $i_X = j_X$. Therefore, for any space X, the algebra $F^\alpha(X,K)$ exists, is unique, and presents a discrete space. There always exists a continuous onto homomorphism $k_X : F^\alpha(X,K) \longrightarrow F(X,K)$ such that $i_X = j_X \cdot k_X$. If k_X is an isomorphism then the algebra $F(X,K)$ is referred to as abstractly free. For any Tychonoff space X, the algebra $F(X,K)$ is abstractly free [42, 193]. The following fact plays an important role in studying free objects [46, 193].

Lemma 2.4.4 (Change-of-variable lemma). *Consider a nontrivial full T_i-quasivariety K of topological E-algebras, a Tychonoff space X, and an n-ary term u. Let $u(x_1,...,x_n) = u(y_1,...,y_n)$ for some $x_1, y_1, x_2, y_2, ..., x_n, y_n \in X \subset F(X,K)$. Suppose that $y_{i_1} = y_{i_2} = ... = y_{i_k}$ and $y_i \neq y_{i_1}$ for all $i \notin \{i_1, i_2, ..., i_k\}$ and $y_{i_1} \notin \{x_1, x_1, ..., x_n\}$. If $z \in F(X,K), y_i = z_i$ for $i \notin \{i_1, i_2, ..., i_k\}$ and $z = z_i$ for $i \in \{i_1, i_2, ..., i_k\}$, then $u(x_1, x_2, ..., x_n) = u(z_1, z_2, ..., z_n)$.*

2.5. Uniform Algebras

We denote the set X with some separated uniformity μ by μX. If X is a Tychonoff space, then we denote by uX the set X with a uniformity u compatible with the topology in X. Let u_X be the maximal uniformity on the space X and U_0X be the X with the uniformity u_X. Denote the Weyl completion of the uniform space μX by $\beta\mu X$. If $Y \subset X$, then $\mu|Y$ stands for the restriction of the uniformity μ onto Y. Fix a signature E with the discrete uniformity u_E. A E-

algebra A with a uniformity μ is called uniform if the mappings $e_{nA} : E_n \times A_n \longrightarrow A$ are uniformly continuous.

Definition 2.5.1 *A class K of uniform E-algebras presents a full variety, provided that:*
1. the class K is closed under Cartesian products and under taking subalgebras;
2. the class K is closed under passing to uniformly continuous homomorphic images;
3. if $\mu A \in K$ K and the algebra A is uniform with respect to a uniformity η, then $\eta A \in K$.

Fix a nontrivial full variety K of uniform algebras.

Definition 2.5.2 *A pseudometric d on an algebra $A \in K$ is stable whenever $d(x,y) \leq 1$ and, for all $n \geq 1, \omega \in E_n$, and $x_1, y_1, ..., x_n, y_n \in A$, we have $d(\omega(x_1,...,x_n), \omega(y_1,...,y_n)) \leq \sum\{d(x_i,y_i) : i \leq n\}$.*

Stable pseudometrics were studied in [46]. For a group, the stability of a pseudometric is equivalent to its invariance [46]. Every uniform structure is generated by some family of pseudormetrics.

We will consider only normalized pseudometrics. Fix a set X. Then K, as a class of abstract algebras, is a variety and, for the set X, we can define free objects $F^\alpha(X,K)$ with the properties:
1. $X \subset F^\alpha(X,K)$ and X algebraically generates the algebra $F^\alpha(X,K)$
2. for every mapping $f : X \longrightarrow A \in K$ there exists a homomorphism
$\overline{f} : F^\alpha(X,K) \longrightarrow A$ such that $f = \overline{f} \mid X$.

In the article [46] it was proved that we may assign to any normalized pseudometric d on X a stable normalized pseudometric $\hat{d}$ on $F^\alpha(X,K)$. Moreover, the following statements are valid:

a). $d(x,y) = \hat{d}(x,y)$ for all $x, y \in X$;

b). if ρ is a stable normalized pseudometric on $F^\alpha(X,K)$ and $d(x,y) \geq \rho(x,y)$ for all $x, y \in X$, then $\hat{d}(x,y) \geq \rho(x,y)$ for all $x, y \in F^\alpha(X,K)$ as well;

c). if normalized pseudometrics $\{d_\alpha : \alpha \in A\}$ generate a separated uniformity on X then the pseudometrics $\{\hat{d}_\alpha : \alpha \in A\}$ generate a separated uniformity on $F^\alpha(X,K)$ and all the mappings $e_{nF^\alpha(X,K)}$ are uniformly continuous.

Consider the uniform space μX. We denote all normalized continuous pseudometrics on the uniform space μX by $\Pi(\mu)$. Then the set K of pseudometrics $\hat{\Pi}(\mu) = \{\hat{d} \in \Pi(\mu)\}$ generates on $F^\alpha(X, K)$ the uniformity $\hat{\mu}$ for which $\mu X \subset \hat{\mu}F^\alpha(X, K)$ and $\hat{\mu}F^\alpha(X, K) \in K$. Let $\hat{\mu}$ be the maximal uniformity on $F_\alpha(X, K)$ such that $\mu X \subset \hat{\mu}F^\alpha(X, K)$ and $\hat{\mu}F^\alpha(X, K) \in K$. Designate $\hat{\mu}F^\alpha(X, K) = F(\mu X, K)$. Then $F(\mu X, K)$ is a free uniform algebra on the space μX in the class K.

Theorem 2.5.3 *Every algebra $A \in K$ is uniformly embeddable into a product of metric algebras in K.*

Proof. The necessary reasoning is analogous to that of the theorem on embedding a uniform space into a product of metric spaces [42].

Definition 2.5.4 *An algebra A with a uniformity μ is weakly uniform if, for every $n \geq 1$ and $\omega \in E_n$, the mapping $e_{nA}|\{\omega\} \times A^n : \{\omega\} \times A^n \longrightarrow A$ is uniformly continuous.*

If all the sets E_n are finite, then any weakly uniform algebra is uniform.

Example 2.5.5 Let $Z = \{\pm n : n \in N\}$ be the discrete ring of integers and $H = \{(x, y) \in R^2 : x^2 + y^2 = 1\}$ be the commutative compact group of plane rotations. We denote by $\{0, -, +\}$ the signature of the commutative groups $E_0 = \{0\}, E_1 = \{-\} \cup Z, E_2 = \{+\}$, and $E = E_0 \cup E_1 \cup E_2$. Every topological commutative group G is a Z-module and a topological E-algebra with $e_{1G}(n, x) = nx$ for all $x \in G$ and $n \in Z$. Thus, H is a topological E-algebra. The E-algebra H is weakly uniform with respect to the metric $d((x_1, y_1), (x_2, y_2)) = |x_2 - x_1| + |y_2 - y_1|$, since H is compact and every compact algebra is weakly uniform. Fix $0 < \delta < \varepsilon < 4^{-1}$. Find a point $x = (cos(\varphi), sin(\varphi)) \in H$ and a number $n \in N$ such that $|1 - cos\varphi| + |sin\varphi| < \delta$ and $|1 + cos\varphi| + |sin\varphi| < \delta$. Then $nx = (cos(n\varphi), sin(n\varphi)) < \delta, d((1, 0), nx) < \delta$ and $d((1, 0), nx) > 2^{-1}$. Thus, the algebra H is not uniform.

Example 2.5.6 The discrete metric, $d(x, x) = 0$ and $d(x, y) = 1$, is stable on every algebra. Discrete uniform algebras are uniform. By analogy with Definition 5.1, we can define a full variety of weakly uniform algebras. Let K be a nontrivial full variety of weakly uniform E-algebras. Then, for every uniform space μX there exists a maximal uniformity $\bar{\mu}$ on $F^\alpha(X,K)$ such that $\mu X \subset \bar{\mu}F^\alpha(X,K) \in K$. Thus, $F(\mu X, K) = \bar{\mu}F^\alpha(X,K)$ is a free object of the apace μX in the class K.

Theorem 2.5.7 *Let the set E be countable. Then every weakly uniform algebra is uniformly embeddable into a product of weakly uniform metric algebras.*

Proof. The reasoning is similar to that of the theorem on embedding of uniform spaces into products of metric spaces.

Definition 2.5.8 *A class K is uniformly correct if, for any uniform space μX, the uniformity $\hat{\mu}$ generates on $F^\alpha(X,K)$ the uniformity of the space $F(\mu X, K)$.*

Example 2.5.9 *Every variety of uniform groups is uniformly correct. The claim follows from the fact that the uniformity of a uniform group is generated by invariant pseudometrics.*

Theorem 2.5.10 *Let K be a uniformly correct full nontrivial variety of uniform or weakly uniform algebras. If $\eta Y \subset \mu X$, then $F(\eta Y, K) \subset F(\mu X, K)$.*

Proof. The claim follows from the equalities $\Pi(\eta) = \{d|Y : d \in \Pi(\mu)\}$ and $\hat{\Pi}(\eta) = \{d|F^\alpha(X,K) : d \in \hat{\Pi}(\mu)\}$. Fix $i \in \{-1,0,1,2,3,3,5\}$ and a nontrivial full T_i-variety K of topological E-algebras. Designate $K'_u = \{uA \in K\} : uA$ is a uniform algebra. Then K'_u is a full variety of uniform algebras and K''_u is a full variety of weakly uniform algebras. We have topological inclusions $K'_u \subset K''_u \subset K$, and the classes K, K'_u and K''_u coincide as abstract sets. Thus, the algebras $F(X,K), F(uX,uY)$, and $F(uX,K''_u)$ coincide abstractly for any Tychonoff space X.

Theorem 2.5.11 *For every Tychonoff space X, the space X is C-embedded into $F(X,K)$.*

Proof. Let $f : X \longrightarrow R$ be a continuous function.

Assign $Y = f(X)$ and consider an embedding $g : Y \longrightarrow R$, where $g(y) = y$ for all $y \in Y$. On Y, we consider the metric $d(x,y) = min\{1, |x-y|\}$. The metric d can be extended on $F^\alpha(Y,K)$ to a stable metric $\hat{\mathrm{d}}$. There exists a homomorphism $\varphi : F(X,Y) \longrightarrow F^\alpha(Y,K)$ such that $\varphi(x) = f(x)$ for all $x \in X$. The homomorphism φ is continuous with respect to the topology induced by the metric $\hat{d}$ and, in this topology, the set Y is closed in $F^\alpha(Y,K)$. Thus, the function g admits a continuous extension $h : F^\alpha(Y,K) \longrightarrow R$. Then $f(x) = h(\varphi(x))$ is a continuous extension of f. The theorem is proved.

2.6 T-Uniform Algebras

Consider a signature E. A topological E-algebra A with a uniformity u on it is a T-uniform algebra if is possible to introduce in βuA the structure of a topological E-algebra such that A is a subalgebra of the algebra βuA. In this case we say that the T-uniform algebra βuA is the completion of the T-uniform algebra βuA. Each uniform or weakly uniform algebra is T-uniform. The converse is not always true: every topological group with respect to a two-sided uniformity is T-uniform and is not weakly uniform. A topological algebra A is T-uniformizale if on A there exists a compatible uniformity that induces the structure of a T-uniform algebra on A. If a topological algebra A is Diedonné-complete, then it is T-uniformizable.

Lemma 2.6.1 *If a topological algebra A is T-uniformizable, then there exists a maximal uniformity v_A on A that transforms A into a T-uniform algebra.*

Proof. The claim follows from the fact that the property of being a T-uniform algebra is preserved under Cartesian products as well as under taking subalgebras.

Example 2.6.2 A topological completely regular algebra is not always T-uniformizable. There exists such a pseudocompact, space X that $X \times X$ is not pseudocompact (cf. [10, Example 3.10.19]). Thus there exists a continuous unbounded function $f : X \times X \longrightarrow R$. Assign $A = X \oplus R$ and $x \cdot y = f(x, y)$ if $\{x, y\} \in X$ and $x \cdot y = 0$ if $x, y \setminus X \neq \varnothing$. If u is a uniformity on A, then $[X]_{\rho uA}$ is compact. Therefore, the mapping f is not extendable on the set $[X]_{\rho uA}$ squared. Therefore, the topological groupoid A is not T-uniformizable. Other examples of non -T-uniformizable algebras are constructed in the next section.

Definition 2.6.3 *A class K of T-uniformizable algebras is called a full T-uniformizable variety, provided that:*

1. the class K s closed under Cartesian products and taking subalgebras;

2. if $A \in K$ and A is a T-uniform algebra with respect to a topology τ and a uniformity μ, then $(A, \tau) \in K$;

3. if $A \in K$ and $f : A \longrightarrow B$ is a continuous homomorphism onto a T-uniformizable algebra B then $B \in K$.

Let K a nontrivial full T-uniformizable variety of topological E-algebras. Then, for every algebra $A \in K$, we can define the maximal uniformity v_A that T-uniformizes A. We denote by $\hat{A}$ the completion of the algebra A relative to the uniformity v_A. By the principle of preservation of identities under completion, we obtain $\hat{A} \in K$.

Theorem 2.6.4 *Let K be a nontrivial variety of E-algebras. Then, for every uniform space μX, there exists an algebra $F(\mu X, K) \in K$ such that:*
1. μX is a uniform subspace of the uniform space $(F(\mu X, K), v_{F(\mu X,K)})$;
2. the set X generates $F(\mu X, K)$ algebraically;
3. for every uniformly continuous mapping $f : X \longrightarrow A \in K$, where A is endowed with the uniformity v_A, there exists a continuous homomorphism
$\hat{\hat{f}} : F(\mu X, K) \longrightarrow A$ such that $f = \hat{\hat{f}}|X$;
4. the algebra $F(\mu X, K)$ is algebraically free in the class K.

Proof. We denote by K_1 all the algebras of K endowed with uniformities that make them uniform. Then K_1 is a full variety of uniform algebras. Each algebra $A \in K$ with the discrete uniformity belongs to K_1. Therefore, K and K_1 coincide algebraically. From the topological standpoint, we see that $K_1 \subset K$.
It is easy to construct algebras $F(\mu X, K)$ and uniformly continuous mappings $i_X : X \longrightarrow F(\mu X, K')$ such that $i_X(X)$ generates $F(\mu X, K)$ algebraically and, for every uniformly continuous mapping $f : X \longrightarrow A \in K$, there exists a continuous homomorphism $\hat{f} : F(\mu X, K) \longrightarrow A$ such that $f = i_X \cdot \hat{f}$. Since $F(\mu X, K_1) \in K$, there exists a continuous homomorphism $\pi : F(\mu X, K) \longrightarrow F(\mu X, K_1)$ such that $\pi(i_X(X)) = x$ for all $x \in X$. Then i_X is a uniform embedding. Since $F(\mu X, K_1)$ is algebraically free in K_1 and the classes K and K_1 coincide algebraically, we conclude that π is a continuous isomorphism. All in all, the algebra $F(\mu X, K)$ is algebraically free over $X = i_X(X) \subset F(\mu X, K)$. The proof is complete.

Remark *If a homomorphism $f : A \longrightarrow B$ is continuous, then the mapping f is uniformly continuous with respect to the uniformities v_A and v_B. Let K be a nontrivial full T-uniform variety of topological E-algebras. We assign $F(X, K) = F(u_0 X, K)$.*

2.7. T-Uniformization of Varieties

Let us fix a signature E, $i \in \{-1; 0; 1; 2; 3; 3,5\}$, and a nontrivial full T_i-variety K of topological E-algebras. We denote by K_u the collection of all T-uniformizable algebras of K. Clearly, K_u contains all paracompact, all Diedonné-complete, and all discrete algebras of K. Thus, K_u is a full T-uniform variety of algebras and the classes K and K_u coincide algebraically. Hence, for every Tychonoff space X, there exists a continuous isomorphism $\pi_X : F(X, Y) \longrightarrow F(X, K_u)$ such that $\pi_X = x$ for all $x \in X$ and $\pi_X|X$ is a homeomorphism. Apparently, π_X is not always a topological isomorphism. If $X \subset A$ then we denote by $\langle X \rangle$ the algebra generated by the set X in A. For the varieties of all groups, the following fact is established in [173, 176].

Theorem 2.7.1 *Let $G \in K_u$. Consider a set $X \subset G$ and its closure $[X]$ in G, $\mu = \upsilon_G \mid X$ and $\bar{\mu} = \upsilon_G \mid [X]$. Then:*
1. if $\langle [X] \rangle = F(\bar{\mu}[X], K_u)$ then $\langle X \rangle = F(\mu X, K_u)$;
2. if $\langle X \rangle = F(\mu X, K_u)$ then $\langle [X] \rangle = F(\bar{\mu}[X], K_u)$.

Proof. 1. Let $H \in K_u$ and $f : X \longrightarrow H$ be a (μ, υ_H)-uniform continuous extension of $\bar{f} : [X] \longrightarrow \hat{H}$, where $f = g|[X]$. Designate $\hat{f} = g|\langle X \rangle$. Then $g(\langle X \rangle) \subset H$ for $g(X) = f(X)$.
2. Let $\langle X \rangle = F(\mu X, K_u), H \subset K_u$, and $\varphi : [X] \longrightarrow H$ be a $(\bar{\mu}, \upsilon_H)$-uniform continuous mapping. There exists a uniform continuous homomorphism $\psi : F(\mu X, K_u) \longrightarrow H$ for any $\psi|X$ and $\varphi|X$ as well as a continuous extension $\bar{\psi} = G_1 \longrightarrow \hat{H}$, where $G_1 = [F(\mu X, K_u)]_{G^u}$ since $G_1 \subset F(\widehat{\mu X, K_u})$. Taking into account the equality $\varphi([X]) = \bar{\psi}([X]) \subset H$, we deduce that $\bar{\varphi} = \bar{\psi}|\langle [X] \rangle : \langle \hat{X} \rangle \longrightarrow H$ is a continuous extension of φ. Thus, $\langle [X] \rangle = F(\bar{\mu}[X], K_u)$. The theorem is proved.

Corollary 2.7.2 *A continuous homomorphism $q : F(\mu_0 X, K_u) \longrightarrow F(\beta\mu_0 X, K_u)$ where $q(x) = x$ for all $x \in X$, is a topological embedding.*

Corollary 2.7.3 *If X is pseudocompact then $\pi : F(X, K_u) \longrightarrow F(\beta X, K_u)$ is a topological embedding.*

Corollary 2.7.4 *If $dim X = 0$, X is pseudocompact, and $|E| \leq \aleph_0$ then $ind F(X, K_u) = 0$. In conclusion, we expose examples of non T-uniformizable algebras.*

Example 2.7.5 Let A be an infinite separable compact group. Then A contains a countable subgroup B. The Weyl completion $\hat{B}$ of the group B coincides with A; but $\beta B \neq A$, for the group B is not pseudocompact. Therefore, there exists a continuous mapping $f : B \longrightarrow R$ which is unextendable to A. On the group $G = B \times R$ we consider the unary operation $\omega : B \times R \longrightarrow \{1\} \times R \subset G$ given by $\omega(x, y) = (1, f(x))$ for $(x, y) \in B \times R$. Assign $E = \{\omega\} \cup E'$, where E' is the group signature. If v is the uniform structure of the topological group G, then the E-algebra is not T-uniform with respect to the uniformity v. Since the space G is paracompact; therefore, the E-algebra G is T-uniformizable by the maximal uniformity u_G.

Example 2.7.6 Consider the signature $E' = E_0 \cup E_2$ of semigroups with unity, where E_0 contains the operation of fixing a unit and E_2 contains the multiplication. For every space X, the free semigroup $P(X)$ is homeomorphic to the discrete *sum* $\oplus \{X^n : n \in N\}$. Let a Tychonoff space X be pseudocormpact and let the space X^2 be not pseudocompact. Then, on $P(X) \supset X^2$ there exists a continuous unbounded function $f : P(X) \longrightarrow R$ which is at the same time unbounded on X^2. Consider the semigroup $G = P(X) \times R$. Assume $\{1\} \times R \subset G$. Consider on G the unary operation $\omega : G \longrightarrow G$, where $\omega(x, y) = (1, f(x))$ for all $(x, y) \in G$. Designate $E = E' \cup \{\omega\}$. Then G is a E-algebra. Let u be a uniform structure on G. Suppose that βuG ia a topological E-algebra. The set $Y = [X \times \{0\}]$ is compact and $Y \cdot Y \supset X \times X$. Consequently, the function f cannot be extended to $Y \cdot Y$, for the set $Y \cdot Y$ is compact and the function f is unbounded. It follows that the operation ω is not extendable to $Y \cdot Y$. All in all, the completely regular topological E-algebra G is not T-uniformizable.

2.8. Mapping Pseudocompact Sets in Free Algebras

Let E be a signature, $-1 \leq i \leq 3,5$, and K be a nontrivial full T_i-variety of topological E-algebras. On the sets $\{1, 2, ..., s\}$ we consider only the discrete topology.

Proposition 2.8.1 *Let $n \geq 1$, $s \geq 1$, $Y \subset X$, the space Y^n be pseudocompact, $\Gamma = [Y]_X = \beta Y$, and $u_1, ..., u_s$ be n-ary terms. Consider the mapping $u : \{l, 2, ..., s\} \times X^n \longrightarrow F(X, K_u)$, where $u(i, x) = u_i x$ for all $i \geq s$ and*

$x \in X^n$. Then the mapping $\omega = u \mid \{1, 2, ..., s\} \times Y^n : \{1, 2, ..., s\} \times Y^n \longrightarrow Z = u(\{1, 2, ..., s\}) \times Y^n \subset F(X, K_u)$ is dense and pseudocompact and, moreover, $\beta Z = [Z]_{F(XK_u)}$.

Proof. Let $L \subset X$, $A = \{a_1, ..., a_k\} \subset L$, and $n > k \geq 0$. Points $x = (x_1, ..., x_n)$ and $y = (y_1, ..., , y_n) \in L^n$ are A-similar if $x_i = x_j$ implies $y_i = y_j$ and $x_i \in A$ implies $y_i = x_i$. Let $P(x, L^n/A)$ be the collection of all the points in L^n that are A-similar to a point $x \in L^n$. The set $P(x, L^n/A)$ is homeomorphic with the set of the form L^m, where $m \leq n - k$. If $x \in H \subset L$ then $P(x, H^n/A) = H^n \cap H(x, L^n/A)$. Moreover, if H is everywhere dense in L, then $P(x, H^n/A)$ is also dense in $P(x, L^n/A)$.

Assign $\Delta(Z) = Z$ and $\Delta(Z^k) = \{(z, z, ..., z) : z \in Z\} \subset Z^k$ for all $k > 1$. Then the set $P(x, H^n/A)$ is of the form $\Delta(H^{m_1}) \times \Delta(H^{m_2}) \times ... \times \Delta(H^{m_k}) \times (a_1, ..., a_m)$ up to a permutation of coordinates in X^n, where $A = \{a_1, ..., a_m\}$ and $m + m_1 + m_2 + ... + m_k = n$. Thus, from the pseudocompactness of the space Y^n and the equality $[Y]_X = \beta Y$, it follows that the union Z of a finite number of the sets in Y^n of the form $P(X, Y^n/A)$ is C-embedded into X^n and, moreover, $[Z]_{X^n} = \beta Z$.

For every point $b \in F(X, K_u)$, we define its support as $Sup(b) \subset X : x \in Sup(b)$ whenever the subalgebra generated by the set $X \setminus \{x\}$ contains the point b. The case $Sup(b) = \varnothing$ is not excluded.

Fix $b \in Z$. Then the set $A = Sup(b) \subset Y$ contains at most n points. Let $L = \{i \leq s : \omega^{-1}(b) \cap (\{1\} \times Y^n) \neq \varnothing\}$ If $i \in L$ and $(i, x) \in \omega^{-1}(b)$, then $\{i\} \times P(x, Y^n/A) \subset \omega^{-1}(b)$. Thus, $\omega^{-1}(b)$ is the union of a finite number of the sets of the form $\{i\} \times P(x, Y^n/A)$. Since $[Y]_X = \beta Y$, $[Y^m]_{X^m} = \beta(Y^m) = \Gamma^m$ for all $m \leq n$, and $P(x, Y^n/A) = P(x, X^n/A) \cap Y^n$; therefore, the set $\omega^{-1}(b)$ is C-embedded into X^n and Y^n, and $\beta\omega^{-1}(b) = [\omega^{-1}(b)]_{X^n} = [\omega^{-1}()b]_{\beta(X^n)}$ and $\beta(Y^n) = \beta(Y)^n = [Y^n]_{X^n}$. Consequently, the mapping ω is dense and pseudocompact.

Consider a continuous function f on Z. The function $g = \omega f$ is continuous Y^n; we denote by h its continuous extension on $P = \{1, 2, ..., s\} \times \beta(Y^n) = \{1, 2, ..., s\} \times [Y^n]_{X^n}$. Fix a point $d \in u(\{1, ..., k\} \times P)$ and assign $v = u|P$. Let us prove that the function h is constant on $v^{-1}(d)$. For $d \in Z$, the fact follows from the density of the mapping ω. Let $m = (i, b_1, ..., b_n) \in v^{-1}(d)$, $c = (j, c_1, ..., c_n) \in v^{-1}(d)$, and $|h(b) - h(c)| > 3\varepsilon > 0$. In $\{i, j\} \times Y^n$ there exist

points $b' = (i, b'_1, ..., b'_n)$ and $c' = (j, c'_1, ..., c'_n)$ such that:
1. $|h(b) - h(c')| < \varepsilon$, $|h(c) - h(c')| < \varepsilon$,
2. if $b_i = b_j$ then $b'_i = b'_j$,
3. if $c_i = c_j$ then $c'_i = c'_j$,
4. if $b_i = c_j$ then $b'_i = c'_j$.

Then from $v(b) = v(c)$ it follows $v(b') = v(c')$. Therefore, $g'(b') = g(c'), v(b') = \omega(b') \in Z$, and the function h is constant on $v^{-1}(v(b'))$. Hence,
$|h(b) - h(c)| = |h(b) - h(b') + h(c') - h(c)| \leq |h(b) - h(b')| + |h(c') - h(c)| \leq 2\varepsilon$.
All in all, the function h is constant on $v^{-1}(x)$ for all $x \in Z$. Since the mapping $v : P \longrightarrow [Z]_{F(X,K)}$ is perfect, the function $\varphi(x) = h(v^{-1}(x))$ is a continuous extension of f onto $[Z]_{F(X,K)}$. The proposition is proved.

Corollary 2.8.2 *Let $i = 3, 5, n \geq 1, Y \subset X$, the space Y^n be pseudocompact, $[Y]_X = \beta Y$, and $u_1, ..., u_k$ n-ary terms. Then the mappings $u : Z = \{1, 2, ..., k\} \times Y^n \longrightarrow H = u(Z) \subset F(X, K)$, where $u(i, x_1, ..., x_n) = u_i(x_1, ..., x_n)$, and $v : Z \longrightarrow H = v(Z) \subset F(X, K_u)$, where $v(i, x_1, ..., x_n) = u_i(x_1, ..., x_n)$ are dense, pseudocompact, z-closed, R-factor, and $[H]_{F(X,K)} = [H]_{F(X,K_u)} = \beta H$. If moreover Y^n is normal then the mappings u and v are closed.*

2.9. Free Algebras of Π_ω-Spaces and C_ω-Spaces

Fix a countable signature $E, i \in \{-1, 0, 1, 2, 3, 3, 5\}$, and a full T_i-variety K of topological E-algebras. Let $\{u_n : n \in N\}$ be all the terms of signature E and let m_n be the arity of the term u_n.

Theorem 2.9.1 *If X is a k_ω-space then $F(X, K) = F(X, K_u)$.*

Proof. Consider a monotonic k_ω-decomposition $\{X_n : n \in N\}$ of the space X. Assign $Z_n = \{u_n(x_1, ..., x_{m_n}) : (x_1, ..., x_{m_n}) \in X^{m_n}\} \subset F(X, K)$. The set Z_n is compact. There exists a continuous isomorphism $\pi : F(X, K) \longrightarrow F(X, K_u)$ such that $\pi(x) = x$ for all $x \in X$. The set $\{H_n \cup \{Z_i : i \leq n\} : n \in N\}$ constitutes a k_ω-decomposition of $F(X, K_u)$. Since $\pi|Z_n$ is a homeomorphism for every $n \in N$, the topology on $F(X, K_u)$ cannot be strengthened and π is a topological isomorphism. The proof is complete.

Theorem 2.9.2 *Let $i = 3, 5$. For a Tychonoff space X, the following conditions are equivalent:*

1. X is a Π_ω-space,

2. $F(X,K)$ is a Π_ω-space,

3. $F(X,K_u)$ is a Π_ω-space homeomorphic to $F(X,K)$.

Proof. Let $\{X_n : n \in N\}$ be a monotonic Π_ω-decomposition of the space X and $Y = vX$. Then $\{Y_n = [X_n]_Y : n \in N\}$ is a monotonic k_ω-decomposition of Y. By Theorem 9.1, $\{H_n = \cup\{u_i(X^m_{i_i}) : i \leq n\} : n \in N\}$ is a monotonic k_ω-decomposition of $F(X,K)$, and the algebras $F(X,K)$ and $F(X,K_u)$ are topologically isomorphic. Let us fix $n \geq 1$. Assign $k = max\{m_i : i \leq n\}$. Since the projections $\pi_{mi} : X^m \longrightarrow X$ are terms and allow us to increase the arity of terms, there exist k-ary terms $v_1, v_2, ..., v_n$ such that $v_i(X^k) = u_i(Y^{m_i})$ for all $i \leq n$. Consider the mappings $v : \{0,1,...,n\} \times X^k \longrightarrow Z_n = v(\{0,1,...,n\} \times K^n) \subset F(Y,K_u)$, where $(i, x_1, ..., x_k) = v_i(x_1, ..., x_k)$. By virtute of Corollary 8.2, $[Z_n]_{F(Y,K_u)} = H_n = \beta Z_n$. The algebra $G = \cup\{H_n : n \in N\}$ is generated by the set X in $F(Y,K_u)$ and there exists a continuous isomorphism $\psi : F(X,K) \longrightarrow G$ such that $\psi(u) = x$ for all $x \in X$. By Theorem 2.13, $\{Z_n : n \in N\}$ is a monotonic Π_ω-decomposition of the space G. Thus, G is a Π_ω-space and ψ is a topological isomorphism. This proves the implications $1 \longrightarrow 2$ and $1 \longrightarrow 3$. Suppose that $F(X,K)$ is a Π_ω-space. By Theorem 5.11, X is C-embedded and closed in $F(X,K)$ and $F(X,K_u)$. Therefore, X is a Π_ω-space. The proof is complete.

Theorem 2.9.3 *For a space X, the following conditions are equivalent:*

1. X is a C_ω-space,

2. $F(X,K)$ is a C_ω-space,

3. $F(X,K_u)$ is a C_ω-space.

Proof. The claim follows from Theorems 2.9.2 and 2.3.5 and Corollary 2.8.2.

Corollary 2.9.4 *Let X be either a C_ω-space or $i = 3, 5$ and let X be a Π_ω-space. Then:*

1. $F(X,K)$ is topologically embeddable into $F(vX,K)$,

2. $vF(X,K) = F(vX,K)$,

3. every continuous function on $F(X,K)$ has a continuous extension on $F(vX,K)$,

4. $\beta F(X,K) = \beta(F(vX,K))$,

5. if $dimX = 0$ then $dimF(X,K) = dimF(vX,K) = 0$,

6. $dimF(X,K) = dimF(vX,K)$.

2.10. Diedonné Completeness for Free Algebras

Fix a countable signature E and a nontrivial full $T_{3,5}$-variety K of topological E-algebras.

Lemma 2.10.1 *Let μX be a uniform space, $X = Y \cup Z, \eta = \mu \mid Y$, the subspace ηY be a complete, and, for any subspace $F \subset Z$ closed in X, the uniformity $\mu|F$ be complete. Then the uniform space μX is complete.*

Proof. Let $\mathcal{F} = \{F_\alpha : \alpha \in A\}$ be a Cauchy filter formed by sets closed in X. If $Y \in \mathcal{F}$ then the filter $\mathcal{F}$ converges in X. Let $T \notin \mathcal{F}$. Then $Y \cap F_\alpha = \varnothing$ for some $alpha \in A$. Clearly, $F_\alpha \subset Z$. Thus, $\mathcal{F}$ converges in F_α. The proof is complete.

Definition 2.10.2 *A uniform space μX is a σ-complete, provided that there exist sets $\{X_n : n \in N\}$ such that:*
l. $X = \cup\{X_n : n \in N\}$,
2. the uniformity $\mu|X_n$ is complete on X_n for all $n \in N$.

Theorem 2.10.3 *Let μX be a σ-complete uniform space. Then there exists a σ-bounded uniformity η on X such that:*
1. the uniformity $\xi = \mu \vee \eta$ is complete on X,
2. the uniformities μ, η and ξ induce the same topology on X.

Proof. We assume that μ is a uniformity on a Tychonoff space X. There exist subsets $\{X_n : n \in N\}$ such that μ is complete on X_n for all $n \in N$. Then the sets X_n are closed in X. Assign $Y = \beta\mu X$. In βY, we consider the set $Y_n = [X_n]_{\beta Y}$. It is clear that $Y_n \cap Y = X_n$ The set $Z = \cup\{Y_n : n \in N\}$ is σ-compact. Therefore, the uniformity u_Z is complete and σ-compact. Let $\eta = u_Z|X$. It is plain that the sets X_n are bounded with respect to the uniformity η; and the uniformities μ, η, and $\xi = \mu \vee \eta$ induce the same topology on X. By construction, $Z \cap Y = X$. In the product $Y \times Z$, the uniformity $\beta\mu \times \eta$ induces on $X = \{(x, x) : x \in X\} \subset Y \times Z$ the uniformity ξ. The set $\{(x, x) : x \in X\}$ is closed in $Y \times Z$. Therefore, the uniformity ξ is complete on X. The theorem is proved.

Corollary 2.10.4 *If a Tychonoff space X admits of a σ-complete uniformity, then the space X is Diedonné complete.*

Corollary 2.10.5 *If, for a Tychonoff space X, the uniform space $u_0 X$ is*

σ-complete, then it is complete and the space X is Diedonné-complete.

Theorem 2.10.6 *For a Tychonoff space X, the following assertions are equivalent:*
1. the space X is Diedonné-complete,
2. the space $F(X, K)$ is Diedonné-complete,
3. the space $F(X, K_u)$ is Diedonné-complete.

Proof. The implications $2 \longrightarrow 1$ and $3 \longrightarrow 1$ are immediate. Suppose that the space X is Diedonné-complete. We may order all the terms of the signature E, $\{u_n : n \in N\}$, is such a way that, for every $n \in N$, the mapping $g_n = u_n|(X^{m_n} \setminus \cup\{u_n^{-1}(u_i(X_i^m)) : i < n\})$ is an embedding. In particular, $u_0 : X_m \longrightarrow F(X, K_u)$ is an embedding. The last fact was established in [11]. Therefore, the set $F_0(X, K_u) = u_0(X^{m_0})$ is complete with respect to the uniformity $\xi = u_{F_0(X,K_u)}$. Suppose that the sets $F_i(X, K_u) = u_i(X^{m_i})$ are complete with respect to the uniformity ξ for all $i < n$. If a set, Φ is closed in $F(X, K_u)$ and $\Phi \subset F_n(X, K_u) \setminus \cup\{F_i(X, K_u) : i < n\}$, then the uniformity ξ is complete on Φ in view of the property indicated above. By Lemma 10.1, the set $F_n(X, K_u)$ is complete. Therefore, the uniformity ξ is σ-complete. Corollary 10.5 completes the proof.

2.11. Free Algebras for P-Spaces

Let E be a countable signature and K be a full T_i-variety of topological E-algebras, where $i \in \{-1; 0; 1; 2; 3; 3, 5\}$. If is a P-space, then $F(X, K)$ is a P-space [82].

Lemma 2.11.1 *Let X be a Lindelöf P-space. Then, for every n-ary term u of signature E, the mapping $u : X^n \longrightarrow F(X, K)$ is closed. In particular, the set $u(X^n)$ is closed in $F(X, K)$.*

Proof. If X is a Lindelëf P-space, then X^n is a Lindelöf P-space. In a separated P-space, all Lindelöf sets are closed. Thus, the mapping u is closed.

Theorem 2.11.2 *Let X be a Lindelöf P-space, let $\{u_n : n \in N\}$ be all the terms of signature E, and let m_n stand for the arity of the term u_n. Denote $F_n(X, K) = u_n(X^{m_n})$. Then $F(X, K)$ is the inductive limit of the normal spaces $\{F_n(X, K) : n \in N\}$ and $F(X, K) = F(X, K_u)$; moreover, the space $F(X, K)$ is*

Lindelöf.

Proof. If F is a subset of $F(X,K)$ and $F \cap F_n(X,K)$ is closed in $F_n(X,K)$ for all $n \in N^+$, then F is the union of a countable number of subsets closed in $F(X,K)$; i.e., F is closed in $F(X,K)$.

2.12. The Case of Topological Groups Rings, and Modules

Consider a countable signature E and a full T_i-variety of topological E-algebras. Suppose that there exist operations $\omega_0 \in E_0, \omega_1 \in E_1$ and $\omega_2 \in E_2$ such that the algebras $A \in G$ are groups in the signature $\{\omega_0, \omega_1, \omega_2\}$. Say that K is a full variety of topological E-groups or groups with operators. For every Tychonoff space X, the space $F(X,K)$ is Tychonoff and we may assume that $i = 3, 5$. Note that rings and modules give examples of groups with operators. Thus, Theorems 2.9.2 and 2.9.3 imply.

Corollary 2.12.1 *Let K be a nontrivial full variety of topological groups with operators. Then:*
1. if X is a Π_ω-space, then $F(X,K)$ is a Π_ω-space,
2. if X is a C_ω-space, then $F(X,K)$ is a C_ω-space.

Let K be a nontrivial full variety of topological groups, i.e., $E = \{\omega_0, \omega_1, \omega_2\}$. Then $e_{0G}(\omega_0, G^0) = 1$ coincides with the unit of the group G, and $e_{1G}(\omega_1, x) = x^{-1}$ is the inverse element of $x \in G$, and, moreover, $e_{2G}(\omega_2, y) = x \cdot y$ is the product of elements $x, y \in G$. The terms are defined as follows: the nullary term $u_{00} = 1$; the unary terms $u_{11}(x) = x^{-1}$ and $u_{12}(x) = 1 \cdot x = x$; the binary terms $u_{21}(x,y) = x \cdot y$, $u_{22}(x,y) = xy^{-1}$, $u_{23}(x,y) = x^{-1}y$, $u_{24}(x,y) = x^{-1}y^{-1}$. For $n > 2$, we define 2^n n-ary terms $u_{ni}(x_1, ..., x_n) = x^{i_1} \cdot x^{i_2} \cdot ... \cdot x^{i_n}$, where $\{i_1, i_2, ..., i_n\} \subset \{-1, 0, 1\}$. For $n \geq 1$ consider the mapping $i_n : (X \oplus \{1\} \oplus X^{-1})^n \longrightarrow F(X,K)$, where $i_n(x_1, ..., x_n) = x_1 \cdot x_2 \cdot ... \cdot x_n$ and $1 = x_0$. The set $i_n : (X \oplus \{1\} \oplus X^{-1})^n = F^n(X,K)$ is closed in $F(X,K)$. Let $F_0(X,K) = \{1\}$. Then $F_0(X,K)$ results from applying all the terms of arity at most n. The mapping i_1 is an embedding. Let us study the mapping i_2. The mapping i_2 sends the sets $X \times X$ and $X^{-1} \times X^{-1}, X \times \{1\}, X^{-1} \times \{1\}$ onto disjoint clopen subsets of the space $F_2(X,K)$ homeomorphically. Moreover, $i_2(\{1\} \times X) = i_2(X \times \{1\})$ and $i_2(\{1\} \times X^{-1}) = i_2(X^{-1} \times \{1\}$. Thus, the mappings $i_2|(X \times X^{-1})$ and $i_2|(X^{-1} \times X)$

are of interest. It is clear that $u_{22}(x,y) = i_2(x,y^{-1})$ and $u_{23}(x,y) = i_2(x^{-1},y)$.

Lemma 2.12.2 *If a space X is not normal but the space $X \times X$ is pseudocompact, then the mappings i_2, u_{22}, and u_{23} are not factor nor closed.*

Proof. Let F and Φ be disjoint subsets of X that are closed and functionally undistinguishable. The set $F\times\Phi$ is closed in $X\times X$ and $(F\times\Phi)\cap\Delta(X) = \varnothing$. The mapping $u_{22} : X \times X \longrightarrow F_2^2(X,K) \subset F^2(X,K)$ is such that $u_{22}(\Delta(X)) = \{1\}$ and $u_{22}|(X \times X) \setminus \Delta(X)$ is a embedding. Let the mapping u_{22} be factor. Assign $U = (X\times X)\setminus(F\times\Phi)$ and $u_{22} = V$. Whence, $u_{22}^{-1}V = U$ and the set V is open in $F_1^2(X,K)$. Therefore, there exists a continuous function $f : F_2^2(X,K) \longrightarrow [0,1]$ such that $f(1) = 0$ and $F_2^2(X,K) \setminus V \subset f^{-1}(1)$. The function $g(x) = f(u_{22}(x))$ is continuous on $X \times X, \Delta(X) \subset g^{-1}(0)$, and $F \times \Phi \subset g^{-1}(1)$. Consider the extension βg of the function g on $\beta(X\times X) = \beta X\times\beta X$. Then $\Delta(\beta X) \subset \beta g^{-1}(0)$. Since the sets F and Φ are functionally undistinguishable, there exists a point $x_0 \in [F]_{\beta X} \cap [\Phi]_{\beta X}$. Consequently, $\beta g(x_0,x_0) = 0, (x_0,x_0) \in [F \times \Phi]_{\beta X\times\beta X}$, and $F \times \Phi \subset \beta g^{-1}(1)$. The contradiction obtained completes the proof.

Corollary 2.12.3 *Let X and X^2 be pseudocompact and not normal spaces. Then, for every $n \geq 2$, the mapping $i_n : (X \oplus \{1\} \oplus X^{-1}) \longrightarrow F_n(X,K)$, as well as the action of an arbitrary n-ary term $u : X^n \longrightarrow u(X^n) \subset F(X,K)$, is not a factor mapping.*

2.13. Conclusions for Chapter 2

In this Chapter we study topologies on free algebras of pseudocompact and countably compact spaces. In the investigation an important role is played by uniform structures. The scientific innovation of this Chapter is determined by the following:

1). there have been elaborated the concepts of Π_ω-spaces and C_ω-spaces.

2). there have been elaborated studying methods of the uniform structures on free algebras, in particular, on free algebras of Π_ω-spaces and C_ω-spaces.

3). there have been created the method of the applying of the uniform structures for the study of free topological algebras. The methods of uniform structures lead to solving the following difficult problems in this area:

- an analog of the Nummela-Pestov theorem [173, 176] for varieties of uniform

algebras is proved.
- the results of M. Tkachenko, T.H.Fay, B.V. Smith-Thomas are generalized to the case of arbitrary varieties formed by topological algebras.

In this way, expanding on the ideas of A. Arhangelskii, E. Nummela, V. Pestov, T.H.Fay, B.V. Smith-Thomas and M. Tkacenko, which were successfully used in the research of free topological groups, made it possible to elaborate and implement new methods of studying topologies on free topological algebras with continuous signature generated by pseudocompact and countable compact spaces.

In this context it is opportune:
- to elaborate alternative methods of research of topologies on free algebra generated by pseudocompact and countable compact spaces;
- to elaborate the methods of constructions the universal covers on topological universal algebras with continuous signature;
- to investigate topological quasigroups which are obtained by using the isotopies of topological groups.
- to identify the conditions when continuous homomorphisms of topological groupoids with a continuous division are open;
- to study on topological groupoids relationship between mediality, paramediality and associativity;
- to elaborate methods of embedding every topological n-groupoid into topological n-groupoid with division.

3. TOPOLOGICAL GROUPOIDS AND QUASIGROUPS WITH MULTIPLE IDENTITIES

In Chapter 3, Sections 3.1 - 3.11, we describe the topological quasigroups with (n, m)-identities, which are obtained by using isotopies of topological groups. Such quasigroups are called (n, m)-homogeneous quasigroups. We extend some affirmations of the theory of topological groups on the class of topological (n, m)-homogeneous quasigroups.

In Sections 3.12 - 3.15, we show that every topological n-groupoid A can be embedding into a topological n-groupoid with division B. We give the conditions when continuous homomorphisms of topological groupoids with continuous homomorphisms of topological groupoids with a continuous division are open. We study one special class of topological groupoids with division, namely the class of medial topological quasigroups. We proved that if P is open compact set of a left identity from left medial topological loop G, then P contains an open compact left medial subloop Q.

In Sections 3.16 - 3.18, we study universal covering algebras. L.S.Pontrjagin [178] showed that a linearly connected space, which covers topological group, has itself a structure of topological group. We obtain the analogical result for the universal algebras with continuous signature. This result, for the case of discrete signature, was also obtained by A.I. Mal'cev [154].

3.1 General Notes

A set G is said to be *a groupoid relative to a binary operation* $(\cdot)$, if for every ordered pair a, b of elements of G, is defined a unique element $a \cdot b$ of G.

If the groupoid G is a topological space and the multiplication operation $(a, b) \to a \cdot b$ is continuous, then G is called *a topological groupoid*.

A groupoid $(G, \cdot)$ is called *a groupoid with division*, if for every $a, b \in G$ the equations $ax = b$ and $ya = b$ have solutions, not necessarily unique.

A groupoid $(G, \cdot)$ is called *reducible* or *cancellative*, if for each equality $xy = uv$ the equality $x = u$ is equivalent to the equality $y = v$.

A groupoid $(G, \cdot)$ is called *a primitive groupoid with the divisions*, if there exist two binary operations $l : G \times G \to G$, $r : G \times G \to G$ such that $l(a, b) \cdot a = b$,

$a \cdot r(a,b) = b$ for all $a, b \in G$. Thus a primitive groupoid with divisions is a universal algebra with three binary operations.

If in a topological groupoid G the primitive divisions l and r are continuous, then we can say that G is *a topological primitive groupoid with continuous divisions*.

A primitive groupoid G with divisions is called a *quasigroup* if every of the equations $ax = b$ and $ya = b$ has unique solution. In the quasigroup G the divisions l, r are unique.

An element $e \in G$ is called an *identity* if $ex = xe = x$ for every $x \in X$. A quasigroup with an identity is called a *loop*.

If a multiplication operation in a quasigroup $(G, \cdot)$ with a topology is continuous, then G is called a *semitopological quasigroup*.

If in a semitopological quasigroup G the divisions l and r are continuous, then G is called a *topological quasigroup*.

A quasigroup G is called *medial* if it satisfies the law $xy \cdot zt = xz \cdot yt$ for all $x, y, z, t \in G$.

If a medial quasigroup G contains an element e such that $e \cdot x = x$ $(x \cdot e = x)$ for all x in G, then e is called a *left (right) identity element* of G and G is called a *left (right) medial loop* (see [17]).

Let $N = \{1, 2, ...\}$ and $Z = \{0, \pm 1, \pm 2, ...\}$.

3.2 Multiple Identities

We consider a groupoid $(G, +)$. For every two elements $a, b \in (G, +)$ denote

$$\begin{aligned} 1(a,b,+) &= (a,b,+)1 = a+b, \\ n(a,b,+) &= a + (n-1)(a,b,+), \\ (a,b,+)n &= (a,b,+)(n-1) + b \end{aligned}$$

for all $n \geq 2$.

If a binary operation $(+)$ is given on a set G, then we shall use the symbols $n(a,b)$ and $(a,b)n$ instead of $n(a,b,+)$ and $(a,b,+)n$.

Definition 3.2.1. *Let $(G, +)$ be a groupoid, $n \geq 1$ and $m \geq 1$. The element e of a groupoid $(G, +)$ is called an (n,m)-zero of G if $e + e = e$ and $n(e,x) =$*

$(x, e)\,m = x$ for every $x \in G$. If $e + e = e$ and $n\,(e, x) = x$ for every $x \in G$, then e is called an (n, ∞)-zero. If $e + e = e$ and $(x, e)\,m = x$ for every $x \in G$, then e is called an (∞, m)-zero. It is clear that $e \in G$ is an (n, m)-zero, if it is an (n, ∞)-zero and an (∞, m)-zero.

Remark 3.2.2. *In the multiplicative groupoid $(G, \cdot)$ the element e is called an (n, m)-**identity**. The notion of the (n, m)-**identity** was introduced in [57].*

Theorem 3.2.3. *Let $(G, \cdot)$ be a multiplicative groupoid, $e \in G$ and the following conditions hold:*

1. *$ex = x$ for every $x \in G$;*
2. *$x^2 = x \cdot x = e$ for every $x \in G$;*
3. *$x \cdot yz = y \cdot xz$ for all $x, y, z \in G$;*
4. *For every $a, b \in G$ there exists a unique point $y \in G$ such that $ay = b$.*

   Then e is a $(1, 2)$-identity in G.

Proof. Fix $x \in G$. Pick $y \in G$ such that $xe \cdot y = x$. By virtue of the condition 2 we have $x \cdot (xe \cdot y) = x \cdot x = e$, i.e. $x \cdot (xe \cdot y) = e$. From the condition 3 it follows that $xe \cdot xy = e$. It is clear that $xe \cdot xe = e$. Thus $xe \cdot xy = xe \cdot xe$, $xy = xe$ and $y = e$. Therefore $(x \cdot e) \cdot e = (x \cdot e) \cdot y = x$ and e is a $(1, 2)$-identity. The proof is complete.

Example 3.2.3. Let $(G, +)$ be a commutative additive group with a zero 0. Consider a new binary operation $x \cdot y = y - x$. Then $(G, \cdot)$ is a medial quasigroup with a $(1, 2)$-identity 0. If $x + x \neq 0$ for some $x \in G$, then 0 is not an identity in $(G, \cdot)$.

Theorem 3.2.4. *Let $(G, \cdot)$ be a multiplicative groupoid, $e \in G$ and the following conditions hold:*

1. *$ex = x$ for every $x \in G$;*
2. *$x \cdot x = e$ for every $x \in G$;*
3. *$xy \cdot uv = xu \cdot yv$ for all $x, y, u, v \in G$;*
4. *If $xa = ya$, then $x = y$.*

   Then G is a medial quasigroup with a $(1, 2)$-identity e.

*Proof.*If $x \in G$, then $xe \cdot e = xe \cdot xx = xx \cdot ex = e \cdot ex = x$. Thus e is a $(1,2)$-identity.

Consider the equation $xa = b$. Then $xa \cdot e = b \cdot e$, $xa \cdot ee = be$ and $xe \cdot ae = be$. Thus $(xe \cdot ae) \cdot (be) = e$, $(xe \cdot b) \cdot (ae \cdot e) = e$, $(xe \cdot b)\, a = e$, $(xe \cdot b) \cdot (ea) = e$, $(xe \cdot e) \cdot (ba) = e$ and $x \cdot ba = e$. Therefore $x \cdot ba = ba \cdot ba$ and $x = ba$. Since $ba \cdot a = ba \cdot ea = be \cdot aa = be \cdot e = b$, the element $x = ba$ is a unique solution of the equation $xa = e$. Now we consider the equation $ay = b$. In this case $be = ay \cdot e = ay \cdot aa = aa \cdot ya = e \cdot ya = ya$. Thus $y = be \cdot a$ is a unique solution of the equation $ay = b$. The proof is complete.

Corollary 3.2.5. Let $(G, \cdot)$ be a left medial loop, $e \in G$ and $x^2 = e$ for every $x \in G$. Then e is a $(1,2)$-identity.

3.3 Homogeneous Isotopes

Definition 3.3.1. *Let $(G, +)$ be a topological groupoid. A groupoid $(G, \cdot)$ is called a homogeneous isotope of the topological groupoid $(G, +)$ if there exist two topological automorphisms $\varphi, \psi : (G, +) \to (G, +)$ such that $x \cdot y = \varphi(x) + \psi(y)$ for all $x, y \in G$.*

If $h : X \to X$ is a mapping, then $h^1(x) = h(x)$ and $h^n(x) = h(h^{n-1}(x))$ for all $x \in X$ and $n \geq 2$.

Definition 3.3.2. *Let $n, m \leq \infty$. A groupoid $(G, \cdot)$ is called an (n, m)-homogeneous isotope of a topological groupoid $(G, +)$ if there exist two topological automorphisms $\varphi, \psi : (G, +) \to (G, +)$ such that:*

1. *$x \cdot y = \varphi(x) + \psi(y)$ for all $x, y \in G$;*
2. *$\varphi\psi = \psi\varphi$;*
3. *If $n < +\infty$, then $\varphi^n(x) = x$ for every $x \in G$.*
4. *If $m < +\infty$, then $\psi^m(x) = x$ for every $x \in G$.*

Definition 3.3.3. *A groupoid $(G, \cdot)$ is called an isotope of a topological groupoid $(G, +)$, if there exist two homeomorphisms $\varphi, \psi : (G, +) \to (G, +)$ such that $x \cdot y = \varphi(x) + \psi(y)$ for all $x, y \in G$.*

Under the conditions of Definition 3.3.3. we shall say that the isotope $(G, \cdot)$ is generated by the homeomorphisms φ, ψ of the topological groupoids $(G, +)$ and denote $(G, \cdot) = g\,(G, +, \varphi, \psi)$.

Theorem 3.3.4 *Let $(G, +)$ be a topological groupoid, $\varphi, \psi : G \to G$ be homeomorphisms and $(G, \cdot) = g\,(G, +, \varphi, \psi)$. Then:*

1. *$(G, +) = (G, \cdot, \varphi^{-1}, \psi^{-1})$;*
2. *$(G, \cdot)$ is a topological groupoid;*
3. *If $(G, +)$ is a reducible groupoid, then $(G, \cdot)$ is a reducible groupoid too;*
4. *If $(G, +)$ is a groupoid with a division, then $(G, \cdot)$ is a groupoid with a division too;*
5. *If $(G, +)$ is a topological primitive groupoid with a division, then $(G, \cdot)$ is a topological primitive groupoid with a division too;*
6. *If $(G, +)$ is a topological quasigroup, then $(G, \cdot)$ is a topological quasigroup too;*
7. *If $n, m, p, k \in N$ and $(G, \cdot)$ is an (n, m)-homogeneous isotop of the groupoid $(G, +)$ and e is a (k, p)-zero in $(G, +)$, then e is an (mk, np)-identity in $(G, \cdot)$.*

Proof. We have $x \cdot y = \varphi\,(x) + \psi\,(y)$. Therefore

$$\varphi^{-1}(x) \cdot \psi^{-1}(y) = \varphi\left(\varphi^{-1}(x)\right) + \psi\left(\psi^{-1}(y)\right) = x + y$$

and $(G, +) = g\,(G, \cdot, \varphi^{-1}, \psi^{-1})$. The assertion 1 is proved. The assertion 2 and 3 are obvious.

Let $(G, +, r, l)$ be a topological primitive groupoid with the divisions, where $l : G \times G \to G$ and $r : G \times G \to G$ be continuous primitive divisions. Then the mappings $l_1(a, b) = \varphi^{-1}\left(l\left(\psi(a), b\right)\right)$ and $r_1(a, b) = \psi^{-1}\left(r\left(\varphi(a), b\right)\right)$ are the divisions of the groupoid $(G, \cdot)$. The divisions l_1, r_1 are continuous if and only if the divisions l, r are continuous. The assertions 4, 5 and 6 are proved.

Let $(G, \cdot)$ be an (n, m)-homogeneous isotope of the groupoid $(G, +)$ and e be a (k, p)-zero in $(G, +)$. We mention that $\varphi^q(e) = \psi^q(e) = e$ for every $q \in N$. If $k < +\infty$, then in $(G, +)$ we have $qk\,(e, x, +) = x$ for each $x \in G$ and for every $q \in N$.

Let $m < +\infty$ and $\psi^m(x) = x$ for all $x \in G$.

Then $1\,(e, x, \cdot) = 1\,(e, \psi(x), +)$ and $q\,(e,x, \cdot) = q\,(e, \psi^q(x), +)$ for every $q \geq 1$. Therefore

$$mk\,(e, x, \cdot) = mk\,\big(e, \psi^{mk}(x), +\big) = mk\,(e, x, +) = x.$$

Analogously we obtain that

$$(e, x, \cdot)\,np = (e, \varphi^{np}(x), +)\,np = (e, x, +)\,np = x.$$

Hence e is an (mk, np)-identity in $(G, \cdot)$. The statement 7 is proved. The proof of Theorem 7.3.4 is complete.

Remark 3.3.5. *Let $(G, +)$ be a topological quasigroup, $a, b \in G$ and φ, ψ be two automorphisms of $(G, +)$. If $x \cdot y = (a + \varphi(x)) + (\psi(y) + b)$, then we denote $(G, \cdot) = g\,(G, +, \varphi, \psi, a, b)$. It is clear that $(G, \cdot)$ is a topological quasigroup too. If $\varphi_1(x) = a + \varphi(x)$ and $\psi_1(x) = \psi(x) + b$, then φ_1, ψ_1 are homeomorphism of $(G, +)$ and $(G, +, \varphi, \psi, a, b) = (G, +, \varphi_1, \psi_1)$.*

3.4 The Homogeneous Isotopes and Congruences

We consider a topological groupoid $(G, +)$.

If α is a relation on G, then $\alpha(x) = \{y \in G : x\alpha y\}$ for every $x \in G$.

An equivalence relation α on G is called a congruence on $(G, +)$ if from $x\alpha u$ and $y\alpha v$ it follows $(x + y)\,\alpha\,(u + v)$. If $(G, +)$ is a primitive groupoid with divisions l and r, then we consider that $l\,(x, y)\,\alpha l\,(u, v)$ and $r\,(x, y)\,\alpha r\,(u, v)$ provided $x\alpha u$ and $y\alpha v$.

Two congruences α and β on G are called conjugate if there exists a topological automorphism $\varphi : G \to G$ such that the relation $x\alpha y$ is equivalent to the relation $\varphi(x)\beta\varphi(y)$.

Let α, β be two conjugate congruences on G and φ be the topological automorphism for which the relation $x\alpha y$ is equivalent to the relation $\varphi(x)\beta\varphi(y)$. Let $\alpha\,(x) = \{y \in G : x\alpha y\}$. Then $\varphi\,(\alpha(x)) = \beta\,(\varphi(x))$. If $\{\beta_\mu : \mu \in M\}$ is a family of congruences on $(G, +)$, then there exists the intersection $\beta = \cap\,\{\beta_\mu : \mu \in M\}$, where $\beta\,(x) = \cap\,\{\beta_\mu\,(x) : \mu \in M\}$. The relation $x\beta y$ is hold, if and only if $x\beta_\mu y$ is hold for every $\mu \in M$.

Theorem 3.4.1. *Let $(G,\cdot) = g(G,+,\varphi,\psi)$ be an isotope of the topological primitive groupoid $(G,+)$ with the divisions $\{r,l\}$, φ,ψ be topological automorphisms of $(G,+)$, and α be a congruence on the groupoid $(G,+,l,r)$. Then:*

1. *If $(G,\cdot)$ is a homogeneous isotope, then there exists a countable set of congruences $\{\beta_n : n \in N\}$ of the groupoid $(G,+)$, conjugate to α, such that $\alpha \in \{\beta_n : n \in N\}$and $\beta = \cap\{\beta_n : n \in N\}$ is a common congruence of the groupoids $(G,+)$ and $(G,\cdot)$.*

2. *If $(G,\cdot)$ is an (n,m)-homogeneous isotope of the groupoid $(G,+)$, and $n,m < +\infty$, then there exists a finite set of congruences $\{\beta_i : i \leq n \cdot m\}$ of the groupoid $(G,+)$, conjugate to α, such that $\beta = \cap\{\beta_i : i \leq n \cdot m\}$ is a common congruence of the groupoids $(G,+)$ and $(G,\cdot)$.*

Proof. Let $Z = \{0, \pm 1, \pm 2, ...\}$ be the set of all integer numbers. If $n = 0$, then $\varphi^0(x) = x$ for all $x \in G$. If $n \in Z$ and $n < 0$, then $\varphi^n = (\varphi^{-1})^{-n}$. Denote by $\{h_n : n \in Z\}$ the set of the all automorphisms

$$\left\{\varphi^{k_1} \circ \psi^{m_1} \circ \varphi^{k_2} \circ \psi^{m_2} \circ ... \circ \varphi^{k_n} \circ \psi^{m_n} : n \in N \, and \, k_1, m_1, ..., k_{n,} m_n \in Z\right\}.$$

If $\varphi\psi = \psi\varphi$, then

$$\{h_n : n \in Z\} = \left\{\varphi^k \circ \psi^m : k, m \in Z\right\}.$$

For each $n \in N$ we define the congruence $\beta_n(x) = h_n(\alpha(x))$ for all $x \in G$.

Denote $\beta = \cap\{\beta_k : k \in N\}$. Then $\varphi(\beta(x)) = \psi(\beta(x)) = \beta(x)$ for each $x \in G$. Hence β is a common congruence of groupoids $(G,+)$ and $(G,\cdot)$. Suppose that automorphisms φ and ψ satisfy the Definition 3 and $(G,\cdot)$ is an (n,m)-isotope of groupoid $(G,+)$. In this case we have

$$\varphi^{k_1} \cdot \psi^{q_1} \cdot \varphi^{k_2} \cdot \psi^{q_2} \cdot ... \cdot \varphi^{k_n} \cdot \psi^{q_n} = \left(\varphi^{k_1+...+k_n}\right) \cdot \left(\psi^{q_1+...+q_n}\right)$$

Therefore

$$\{h_k : k \in N\} = \left\{\varphi^i \cdot \psi^j : i = 1, ..., n; j = 1, ..., m\right\} = \{h_k : k \leq n \cdot m\}$$

and the set $\{\beta_n : n \in N\}$ is finite and contains no more than $n \cdot m$ distinct elements. The proof is complete.

Remark 3.4.2. *Let α and β be two conjugate congruences on a topological groupoid G.*

Then:

1. *The sets* $\alpha(x)$ *are* G_δ*-sets if and only if the sets* $\beta(x)$ *are* G_δ*-sets in* G.

2. *The sets* $\alpha(x)$ *are closed in* G *if and only if the sets* $\beta(x)$ *are closed in* G.

3. *The sets* $\alpha(x)$ *are open in* G *if and only if the sets* $\beta(x)$ *are open in* G.

Remark 3.4.3. *Let* $\{\beta_n : n \in N' \subset N\}$ *be a family of congruences on a topological goupoid* G *and* $\beta = \cap\{\beta_n : n \in N\}$. *Then:*

1. *If the sets* $\beta_n(x)$ *are* G_δ*-sets in* G, *then the sets* $\beta(x)$ *are* G_δ*-sets in* G *too.*

2. *If the set* N' *is finite and the sets* $\beta_n(x)$ *are open, then the sets* $\beta(x)$ *are open in* G.

3.5 General Properties of Medial Quasigroups

Let $(G, \cdot)$ be a topological medial quasigroup. By virtue of Toyoda's Theorem [199] there exist a binary operation $(+)$ on G, two elements $0, c \in G$ and two topological automorphisms $\varphi, \psi : (G, +) \to (G, +)$ such that $(G, +)$ is a topological commutative group, 0 is the zero of $(G, +)$ and $(G, \cdot) = g(G, +, \varphi, \psi, 0, c)$ is a homogeneous isotope of $(G, +)$. In particular, $\varphi\psi = \psi\varphi$.

In [18] G.B. Beleavskaya has proved a generalization of Toyoda's Theorem.

Theorem 3.5.1. *Let* $(G, +)$ *be a topological quasigroup,* $0 \in G, 0+0 = 0, \varphi, \psi$ *be two automorphisms of* $(G, +)$ *and* $(G, \cdot) = (G, +, \varphi, \psi)$. *Then:*

1. $\{0\}$ *is a subquasigroup of the quasigroups* $(G, +)$ *and* $(G, \cdot)$.

2. *If* $n < +\infty$, *then* 0 *is an* (n, ∞)*-identity of* $(G, \cdot)$ *if and only if* $\varphi^n(x) = x$ *for every* $x \in G$.

3. *If* $m < +\infty$, *then* 0 *is an* (∞, m)*-identity of* $(G, \cdot)$ *if and only if* $\psi^m(x) = x$ *for every* $x \in G$.

4. *If* $n, m < +\infty$, *then* 0 *is an* (n, m)*-identity of* $(G, \cdot)$ *if and only if* $\varphi^n(x) = \psi^m(x) = x$ *for every* $x \in G$.

Proof. Let n $< +\infty$.If $\varphi^n(x) = x$ for every $x \in G$, then from Theorem 4 it follows that 0 is an $(n, +\infty)$-identity in $(G, \cdot)$.

Let 0 be an (n, ∞)-identity in $(G, \cdot)$. By construction, $\varphi(0) = \psi(0) = 0$ and $x \cdot y = \varphi(x) + \psi(y)$. Then $(x, 0)\,k = \varphi^k(x)$ and $(0, x)\,k = \varphi^k(x)$ for every $k \in N$. Since $(x, 0)\,n = x$ we obtain that $\varphi^n(x) = x$. The proof is complete.

Consider on G some equivalence relation α. Denote by G/α the collection of classes of equivalence $\alpha(x)$ and $\pi_\alpha : G \to G/\alpha$ is the natural projection. On G/α we consider the quotient topology. The mapping π_α is continuous. If α is a congruence on $(G, \cdot)$ (or on $(G, +)$), then the mapping π_α is open.

An equivalence relation α on G is called compact if the sets $\alpha(x)$ are compact.

Theorem 3.5.2. *Let $(G, +)$ be a commuttive topological group, 0 be a zero of $(G, +)$, $c \in G$, φ and ψ be two automorphisms of the topological group $(G, +)$ and $(G, \cdot) = g(G, +, \varphi, \psi, 0, c)$. If the space G contains a non-empty compact subset F of countable character, then for every open subset U of G containing 0 there exists a compact equivalence relation α_U on G such that:*

1. *$\alpha_U(0) \subseteq U$.*
2. *α_U is a congruence on $(G, \cdot)$.*
3. *α_U is a congruence on $(G, +)$.*
4. *The natural projection $\pi_U = \pi_{\alpha_U} : G \to G/\alpha_U$ is an open perfect mapping.*
5. *The space G/α_U is metrizable.*

Proof. We consider that $0 \in F \subseteq U$. Fix a sequence $\{U_n : n \in N\}$ of open subsets of G such that for every open set V containing F there exists $n \in N$ such that $F \subseteq U_n \subseteq V$. Suppose that $F \subseteq U_n$ and $U_{n+1} \subseteq U_n$ for every $n \in N$.

Then there exists a sequence $\{V_n : n \in N\}$ of open sets of G such that:

- $V_{n+1} + V_{n+1} \subseteq V_n \subseteq U_n$, $\quad cl_G V_{n+1} \subseteq V_n$ and $V_n = -V_n$ for every $n \in N$,

- $\varphi(V_{n+1}) \cup \psi(V_{n+1}) \subseteq V_n$ for every $n \in N$.

We put $H = \cap\{V_n : n \in N\}$. By construction, H is a compact subgroup and the natural projection $\pi : G \to G/H$ is open and perfect. Let $\alpha(x) = x + H$ for every $x \in G$. Then α is a congruence on $(G, +)$. Suppose that $x\alpha z$ and $y\alpha v$. Then

$$\begin{aligned} x \cdot y &= \varphi(x) + \psi(y) + c, \\ z \cdot v &= \varphi(z) + \psi(v) + c, \\ \varphi(x) - \varphi(z) &\in H, \ \psi(y) - \psi(v) \in H. \end{aligned}$$

Thus

$$\begin{gathered} (x \cdot y) - (z \cdot v) = \\ = (\varphi(x) + \psi(y)) - (\varphi(z) + \psi(v)) = \\ = (\varphi(x) - \varphi(z)) + (\psi(y) - \psi(v)) \in H. \end{gathered}$$

Therefore α is a congruence on $(G, \cdot)$ too.

It is clear that the space G/H is metrizable. The proof is complete.

Corollary 3.5.3. *A first countable topological medial quasigroup is metrizable.*

A space X is called a paracompact p-space if there exists a perfect mapping $g : X \to Y$ onto some metrizable space Y [6].

Corollary 3.5.4. *If a topological medial quasigroup contains a non-empty compact subset of countable character then it is a paracompact space p-space and admits an open perfect homomorphism onto a medial metrizable quasigroup.*

Corollary 3.5.5. *A Čech complete topological medial quasigroup is paracompact and admits an open perfect homomorphism onto a complete metrizable medial quasigroup.*

Corollary 3.5.6. *A locally compact medial quasigroup is paracompact and admits an open perfect homomorphism onto a metrizable locally compact medial quasigroup.*

3.6 On Haar Measures on Medial Quasigroups

By $B(X)$ denote the family of all Borel subsets of the space X.

A non-negative real-valued function μ defined on the family $B(X)$ of Borel subsets of a space X is said to be a Radon measure on X if it has the following properties:

- $\mu(H) = \sup\{\mu(F) : F \subseteq H, F$ is a compact subset of $H\}$ for every $H \in B(X)$;

- for every point $x \in X$ there exists an open subset V_x such that $x \in V_x$ and $\mu(V_x) < \infty$.

Definition 3.6.1. *Let $(A, \cdot)$ be a topological quasigroup with the divisions $\{r, l\}$. A Radon measure μ on A is called:*

*- a **left invariant Haar measure**, if $\mu(U) > 0$ and $\mu(xH) = \mu(H)$ for every non-empty open set $U \subseteq A$, a point $x \in A$ and a Borel set $H \in B(A)$;*

*- a **right invariant Haar measure**, if $\mu(U) > 0$ and $\mu(Hx) = \mu(H)$ for every non-empty open set $U \subseteq A$, a point $x \in A$ and Borel set $H \in B(A)$;*

*- an **invariant Haar measure** if $\mu(U) > 0$ and $\mu(xH) = \mu(Hx) =$*

$\mu(l(x,H)) = \mu(r(H,x)) = \mu(H)$ *for every non-empty open set* $U \subseteq A$*, a point* $x \in A$ *and a Borel set* $H \in B(A)$;

Definition 3.6.2. *We say that on a topological quasigroup* $(A,\cdot)$ ***there exists a unique left (right) invariant Haar measure****, if for every two left (right) invariant Haar measures* μ_1*,* μ_2 *on* A *there exists a constant* $c > 0$ *such that* $\mu_2(H) = c \cdot \mu_1(H)$ *for every Borel set* $H \in B(A)$.

If $(G,+)$ is a locally compact commutative group, then on G there exists a unique invariant Haar measure μ_G [114].

Theorem 3.6.3. *Let* $(G,\cdot)$ *be a locally compact medial quasigroup,* $(G,+)$ *be a commutative topological group,* $\varphi, \psi : G \to G$ *be automorphisms of* $(G,+), b \in G$ *and* $(G,\cdot) = g(G,+,\varphi,\psi,0,b)$. *On the group* $(G,+)$ *consider the invariant Haar measure* μ_G*. Then :*

1. *On* $(G,\cdot)$ *the right (left) invariant Haar measure is unique.*
2. *If* μ *is a left (right) invariant Haar measure on* $(G,\cdot)$, *then* μ *is a left (right) invariant Haar measure on* $(G,+)$ *too.*
3. *On* $(G,\cdot)$ *there exists some right invariant Haar measure if and only if* $\mu_G(\varphi(H)) = \mu_G(H)$ *for every* $H \in B(A)$.
4. *If* $n < +\infty$, *and on* G *there exists some* $(n, +\infty)-$*identity, then on* $(G,\cdot)$ *the measure* μ_G *is a unique right invariant Haar measure.*
5. *If* $m < +\infty$, *and on* G *there exists some* $(+\infty, m)-$*identity, then on* $(G,\cdot)$ *the measure* μ_G *is a unique left invariant Haar measure.*
6. *If* $n, m < +\infty$, *and on* G *there exists some* $(n, m)-$*identity, then on* $(G,\cdot)$ *the measure* μ_G *is a unique invariant Haar measure.*

Proof. Let μ be a right invariant Haar measure on $(G,\cdot)$. Since $x \cdot y = \varphi(x) + \psi(y) + b$ for all $x, y \in G$, then $Hx = \varphi(H) + \psi(H) + b$. Thus μ is an invariant Haar measure on $(G,+)$ and there exists a constant $c > 0$ such that $\mu(H) = c \cdot \mu_G(H)$. Thus μ_G is a right invariant Haar measure on $(G,\cdot)$. The assertions 1,2 and 3 are proved.

Consider some topological automorphism h of $(G,+)$. Then $\mu_h(H) = \mu_G(h(H))$ is an invariant Haar measure on $(G,+)$. There exists a constant $c_h > 0$ such that

$\mu_h(H) = \mu_G(h(H)) = c_h \cdot \mu_G(H)$ for every Borel subset $H \in B(G)$. In particular, $\mu_G\left(h^k(H)\right) = c_h^k \mu_G(H)$ for every $k \in N$. If $n < +\infty$ and 0 is an $(n, +\infty)-$identity, then $\varphi^n(x) = x$ for every $x \in G$ and $c_\varphi^n = 1$. Thus $c_\varphi = 1$, $\mu_G(H) = \mu_G(h(H))$ and μ_G is a right invariant Haar measure on $(G, \cdot)$. The assertions 4, 5 and 6 are proved. The proof is complete.

In this way we can prove the following results.

Theorem 3.6.3. *Let $(G, +)$ be a topological quasigroup and $(G, \cdot)$ be an (n, m)-homogeneous isotope of $(G, +)$. Then:*

1. *On $(G, +)$ there exists a left (right) invariant Haar measure if and only if on $(G, \cdot)$ there exists a left (right) invariant Haar measure.*
2. *If on $(G, +)$ the a left (right) invariant Haar measure is unique, then on $(G, \cdot)$ the a left (right) invariant Haar measure is unique too.*

Theorem 3.6.4. *On a compact medial quasigroup G there exists a unique Haar measure μ for which $\mu(G) = 1$.*

Theorem 3.6.5. *Let $(G, +)$ be a locally compact group, μ_G be the left invariant Haar measure on $(G, +)$ and $\varphi, \psi : G \to G$ be the topological automorphism of $(G, +)$. Fix $c \in G$ and consider the binary operation $x \cdot y = \varphi(x) + \psi(y) + c$. Then:*

1. *$(G, \cdot)$ is a topological quasigroup.*
2. *If $\mu_G(\psi(H)) = \mu_G(H)$ for every Borel subset $H \in B(G)$, then μ_G is a left invariant Haare measure on $(G, \cdot)$.*
3. *If $m \in N$ and $\psi^m(x) = x$ for every $x \in G$, then μ_G is a left invariant Haar measure on $(G, \cdot)$.*
4. *If $(G, +)$ is a compact group, then μ_G is an invariant Haar measure on $(G, \cdot)$.*

3.7 Examples of Quasigroups with Multiple Identities

Example 3.7.1. Let $(R, +)$ be a topological commutative group of real numbers, $a > 0$, $b > 0$, $\varphi(x) = ax$, $\psi(y) = bx$ and $x \cdot y = \varphi(x) + \psi(y)$. Then

$(R, \cdot)$ is a commutative locally compact medial quasigroup. If $H = [c, d]$, then $0 \cdot H = [ac, ad]$ and $H \cdot 0 = [bc, bd]$. Thus:

- on $(G, \cdot)$ there exists some right invariant Haar measure if and only if $a = 1$;
- on $(G, \cdot)$ there exists some left invariant Haar measure if and only if $b = 1$;
- if $a \neq 1$ and $b \neq 1$, then on $(G, \cdot)$ does not exist any left or right invariant Haar measure.

Example 3.7.2. Denote by $Z_p = Z/pZ = \{0, 1, ..., p-1\}$ the cyclic Abelian group of order p. Consider the Abelian group $(G, +) = (Z_5, +)$ and $\varphi(x) = 2x$, $\psi(x) = 4x$. Then $(G, \cdot) = g(G, +; \varphi, \psi)$ is a medial quasigroup and each element from $(G, \cdot)$ is $(2, 4)$-identity in G.

Example 3.7.3. Consider the Abelian group $(G, +) = (Z_5, +)$ and $\varphi(x) = \psi(x) = 3x$. Then $(G, \cdot) = g(G, +; \varphi, \psi)$ is medial quasigroup and all elements from $(G, \cdot)$ are the $(4, 4)$-identities in G.

Example 3.7.4. Consider the commutative group $(G, +) = (Z_5, +)$, $\varphi(x) = 2x, \psi(x) = 2x + 1$ and $x \cdot y = 2x + 2y + 1$. Then $(G, \cdot) = g(G, +; \varphi, \psi, 0, 1)$ is a commutative medial quasigroup and $(G, \cdot)$ does not contain (n, m)-identities.

Example 3.7.5. Consider the commutative group $(G, +) = (Z, +)$, $\varphi(x) = x, \psi(x) = x + 1$ and $x \cdot y = x + y + 1$. Then $(G, \cdot) = g(G, +; \varphi, \psi)$ is a medial quasigroup and $(G, \cdot)$ does not contain (n, m)-identities. On $(G, \cdot)$ there exists an invariant Haar measure.

Example 3.7.6. Let $(G, +)$ be an Abelian group and $x + x \neq 0$ for each $x \in G$. For example $(G, +) \in \{(Z_p, +) : p \in N, p \geq 2\}$. Denote $\varphi(x) = x$ and $\psi(x) = -x$ for each $x \in G$. Then $(G, \cdot) = g(G, +; \varphi, \psi)$ is a medial quasigroup and $(G, \cdot)$ contains the unique $(1, 2)$-identity, which coincide with the zero element in $(G, +)$.

Example 3.7.7. Let $(G, +) = (Z_7, +)$, and $\varphi(x) = 3x$ and $\psi(x) = 5x$. Then $(G, \cdot) = g(G, +; \varphi, \psi)$ is a medial quasigroup. In this case 0 and 3 are $(12, 6)$-identities.

3.8. On Medial and Paramedial Topological Groupoids

In Sections 3.8 - 3.11 we study the (n, m)-homogeneous isotopies of topological groupoids with multiple identities and relation between paramediality and associativity. The concept of multiple identities and homogeneous isotopies, introduced in [63], facilitates the study of topological groupoids with (n, m)-identities and homogeneous quasigroups, which are obtained by using isotopies of topological groups.

The results established are related to the results of M. Choban and L. Kiriyak [63] and to the research papers [57, 73, 74, 139, 142, 143]. We prove that if $(G, +)$ is a medial topological groupoid and e is a (k, p)-zero, then every (n, m)-homogeneous isotope $(G, \cdot)$ of $(G, +)$ is medial, with (mk, np)-identity e in $(G, \cdot)$. We present some interesting properties of a class of (n, m)-homogeneos quasigroups.

K. Sigmon, continuing the work of Professor A.D. Wallace, has shown that whenever a medial topological groupoids contains a bijective idempotent, it can be obtained from some commutative topological semigroup [194]. In Section 3, we obtain these and some other results in the case of paramedial topological groupoids. The relationship between mediality, paramediality and associativity was also studied in [75, 194]. In Section 4 we extended one well-known statement of the theory of topological groups on the class of topological (n, m)-homogeneous primitive groupoids with divisions.

A groupoid G is called paramedial if it satisfies the law $xy \cdot zt = ty \cdot zx$ for all $x, y, z, t \in G$. A groupoid G is said to be hexagonal if it is idempotent, medial and semisymmetric, i.e. the equalities $x \cdot x = x$, $xy \cdot zt = xz \cdot yt$, $x \cdot zx = xz \cdot x = z$ hold for all of its elements.

If a medial (paramedial) guasigroup G contains an element e such that $e \cdot x = x(x \cdot e = x)$ for all x in G, then e is called a left (right) identity element of G and G is called a left (right) medial (paramedial) loop.

Let $N = \{1, 2, ...\}$ and $Z = \{..., -2, -1, 0, 1, 2, ...\}$.

Example 3.8.1. Let $(G, \cdot)$ be a paramedial groupoid, $e \in G$ and $xe = x$ for every $x \in G$. Then $(G, \cdot)$ is paramedial groupoid with $(2, 1)$-identity e in G. Actually, if $x \in G$, then $e \cdot ex = ee \cdot ex = xe \cdot ee = xe \cdot e = xe = x$.

Example 3.8.2. Let $G = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$. We define the binary operation $\{\cdot\}$.

Table 3.1. Example of not-commutative hexagonal quasigroup

$(\cdot)$	1	2	3	4	5	6	7	8	9
1	1	8	6	2	9	4	3	7	5
2	4	2	9	5	3	7	6	1	8
3	7	5	3	8	6	1	9	4	2
4	6	1	8	4	2	9	5	3	7
5	9	4	2	7	5	3	8	6	1
6	3	7	5	1	8	6	2	9	4
7	8	6	1	9	4	2	7	5	3
8	2	9	4	3	7	5	1	8	6
9	5	3	7	6	1	8	4	2	9

Then $(G, \cdot)$ is a not-commutative hexagonal quasigroup and each element from $(G, \cdot)$ is a $(6, 6)$-identity in G.

Example 3.8.3. Let $(R, +)$ be a topological Abelian group of real numbers.
1. If $\varphi(x) = x$, $\psi(x) = 2x$ and $x \cdot y = x + 2y$, then $(R, \cdot) = g(R, +, \varphi, \psi)$ is a commutative locally compact medial quasigroup. By virtue of Theorem 7 from [63], there exists a right invariant Haar measure on $(R, \cdot)$.
2. If $\varphi(x) = x$, $\psi(x) = x + 7$ and $x \cdot y = x + y + 7$, then $(R, \cdot) = g(R, +, \varphi, \psi)$ is a commutative locally compact medial quasigroup and $(R, \cdot)$ does not contain (n, m)-identities. As above, by virtue of Theorem 7 from [63] there exist an invariant Haar measure on $(R, \cdot)$.

Example 3.8.4. Denote by $Z_p = Z/pZ = \{0, 1, ..., p-1\}$ the cyclic Abelian group of order p. Consider the commutative group $(G, +) = (Z_7, +)$, $\varphi(x) = 3x$, $\psi(x) = 4x$ and $x \cdot y = 3x + 4y$. Then $(G, \cdot) = g(G, +, \varphi, \psi)$ is a medial quasigroup with $(3, 6)$-identity in $(G, \cdot)$, which coincides with the zero element in $(G, +)$.

Example 3.8.5. Consider the commutative group $(G, +) = (Z_5, +)$, $\varphi(x) = 2x$, $\psi(x) = 3x$ and $x \cdot y = 2x + 3y$. Then $(G, \cdot) = g(G, +, \varphi, \psi)$ is a medial quasigroup and the zero from $(G, \cdot)$ is a $(4, 4)$-identity in G.

Example 3.8.6. Consider the Abelian group $(G, +) = (Z_5, +)$, $\varphi(x) = 4x$, $\psi(x) = 2x$ and $x \cdot y = 4x + 2y$. Then $(G, \cdot) = g(G, +, \varphi, \psi)$ is a medial quasigroup and each element from $(G, \cdot)$ is a $(4, 2)$-identity in G.

3.9. Some properties of (n, m)-homogeneous isotopies

Proposition 3.9.1 *If* $(G, +)$ *is medial topological groupoid, and* e *is* (k, p)*-zero, then every* (n, m)*-homogeneous isotope* $(G, \cdot)$ *of the topological groupoid* $(G, +)$ *is medial with* (mk, np)*-identity* e *in* $(G, \cdot)$ *and* $(x\cdot y)+(u\cdot v) = (x+u)\cdot(y+v)$ *for all* $x, y, u, v \in G$ *and* $n, m, p, k \in N$.

Proof. The mediality of the (n, m)-homogeneous isotope $(G, \cdot)$ and the fact that e is (mk, np)-identity in $(G, \cdot)$ follows from [63]. Using the algorithm from [194] we will show that $(x \cdot y) + (u \cdot v) = (x + u) \cdot (y + v)$. Let $x \cdot y = \varphi(x) + \psi(y)$ and $u \cdot v = \varphi(u) + \psi(v)$. Then

$$(x \cdot y) + (u \cdot v) = [\varphi(x) + \psi(y)] + [\varphi(u) + \psi(v)] =$$
$$= [\varphi(x) + \varphi(u)] + [\psi(y) + \psi(v)] = \varphi(x + u) + \psi(y + v) = (x + u) \cdot (y + v).$$

In this way we have that $(x \cdot y) + (u \cdot v) = (x + u) \cdot (y + v)$. The proof is complete.

Corollary 3.9.2 *If* $(G, +)$ *is medial topological groupoid, then every homogeneous isotope* $(G, \cdot)$ *where* $\varphi\psi = \psi\varphi$ *of topological groupoid* $(G, +)$ *is medial and* $(x \cdot y) + (u \cdot v) = (x + u) \cdot (y + v)$.

Definition 3.9.3 *A topological quasigroup* $(G, \cdot)$ *is called:*

- *homogeneous, if* $(G, \cdot)$ *is a homogeneous isotope of the topological group* $(G, +)$.

- (n, m)*-homogeneous, if* $(G, \cdot)$ *is a* (n, m)*-homogeneous isotope of the topological group* $(G, +)$.

We denote by:

- T the class of all medial quasigroup.

- $Q(n, m)$ the class of all (n, m)-homogeneous quasigroup.

We consider the class: $M(n, m) = T \cap Q(n, m)$.

The class $M(1, 1)$ coincides with the class of topological Abelian groups.

Example 3.9.4 Let $(G, \cdot)$ be a topological medial quasigroup, $e \in G$ and $ex = x$, $xx = e$ for each $x \in G$. Then $(G, \cdot) \in M(1, 2)$. Hence $(G, \cdot)$ is a topological medial quasigroup with $(1, 2)$-identity e in G.

Theorem 3.9.5 *Let* $Q(n, m)$ *be a class of* (n, m)*-homogeneous quasigroups. Then:*

1. For each $G \in Q(n,m)$ *there exists an* (n,m)*-identity* $e \in G$ *with properties:*

1.1 $e \cdot e = e$*;*

1.2 $n(e,x) = x$*;*

1.3 $(x,e)m = x$*;*

1.4 $ex \cdot e = e \cdot xe$*;*

2. If $\varphi(x) = ex$ *and* $\varphi^n(x) = n(e,x) = x$*, then* $\varphi^{-1}(x) = (n-1)(e,x)$*;*

3. If $\varphi^{-1}(x) = (n-1)(e,x)$ *and* $\varphi^n(x) = n(e,x) = x$*, then* $(n-1)(e,ex) = x$*;*

4. If $\psi(x) = xe$ *and* $\psi^n(x) = (x,e)m = x$*, then* $\psi^{-1}(x) = (x,e)(m-1)$*;*

5. If $\psi^{-1}(x) = (x,e)(m-1)$ *and* $\psi^m(x) = (x,e)m = x$*, then* $(xe,e)(m-1) = x$*.*

Proof. 1. Let $(G,+)$ be a group and $\varphi, \psi : G \longrightarrow G$ are automorphisms of this group, such that $\varphi^n(x) = \psi^m(x) = x, \varphi \cdot \psi = \psi \cdot \varphi$, for each $x \in G$ and $(G,\cdot) = g(G,+,\varphi,\psi)$. Let e be a zero in $(G,+)$. According to Theorem 3 from [63], e is an (n,m)-identity in $(G,\cdot)$. Hence, $e \cdot e = e$, $n(e,x) = x$ and $(x,e)m = x$. Thus assertion 1.1, 1.2 and 1.3 are proved. It is easy to see that $\varphi(x) = ex$ and $\psi(x) = xe$. From the equality $\varphi\psi = \psi\varphi$ we have $\varphi\psi = \varphi(xe) = e \cdot xe$ and $\psi\varphi = \psi(ex) = ex \cdot e$. Therefore $e \cdot xe = ex \cdot e$. Assertion 1 is proved.

2. We will show that, if $\varphi(x) = ex$ and $\varphi^n(x) = n(e,x) = x$, then $\varphi^{-1}(x) = (n-1)(e,x)$.

We have $\varphi(x) = ex$. Hence $\varphi(\varphi^{-1}(x)) = e \cdot \varphi^{-1}(x)$. But $\varphi(\varphi^{-1}(x)) = x$. Then $e \cdot \varphi^{-1}(x) = x$. According to condition $n(e,x) = x$. Then

$$e \cdot (\varphi^{-1}(x)) = n(e,x) \tag{3.1}$$

By definition of multiple identies we have

$$e \cdot (n-1)(e,x) = n(e,x) \tag{3.2}$$

From (3.1) and (3.2) we obtain $\varphi^{-1}(x) = (n-1)(e,x)$. Assertion 2 is proved.

3. We will prove that, if $\varphi^{-1}(x) = (n-1)(e,x)$ and $\varphi^n(x) = n(e,x) = x$ then $(n-1)(e,ex) = x$.

Let be $(n-1)(e,ex) = t$. Then

$$e \cdot (n-1)(e,ex) = et \tag{3.3}$$

By definition of multiple identies

$$e \cdot (n-1) \cdot (e,ex) = n(e,ex) = ex \tag{3.4}$$

From (3.3) and (3.4) it follows $ex = et$ and $t = x$. Hence $(n-1)(e, ex) = x$. The assertions is proved.

4. Analogously to properties 2 we obtain that if $\psi(x) = xe$ and $\psi^m(x) = (x, e)m$, then $\psi^{-1}(x) = (x, e)(m-1)$.

5. Similarly to properties 3 we prove if $\psi^{-1}(x) = (x, e)(m-1)$ and $\psi^m(x) = (x, e) \cdot m = x$, then $(xe, e)(m-1) = x$.

The proof of Theorem is complete.

Corollary 3.9.6 *A class $Q(n, m)$ of (n, m)-homogeneous quasigroup forms a variety.*

Corollary 3.9.7 *A class $M(n, m)$ of topological medial quasigroup with (n, m)-identities forms a variety.*

3.10 Paramedial topological groupoids

We show one example of a paramedial groupoid which is not medial.

Example 3.10.1. Let $G = \{1, 2, 3, 4\}$. We define the binary operation $\{\cdot\}$.

Table 3.2. Example of paramedial quasigroup which is not-medial

$(\cdot)$	1	2	3	4
1	1	2	4	3
2	3	4	2	1
3	2	1	3	4
4	4	3	1	2

Then $(G, \cdot)$ is a paramedial quasigroup but it is not-medial. For example, $(2 \cdot 3) \cdot (1 \cdot 4) \neq (2 \cdot 1) \cdot (3 \cdot 4)$.

An element e is called idempotent if $ee = e$, bijective if the maps $x \to xe$ and $x \to ex$ are homeomorphisms.

Theorem 3.10.2. *Let $(G, \cdot)$ be a paramedial topological groupoid and e, e_1 and e_2 are elements of G for which:*

1. $ee_1 = e_1$ and $e_2 e = e_2$;

2. The maps $x \to e_1 x$ and $x \to x e_2$ are homeomorphisms of G onto itself;

3. The map $x \to xe$ is surjective;

If there exists on G a binary operation $\{\circ\}$ such that $(e_1 x) \circ (y e_2) = yx$ then $(G, \circ)$ is a commutative topological semigroup having $e_1 e_2$ as identity.

Proof. Since $x \to e_1 x$ and $x \to x e_2$ are homeomorphism it is clear that $\{\circ\}$ is continuous.

Using the surjectivity and that

$$(e_1 e_2) \circ (y e_2) = y e_2$$
$$(e_1 x) \circ (e_1 e_2) = e_1 x$$

we see that $e_1 e_2$ is an identity for $(G, \circ)$. Observe that $x e_1 \cdot e_2 = $ $= x e_1 \cdot e_2 e = e e_1 \cdot e_2 x = e_1 \cdot e_2 x$.

We observe that

$$\begin{aligned} x e_1 \cdot zt &= (e_1 \cdot zt) \circ (x e_1 \cdot e_2) = \\ &= (e e_1 \cdot zt) \circ (x e_1 \cdot e_2) = \\ &= (t e_1 \cdot ze) \circ (x e_1 \cdot e_2) = \\ &= [(e_1 \cdot ze) \circ (t e_1 \cdot e_2)] \circ (x e_1 \cdot e_2); \end{aligned}$$

$$\begin{aligned} t e_1 \cdot zx &= (e_1 \cdot zx) \circ (t e_1 \cdot e_2) = \\ &= (e e_1 \cdot zx) \circ (t e_1 \cdot e_2) = \\ &= (x e_1 \cdot ze) \circ (t e_1 \cdot e_2) = \\ &= [(e_1 \cdot ze) \circ (x e_1 \cdot e_2)] \circ (t e_1 \cdot e_2). \end{aligned}$$

From paramediality we have

$$[(e_1 \cdot ze) \circ (t e_1 \cdot e_2)] \circ (x e_1 \cdot e_2) = [(e_1 \cdot ze) \circ (x e_1 \cdot e_2)] \circ (t e_1 \cdot e_2).$$

We put $z = e_2$ and since $e_2 e = e_2$ and $e_1 e_2$ is identity it follows that:

$$[e_1 e_2 \circ (t e_1 \cdot e_2)] \circ (x e_1 \cdot e_2) = [e_1 e_2 \circ (x e_1 \cdot e_2)] \circ (t e_1 \cdot e_2)$$

and

$$(t e_1 \cdot e_2) \circ (x e_1 \cdot e_2) = (x e_1 \cdot e_2) \circ (t e_1 \cdot e_2).$$

Hence, $(G, \circ)$ is a commutative topological groupoid and then the associativity is immediate.

$$[(te_1 \cdot e_2) \circ (e_1 \cdot ze)] \circ (xe_1 \cdot e_2) = (te_1 \cdot e_2) \circ [(e_1 \cdot ze) \circ (xe_1 \cdot e_2)] .$$

The proof is complete.

Theorem 3.10.3. *Let $(G, \cdot)$ be a paramedial topological groupoid satisfying the following conditions:*

1. It contains an idempotent e;

2. The maps $x \to xe$ and $x \to ex$ are homeomorphisms of G onto itself;

3. There exists a binary operation $\{\circ\}$ on G such that $(ex) \circ (ye) = yx$.

Then $(G, \circ)$ is a commutative semigroup having e as identity. Furthermore, the maps $x \to xe$ and $x \to ex$ are antihomomorphisms of $(G, \circ)$ and $xe \cdot e = e \cdot ex$.

Proof. The first part of the Theorem 3.10.3 follows from Theorem 3.10.2 with $e = e_1 = e_2$. Indeed, we have $xe \cdot e = xe \cdot ee = ee \cdot ex = e \cdot ex$. Since

$$(ex \circ ye)\, e = yx \cdot e = yx \cdot ee = ex \cdot ey = (e \cdot ey) \circ (ex \cdot e) = (ye \cdot e) \circ (ex \cdot e) ,$$

we see that $x \to xe$ is an antihomomorphism of $(G, \circ)$. Similarly

$$e\, (ex \circ ye) = e \cdot yx = ee \cdot yx = xe \cdot ye = (e \cdot ye) \circ (xe \cdot e) = (e \cdot ye) \circ (e \cdot ex).$$

We obtain that $x \to ex$ is an antihomomorphism of $(G, \circ)$. The proof is complete.

A topological groupoid $(G, \circ)$ is called radical if the map $s : G \to G$ defined by $s\,(x) = x \circ x$ is a homeomorphism.

If $(G, \circ)$ is paramedial and radical then s, and hence s^{-1} is an antihomomorphism of $(G, \circ)$. A topological groupoid $(G, \cdot)$ where $\{\cdot\}$ is defined by

$$x \cdot y = s^{-1}\,(x) \circ s^{-1}\,(y) = s^{-1}\,(y \circ x)$$

is called the radical isotope of $(G, \circ)$.

A radical isotope $(G, \cdot)$ of $(G, \circ)$ is idempotent since,

$$x \cdot x = s^{-1}\,(x \circ x) = s^{-1}\,(s\,(x)) = x$$

for each $x \in G$.

Theorem 3.10.4. *If $(G, \circ)$ is a topological groupoid with unit e, $(G, \cdot)$ is commutative, idempotent topological groupoid and*

$$(x \circ y) \cdot (z \circ t) = (ty) \circ (zx),$$

then $(G, \circ)$ *is a commutative radical semigroup.*

Proof. If we define $t : G \to G$ by $t(x) = ex$ then t is an antihomomorphism of $(G, \circ)$. Indeed, for all $x, y \in G$ we have,

$$t(x \circ y) = e(x \circ y) = (e \circ e)(x \circ y) = (ye) \circ (xe) = (ey) \circ (ex) =$$
$$= t(y) \circ t(x).$$

In particular, we obtain

$$t(s(x)) = t(x \circ x) = t(x) \circ t(x) = s(t(x));$$

where $s : G \to G$ is defined by $s(x) = x \circ x$.

Also, for each $x, y \in G$, and e unit in $(G, \circ)$

$$xy = (e \circ x) \cdot (e \circ y) = (e \circ x) \cdot (y \circ e) = (ex) \circ (ye) =$$
$$= (ex) \circ (ey) = t(x) \circ t(y) = t(y \circ x).$$

Hence $t(s(x)) = t(x \circ x) = xx = x$.

It follows that t is a continuous inverse for s so that that $(G, \circ)$ is the radical. Since $(G, \cdot)$ is commutative and $x \circ y = s(xy) = s(yx) = y \circ x$ then $\{\circ\}$ is commutative. Since $xy = t(y \circ x)$ and $t = s^{-1}$ then $(G, \cdot)$ is the radical isotope of $(G, \circ)$.

It only remains to show that $\{\circ\}$ is associative.

Since t is bijective and

$$t[(x \circ y) \circ z] = z \cdot (x \circ y) = (e \circ z) \cdot (x \circ y) = (yz) \circ (xe) =$$
$$= (yz) \circ (ex) = t(z \circ y) \circ t(x) = t[x \circ (z \circ y)] = t[x \circ (y \circ z)].$$

Hence, $(G, \circ)$ is a commutative radical semigroup. The proof is complete.

3.11 On subquasigroup of the topological quasigroup

We consider a topological groupoid $(G, +)$. If α is a relation on G, then $\alpha(x) = \{y \in G : x\alpha y\}$ for every $x \in G$. An equivalence relation α on G is called a congruence on $(G, +)$ if from $x\alpha u$ and $y\alpha v$ it follows $(x + y)\alpha(u + v)$. If $(G, +)$ is a primitive groupoid with divisions l and r, then we consider that $l(x, y)\alpha l(u, v)$ and $r(x, y)\alpha r(u, v)$ provided $x\alpha u$ and $y\alpha v$.

Let $(G, +)$ be a topological quasigroup with (k, p)-zero e and $(G, \cdot) = g(G, +, \varphi, \psi)$ is (n, m)-homogeneous isotope. Then, by virtute of Theorem 3.3.5 from [63], e is (mk, np)-identity of the topological quasigroup $(G, \cdot)$.

Definition 3.11.1 *A subquasigroup H of the quasigroup $(G, +)$ is called a normal subquasigroup, if $e \in H$ and $H = G(\alpha)$ for some congruence α.*

Lemma 3.11.2 *Let α be a congruence of the topological guaisgroup $(G, +)$. Then there exists an unique normal subquasigroup $G(\alpha)$, which is called quasigroup defined by congruence α such that $e \in G(\alpha)$.*

Proof. The set $G(\alpha) = \alpha(e)$ is the desired subquasigroup.

Definition 3.11.3 *A subquasigroup H_1 and H_2 of the topological quasigroup $(G, +)$ are called conjugate, if $H_2 = h(H_1)$ for some topological automorphism $h : G \to G$.*

Theorem 3.11.4 *Let H be a subquasigroup of the topological quasigroup $(G, +)$ and $e \in H$. Then there exists such subquasigroup Q of the quasigroup $(G, +)$ and $(G, \cdot)$ for which:*

1. $e \in Q \subseteq H$.

2. Q is the intersection of a finite number of the subquasigroups conjugate to H of the quasigroup $(G, +)$.

3. If H is closed set, then Q is closed too.

4. If H is a G_δ-set, then Q is G_δ-set too.

5. If H is a open set, then Q is open too.

6. If H is a normal subquasigroup, then Q is a normal subquasigroup $(G, +)$ and $(G, \cdot)$.

Proof. We put $\{h_p : p \leq n \cdot m\} = \{\varphi^i \cdot \psi^j \leq n, j \leq m\}$, $H_p = h_p(H)$ and $Q = \cap\{H_p : p \leq n \cdot m\}$.

Fix $i \leq n$ and $j \leq n$. Let $h_p = \varphi^i \cdot \psi^j$. It is clear that h_p is automorphism of

$(G, +)$. Thus $H_p = h_p(H)$ is a subquasigroup of $(G, +)$ conjugate to H in $(G, +)$. Therefore Q is a subquasigroup of $(G, +)$. The assertions 1-5 are proved. In the first we prove that Q is a subquasigroup of $(G, \cdot)$. Let $x, y, b \in Q$. Then $xy = \varphi(x) + \psi(y)$ and $\varphi(x), \psi(y) \in H_i$ for any i. Thus $xy \in Q$. If $ax = b$, then $a \in H_i$ for every and $a \in Q$. Hence Q is a subquasigroup of $(G, \cdot)$. Let α be a congruence of the $(G, +)$. Then, by virtute of **Lemma 3.11.2**, there exists an unique normal subquasigroup $H = G(\alpha)$ and $e \in H$. Because h_p is topological automorphism of $(G, +)$, then $H_p = h_p(H)$ is a normal subquasigroups of $(G, +)$ conjugate to normal subquasigroup H. Therefore Q is a normal subquasigroup of $(G, +)$. But Q is a subquasigroup of $(G, \cdot)$. Thus Q is a normal subquasigroup of $(G, \cdot)$. The assertion 6 is proved. The proof is complete.

3.12. Embedding Topological Groupoids

Definition 3.12.1. *A non-empty set A is said to be an n-groupoid relative to an n-ary operation denoted by ω, if for every ordered elements $a_1, ..., a_n \in A$, a unique element $\omega(a_1, ..., a_n) \in A$ is defined.*

An n-groupoid A is called an n-groupoid with division or an nD-groupoid, if the equation $\omega(a_1, ..., a_{i-1}, x, a_{i+1}, ..., a_n) = b$ has a solution (not necessarily unique), for every $a_1, ..., a_n, b \in A$ and any $1 \leq i \leq n$.

If the n-ary operation ω of the n-groupoid (A, ω) with a topology is continuous, then A is called a topological n-groupoid.

An n-groupoid A is called commutative, if $\omega(a_1, ..., a_n)$ does not depend on the order of the elements $a_1, ..., a_n \in A$.

Theorem 3.12.2. *Every topological n-groupoid A can be embedded into a topological nD-groupoid B so that:*

1. *The following topological properties are invariant : paracompactness, sequentiality, Suslin's number , the axioms of separations and the property of being a Lindelöf and a k-space.*

2. *If the n-groupoid A is commutative, then the nD-groupoid is also commutative.*

3. *$w(B) = w(A)$, $\chi(B) = \chi(A)$, $d(B) = d(A)$.*

Proof. Let A be a topological n-groupoid and let $\omega_0 : A^n \to A$ be a continuous mapping. Let $\mathcal{H}A$ be the duplicate (copy) of the topological n-groupoid A. Consider the identity mapping $\mathcal{H} : A \to \mathcal{H}A$ and the discrete sum $A_1 = A \oplus \mathcal{H}A$.

Let

$$A_1' = \{(x_1, ..., x_n) \in A_1^n : |\{i : x_i \in \mathcal{H}(A)\}| = 1\};$$

$$A^n = \{(x_1, ..., x_n) \in A_1^n : |\{i : x_i \in \mathcal{H}(A)\}| = \varnothing\};$$

$$A_1'' = \{(x_1, ..., x_n) \in A_1^n : |\{i : x_i \in \mathcal{H}(A)\}| \geq 2\}.$$

Thus $A_1^n = A^n \oplus A_1' \oplus A_1''$. We define the mapping $\omega_1 : A_1^n \to A$ in the following way:

C_1. If $x_1, ..., x_n \in A$, then $\omega_1(x_1, ..., x_n) = \omega_0(x_1, ..., x_n)$.

C_2. If $u = (x_1, ..., x_{i-1}, \mathcal{H}(x_i), x_{i+1}, ..., x_n) \in A_1'$, then

$$\omega_1(u) = \omega_1(x_1, ..., x_{i-1}, \mathcal{H}(x_i), x_{i+1}, ..., x_n) = x_i$$

for all $u \in A_1'$ and every $i = 1, ..., n$.

C_3. We fix $a \in A$ and put $\omega_1(u) = a$ for all $u = (x_1, ..., x_n) \in A_1''$.

Consider the projection $\pi : A^{i-1} \times \mathcal{H}(A) \times A^{n-i} \to \mathcal{H}(A)$. In this case the projection π is continuous. If $x \in A^{i-1} \times \mathcal{H}(A) \times A^{n-i}$, then $\omega_1(x) = \mathcal{H}^{-1}(\pi(x))$ and ω are continuous.

Therefore, in the cases $C_1 - C_3$ the n-ary operation ω_1 is continuous, since the projection π of products of topological spaces onto the coordinates space is continuous. Then $A \subset A_1$, and A_1 is a topological extension of the groupoid A, where all equations

$$\varphi = \{\omega_1(a_1, ..., a_{i-1}, x_i, a_{i+1}, ..., a_n) = a_i : i = 1, ...n\}$$

have solutions (not necessarily unique) for all $a_1, ..., a_n \in A$.

By repeating this construction we obtain

$$A = A_0 \subset A_1 \subset A_2 \subset ... \subset A_m \subset ...$$

for which:

1. A_m is a n-subgroupoid of the n-groupoid A_{m+1} and $\omega_m = \omega_{m+1}|A_m^n$ for all $m \in N$.

2. A_m is an open-and-closed subspace of A_{m+1}.

3. If $a_1, ..., a_n \in A_m$, then the equation $\omega_m(a_1, ..., x_i, ..., a_n) = a_i$ has solutions in A_{m+1} for all $i = 1, ..., n$.

Therefore, there exists a n-groupoid $B = \cup\{A_m : m \in M\}$ with the properties:

(A1). A_m is an open-end-closed in B;

(A2). The mapping $\omega : B^n \to B$ is such that $\omega|A_m^n = \omega_m$ for each $m \in N$.

(A3). All equations $\varphi\{\omega_m(a_1, ..., x_i, ..., a_n) = a_i : i = 1, ..., n\}$ have solutions in B for every $a_1, ..., a_n \in A_m$.

Thus, B is the desired topological n groupoid with divisions. The proof is complete.

3.13.On homomorphisms of abstract and semitopological quasigroups

Let $E = E_2 = \{\cdot\}$. Then E-algebras from $V(E)$ are called gruppoids. If $\varphi = \{ax = b, ya = b\}$, then algebras from $V(E, \varphi)$ are called gruppoids with division.

Define a free object of a set X in the class $V(E, \varphi)$ according to next definition of M. Choban.

Definition 3.13.1. *The free E-algebra of a set X in the class $V(E, \varphi)$, or the free groupoid with divisions, is an E-algebra $\Gamma(X) \in V(E, \varphi)$ such that:*

1. *$X \subset \Gamma(X)$ and the set X algebraically generates the E-algebra $\Gamma(X)$, i.e. if $X \subset Y \subset \Gamma(X)$, $Y \neq \Gamma(X)$, and Y is a subalgebra of the algebra $\Gamma(X)$, then $Y \notin V(E, \varphi)$.*

2. *For every mapping $f : X \to A$, where $A \in V(E, \varphi)$, there exists a homomorphism $\hat{f} : \Gamma(X) \to A$ such that $\hat{f} \mid X = f$.*

Thoerem 3.13.2. *The free groupoid with divisions* $\Gamma(X)$ *is a quasigroup.*

Proof. It follows from Theorem 1.5.5.

Nevertheless, we will present a direct proof.

Consider the free quasigroup $Q(X)$, where X is non-empty set. For each set $L \subset Q(X)$, we denote by $s(L)$ the least set for which:

1. $L \subset s(L)$;

2. $x \cdot y \in s(L)$, whenever $x, y \in s(L)$.

We construct the set $s(L)$ in the following way:

(A1). $s_0(L) = L$;

(A2). $s_n(L) = \{x \cdot y : x, y \in s_{n-1}(L)\} \cup s_{n-1}(L)$;

(A3). $s(L) = \cup\{s_n(L) : n \in N\}$.

Let $Q_0(X) = s(X)$. We define the mapping $\varphi_0 : Q_0(X) \to \Gamma(X)$ such that $\varphi_0(x) = x$ for all $x \in X$ and $\varphi_0(a \cdot b) = \varphi_0(a) \cdot \varphi_0(b)$ for all $a, b \in Q_0(X)$. Indeed, we put $\varphi_0(x) = x$ for all $x \in X$. Let $\varphi_0(x)$ be a defined for all $x \in s_n(X)$. If $x, y \in s_n(X)$, then we put $\varphi_0(x, y) = \varphi_0(x) \cdot \varphi_0(y)$.

The image $\varphi_0(x \cdot y)$ is uniquely defined. Indeed, let $x_1, x_2, y_1, y_2 \in s_n(X)$ and $x_1 y_1 = x_2 y_2$. Then x_1 is the product of $a_1, a_2, ..., a_m \in X$, and y is the product of $a_{m+1}, ..., a_n \in X$, where the order of brackets in x_1 and y_1 are uniquely determined.

Denote by $x_1 = \overline{(a_1, a_2, ..., a_m)}$ and by $y_1 = \overline{(a_1, a_2, ..., a_m)}$. Thus $x_1 \cdot y_1 = \overline{(a_1, a_2, ..., a_m)} \cdot \overline{(a_{m+1}, ..., a_n)} = \overline{(a_1, a_2, ..., a_m, a_{m+1}, ..., a_n)}$. If we change the order of brackets in the expression $\overline{(a_1, a_2, ..., a_m, a_{m+1}, ..., a_n)}$ then we obtain another expression $\overline{\overline{(a_1, a_2, ..., a_m, a_{m+1}, ..., a_n)}}$ and $x_1 \cdot y_1 \neq \overline{\overline{(a_1, a_2, ..., a_m, a_{m+1}, ..., a_n)}}$. That is why, $x_1 \cdot y_1 = x_2 \cdot y_2$ implies $x_1 = x_2$ and $y_1 = y_2$. Therefore, the mapping φ_0 is uniquely constructed.

Assume that we have constructed the set $\{Q_\alpha(X) : \alpha < \beta\}$ and the mapping $\{\varphi_\alpha : Q_\alpha(X) \to \Gamma(X) : \alpha < \beta\}$ for which:

1. $s(Q_\alpha(X)) = Q_\alpha(X)$ for every $\alpha < \beta$;

2. $\varphi_\alpha(x \cdot y) = \varphi_\alpha(x) \cdot \varphi_\alpha(y)$ for all $x, y \in Q_\alpha(X)$ and $\alpha < \beta$;

3. If $\lambda < \alpha < \beta$, then $Q_\lambda(X) \subset Q_\alpha(X)$ and $\varphi_\lambda = \varphi_\alpha | Q_\lambda(X)$.

Now we will construct the set $Q_\beta(X)$.

Case 1. Let β be a limit in an ordinal number. Then we put $Q_\beta(X) = \cup\{Q_\alpha(X) : \alpha < \beta\}$ and $\varphi_\beta(x) = \varphi_\alpha(x)$ for every $x \in Q_\alpha(X)$ and $\alpha < \beta$.

Case 2. Let $\beta = \alpha + 1$.

If $Q_\alpha(X) = Q(X)$, then we put $Q_\beta(X) = Q(X)$. Let $Q_\alpha(X) \neq Q(X)$. Then there exist $a, b \in Q_\alpha(X)$, for which the equation $ax = b$ and $ya = b$ doesn't have solutions in $Q_\alpha(X)$.

Let c_1 be a solution of the equation $ax = b$ and c_2 be a solution of $ya = b$. We put $L = \{c_1, c_2\} \cup Q_\alpha(X)$ and $Q_\beta(X) = s(L)$. In $\Gamma(X)$ there exist such elements d_1 and d_2 for which $\varphi_\alpha(a) \cdot d_1 = \varphi_\alpha(b)$ and $d_2 \cdot \varphi_\alpha(a) = \varphi_\alpha(b)$. We put $\varphi_\beta(c_1) = \varphi_\alpha(d_1)$, $\varphi_\beta(c_2) = \varphi_\alpha(d_2)$, and $\varphi_\beta(x) = \varphi_\alpha(x)$ for all $x \in Q_\alpha(X)$.

Notice that the mapping $\varphi_\beta : Q_\beta(X) \to \Gamma(X)$ is constructed in the same way as the mapping φ_0.

In view of the fact that $\Gamma(X)$ is a free groupoid with division we obtain that the mappings φ_β are injective. For some β we have $Q_\beta(X) = Q(X)$. But this means that we have constructed the isomorphic mapping $\varphi_\beta : Q(X) \to \Gamma(X)$. The proof is complete.

Corollary 3.13.3. *(Bruck R.H. [20]). Every groupoid with divisions is the homomorphic image of some free quasigroup.*

Theorem 3.13.4. *For each topological groupoid A there exist a semitopological quasigroup B, a subspace $X \subset B$, and a factor homomorphism $g : B \xrightarrow{\text{onto}} A$ such that:*

1. The mapping $f = g|X : X \to A$ is open and $f(X) = A$.

2. The Hausdorff space B is the sum of a countable number of disjoint closed subspaces.

3. In every open cover of the space B one can refine a σ - discrete open cover.

4. If A is a regular space, then B is a paracompact and perfect normal space.

Proof. According to Junnila's Theorem [126] there exist a σ-discrete paracompact space X and an open continous mapping $f : X \xrightarrow{\text{onto}} A$.

Consider the free topological quasigroup $\Gamma(X)$ of the space X. From Mal'tev's Theorem [151] the space $\Gamma(X)$ is regular. Consider that $X \subset \Gamma(X)$. M. Choban proved in [42] that the space $\Gamma(X)$ is paracompact and $\Gamma(X)$ is the reunion of a countable number of discrete closed subspaces. According to Theorem 13.3.2

a mapping f generates a homomorphism $g : \Gamma(X) \xrightarrow{\text{onto}} A$ such that $f = g|X$. Let τ_1 be a topology on A and τ_2 be a topology on $\Gamma(X)$. Consider the topology $\tau = sup\{\tau_2, g^{-1}\tau_1\} = \{u \cap v : u \in \tau_2,\ v \in g^{-1}\tau_1\}$ on $\Gamma(X)$. Then the mapping $g : (\Gamma(X), \tau) \to (A, \tau_1)$ is continuous and X is a subspace of the space $(\Gamma(X), \tau)$.

The multiplication operation in the quasigroup $\Gamma(X)$ with the topology τ is continuous. This follows from the fact that $(\Gamma(X), \tau)$ is topologically isomorphic to the subgroupoid $\{(x, g(x)) : x \in \Gamma(X)\}$ from the product $(\Gamma(X), \tau_2) \times (A, \tau_1)$. Then $B = (\Gamma(X), \tau)$ is the desired semitopological quasigroup. Indeed, X is a subspace of the space B and $f = g|X$ is open. Considering that the space B is continuously denser onto the paracompact σ-discrete space $\Gamma(X)$, we get that B satisfies conditions (2 - 4) of the Theorem 3.2.3 (see[42]). The proof is complete.

M. Ursul posed the following question: "*Is it possible to represent a topological groupoid with division in the form of a factor-homomorphic image of a semitopological quasigroup ?*". By Theorem 3.13.4, one can see that the answer to the afore mentioned question is positive.

Example 3.13.5. Consider the set of real numbers R. The set R is an additive group relative to a binary operation denoted by "+". We choose its subsets of the form $[a, b) = \{x \in R : a \leq x < b\}$. In this case the family of subsets $[a, b)$ is a base of the topological space (R, τ). Thus, the additive operation in the group $(R, +)$ with topology is continuous. Then R is a semitopological group but is not a topological group. It is clear that the semitopological group R is not a factor-homomorphic image of a topological quasigroup. Indeed, from Mal'tsev's Theorem [151] if a semitopological quasigroup is the factor-homomorphic image of a topological quasigroup, then it is a topological quasigroup.

3.14. Homomorphism of topological groupoids with the continuous division

Some problems about groupoids with division were considered in [54, 134, 137]. L.Pontrjagin [178] has proved that for a large class of topological groups the homomorphism mapping is open. In the paper [38] M.Choban has generated this assertion for topological algebras with a continuous signature. In this section we give the conditions when continuous homomorphisms of topological groupoids with a continuous division are open.

Let us recall that a groupoid $(G, \cdot)$ is called a groupoid with division G, if for every $a, b \in G$ the equation $ax = b$ and $ya = b$ have solutions, not necessarily unique.

If the multiplication operation is continuous in the groupoid G with topology, then G is called a topological groupoid. The division in a topological groupoid is continuous from the right, if for every $a, b, c \in G$ for which $ac = b$ and any neighborhood O_c of point c in G there exist neighborhoods O_a and O_b of points a and b in G, such that for every $a' \in O_a$ and $b' \in O_b$ there exists $c \in O_c$, for which $a' \cdot c' = b'$. Continuity of division from the left is defined similarly. If in topological groupoid G division is continuous from the left and right, then one can say that G is groupoid with a continuous division.

The notion of topological groupoids with a continuous division was introduced by M. Choban.

A mapping $h : X \to Y$ of X onto Y is called almost open if $Inth(U) \neq \emptyset$ for every non-empty open subset U of X.

A mapping $f : A \to B$ of X a set A into a set B is said to be finite-to-one if the set $f^{-1}(y)$ is finite for every $y \in B$.

Theorem 3.14.1. *Let G and G_1 be the topological groupoids with a continuous division. Then every almost open continuous homomorphism $h : G \to G_1$ is open.*

Proof. Let $a \in U$ and U be an open set in G. Then there exists such points $c \in G$ and the open sets V and W in G, such that $a \in V \subseteq U, c \in W, V\ W \subset U$ and for every $x \in V$ and $y \in U$ there exists $z \in W$, such that $xz = y$. We put $H = Inth(V)$. There exist $x' \in V$ and $y' \in W$ for which $h(x') \in H$ and $x'y' = a$. Then $h(x' \cdot y') = h(a)$ and $h(a) = h(x') \cdot h(y') \in H \cdot h(y') \subseteq h(V) \cdot h(y') \subset h(V \cdot y') \subset h(U)$. Therefore $h(a) \in Inth(U)$ and the set $h(U)$ is an open set.

Lemma 3.14.2. *Let $f : X \to Y$ be an open finite-to-one continuous mapping of a T_2-space V onto a space Y with the Baire property. If a set F is nowhere dense in X, then $f(F)$ is nowhere dense in Y.*

Proof. Let F be a nowhere dense in X and $U = Intf(F) \neq \emptyset$. We put $M_n = \{x \in U : y^{-1}(x) = n\}$. Then $\cup\{M_i : i \leqslant n\}$ is a closed set in U for every $n \in N$. The subspace U has the Baire property. Therefore $Int\ M_k \neq \emptyset$ for some $k \in N$. We fix $y \in IntM_k$. Let $f^{-1}(y) = \{x_1, \ldots, x_k\}$. Then there exist such open $O_{x_1}, \ldots, O_{x_n}$ in X sets that the following conditions are met:

1. $x_i \in O_{x_i}$;
2. $O_{x_i} \bigcap O_{x_j} = \emptyset$;
3. $f(O_{x_i}) \subset IntM_n$;

We put $V_y = \bigcap\{f(O_{x_i}) : i = 1, 2, \ldots, n\}$ and $W_{x_i} = O_{x_i} \bigcap f^{-1}(V_y)$. Then $f/W_{x_i} : W_{x_i} \to V_y$ is homomorphism for every $i \leqslant n$. Let

$$F_1 = F \bigcap (W_{x_1} \bigcup \cdots \bigcup W_{x_n})$$

Obviously that $f(F_1) = V_y$. Hence $Int(F_1 \bigcap W_{x_i}) = \emptyset$ for some $i \leqslant n$. Therefore $f(F)$ is nowhere dense in Y.

Theorem 3.14.3. *Let G, G_1 be the topological groupoids with continuous division. Let G be a locally compact and Lindelef space, G_1 be a Baire space and for all $a, b \in G_1$ the set of solution $ax = b$ be finite. Then every continuous homomorphism $g : G \xrightarrow{\text{onto}} G_1$ is open.*

Proof. We consider the equation $a \cdot c = a$ on G. We fix some neighborhood U of the point a. As G is a topological groupoid, then there exist such neighborhoods V of a point a and W of a point c such that $V \cdot W \subset U$. By virtue that G is a groupoid with a continuous division for every $x \in V$ and $y \in U$ there exists a point $z \in W$ such that $x \cdot z = y$. Let us show that for every neighborhood U of a point a there exist a neighborhood U' of a point $g(a)$ such that $g(U) \supset U'$. The translations of the kind $L_a : G \to G$, were $L_a(x) = ax$, are open mapping. The family $\{L_{a_n}(V) : a \in G\}$ is an open cover of a set G. Since G is a Lindelef space we choose a countable subcover $\{L_{a_n}(V) : n \in N\}$. Note $F = \bar{V}$. The family $\{F_1', F_2', \ldots\}$, were $F_i' = g(L_{a_i}(F))$, is a countable closed cover of a set G_1'. From Lemma 3.14.2 we have that $H = Intg(F_{i_0}) \neq \emptyset$ for some $i_0 \leqslant n$. From Theorem 3.14.1 it follows that $g(U)$ is an open set. Therefore the homomorphism $g : G \xrightarrow{\text{onto}} G_1$ is open.

3.15. On The Medial Quasigroup

In this Section we study one special class of topological groupoids with a division namely the class of medial topological quasigroups. We proved that if P is open compact set of a left identity of a left medial topological loop G, then P contains open compact left medial subloop Q. This result was obtained by L.Pontrjagin for topological groups (see [178]).

A quasigroup G is called mediall if it satisfies the identity

$$xy \cdot zt = xz \cdot yt \tag{3.5}$$

If a medial quasigroup G contains an element e such that $e \cdot x = x$ for all x in G, then e is called a left identity element of G and G is called a left medial loop.

It is evident that for a left medial loop the following identity holds

$$y \cdot zt = z \cdot yt \tag{3.6}$$

We consider only left medial loop, for which the following identity is valid

$$x^2 = e \tag{3.7}$$

We shall find a solutions for the equations $a \cdot x = b$ and $ya = b$ in a left medial loop G.

It is easy to make sure that $Ix = x^{-1}$ and $Ix^{-1} = x$.

If $z = I(yt)$, then from (3.6) we have

$$yI(yt)t = I(yt)yt,$$

i.e.

$$yI(yt)t = e \tag{3.8}$$

But in the left medial loop, were $x^2 = e$ we have

$$y\ Iy = e \tag{3.9}$$

From (3.9 and (3.8) it follows

$$I(yt)t = I(y) \tag{3.10}$$

or

$$y = I^{-1}(I(yt)t) = I^{-1}(zt) = zt \tag{3.11}$$

Let $ya = b$. Then

$$b\ ya = b^2 \tag{3.12}$$

Using (3.6) and (3.7) in (3.12) we get

$$y\ ba = e \tag{3.13}$$

From (3.11) and (3.13) we obtain

$$y = I^{-1}(ba) = ba \tag{3.14}$$

Let $a\ x = b$. Then $ax\ aa = b\ aa$.
From (3.5) and (3.6) it follows

$$x\ a = b\ e \tag{3.15}$$

Using (3.11) and (3.15) we find that

$$x = (be)\ a \tag{3.16}$$

Lemma 3.15.1. *Let P be a subset of topological left medial loop G and $e \in P$. If $P_1 = P \bigcap Pe$, then:*

1. $P_1\ e = P_1$

2. If P is open, then P_1 is open too.

3. If P is closed, then P_1 is closed too.

4. If P is compact, then P_1 is compact too.

Proof. The mapping $f : G \to G$, where $f(x) = x\ e$, is an homeomorphism and $P_1 = P \bigcap Pe$. That proved the assertions 2, 3 and 4. For every $x \in G$ we have $xe\ e = x$. Therefore $P_1 = Pe \bigcap (Pe\ e) = Pe \bigcap P = P_1$.

Proposition 3.15.2. Let G be a left medial loop. Then the mapping $f : G \to G$, where $f(x) = xe$, is an involutary automorphism, i.e. $f = f^{-1}$ and $f(x \cdot y) = f(x) \cdot f(y)$ for every $x, y \in G$.

Proof. It is obvious that f is an one-to-one mapping. Let $x \in G$. The solution of a equation $y \cdot a = b$ is $y = b \cdot a$. Hence $xy \cdot y = x$ for every $y \in G$. In particular $xe \cdot e = x$ and $f(f(x)) = x$. Hence $f = f^{-1}$. Let $x, y \in G$. Then $f(xy) = xy \cdot e = xy \cdot ee = xe \cdot ye = f(x) \cdot f(y)$.

Remark 3.15.3. Let G be a left medial loop. The relatively operation $x * y = xe \cdot ye$ in G satisfies the following properties:

1. $(G, *)$ is a medial quasigroup.

2. $x * e = x$ for every $x \in G$.

3.$H = \{x \in G : xe = x\}$ is a commutative group and is a subloop of the loops $(G, \cdot)$ and $(G, *)$.

Theorem 3.15.4. *Let G be a left medial topological loop with the identity $x^2 = e$. If P is an open compact subset such that $e \in P$, then P contains an open compact left medial subloop Q of G.*

Proof. In virtue of Lemma we consider that $Pe = P$. Note $Q = \{q \in G : qP \bigcup Pq \subset P\}$. We prove that Q is an open compact left medial subloop. Now we show that Q is the open set. Let q be a fixed point of Q and x be an arbitrary point of P. Since $x\ q \in P$ and P is open set, then there exists such neighborhoods $U_x \ni x$ and $V_x \ni q$, such that $U_x V_x \subset P$. In this case we have $P = \bigcup_{i=1} U_{x_i}$. Because P is a compact set we can extract an open finitely subcovering $U_{x_i}, \ldots, U_{x_k}$ such that $P = \bigcup_{i=1}^{k} U_{x_i}$. Note $V = \bigcap_{i=1}^{k} V_{x_i}$. Then $PV \subset P$. Let us consider $q\ x \in P$. By analogy we prove that there exists such neighborhood $W \ni q$ so that $W\ P \subset P$. Note $V \bigcap W = U$. Then we have that $U\ P \subset P$ and $P\ U = \subset P$. Hence for the open set $U \ni q$ we have $U \subset Q$. Therefore Q is the open set.

Let us show that Q is a closed set. Suppose that $p \notin Q$. Then for some $q \in P$ we have $pq \notin P$ or $qp \notin P$. Let us $pq \notin P$. Then there exist an open set U such that $p \in U$ and $Uq \subset G \backslash P$. Therefore $U \bigcap Q = \emptyset$ and q isn't a limit point of Q. Hence Q is a closed set. If $q \in G$, then $q \in q\ P \bigcap P\ q$. Since if $q\ P \bigcup P\ q \subset P$, then $q \in P$. Therefore $Q \subset P$. By condition of Theorem $eP \bigcup Pe = P \subset P$. Hence $e \in Q$.

We will prove that Q is left medial loop. Fix $a, b \in Q$. Then $P\ ab = a\ Pb \subset a\ P \subset P$ and $ab\ P = ab\ Pe = aP\ be = P\ be = ePbe = eb\ Pe = b\ P \subset P$. Therefore $ab \in Q$ and Q is a subgroupoid of G.

If $a, b \in Q$ then for equation $ax = b$ his solution $x = (be)a$ is in Q and for equation $ya = b$ his solution $y = ba$ is also in Q. Hence Q is a left medial subloop of G.

Corollary 3.15.5. *Let G be a left medial loop. If G contains a non-empty open compact subset, then G contains a open compact left medial subloop.*

Example 3.15.6. Let $(Q, +)$ be a commutative group. We define in Q the operation $(\cdot) : x \cdot y = y - x$ for every $x, y \in Q$. Then $(Q, \cdot)$ is a left medial loop with the identity $x^2 = e$.

3.16. Covering Algebras. Preservation Properties in the Locally Trivial Fibering

L.S. Pontrjagin [178] proved that a linear connected space that covers a topological group admits, in a natural way, a structure of a topological group. In this Chapter we establish a similar result for universal algebras with continuous signature. This result, for the case of a finite discrete signature, was obtained by A.I. Mal'cev [151]. Result from this Chapter is stronger then Mal'cev's Theorem. In particular, the result holds for the topological R-modules, where R is a topological ring.

All spaces are assumed to be T_2-spaces. Fix a continuous signature $E = \oplus\{E_n : n \in N = 0, 1, \ldots\}$. We mention that if J is a set of identities then the totality $V(J)$ of all Hausdorff topological E-algebras, which satisfy the identities J, forms a complete variety of topological E-algebras.

A mapping $f : X \to Y$ is called a locally trivial fibering with the fiber Z, if for every point $y_0 \in Y$ there is such an open set $U \subset Y$, where $y_0 \in U$, and a homeomorphism $\varphi : U \times Z \to f^{-1}U$, such that $\varphi(\{y\} \times Z) = f^{-1}(y)$ for all $y \in U$. If the fibre Z is discrete and the spaces X and Y are linear connected and locally simple connected, then f is called a covering mapping.

A covering mapping $f : X \to Y$ of a simply linearly connected space X onto the linear connected space Y is called universal if the space X is connected. If the covering mapping f is universal, then the space X is called a universal cover of Y.

Theorem 3.16.1. *Let $f : X \to Y$ be a locally trivial fibering with the fiber Z. Then:*

1. *If the spaces Y and Z are T_i-spaces, then X is T_i-space too, where $i \in \{2, 3, 3\frac{1}{2}\}$.*

2. *If the spaces Y and Z are locally compact, then X is locally compact too.*

3. If the space $Y \times Z$ is paracompact, then X is paracompact too.

4. If the spaces Y and Z are metrizable then X is metrizable too.

Proof.1. Let Y and Z be regular spaces. We fix a closed set F in X and a point $x_0 \in F$. There are open sets U and V in Y such that $f(x_0) \in V \subseteq [V] \subset U$ and $f^{-1}U$ and $U \times Z$ are homeomorphic. Then the space $U \times Z$ is regular. Therefore there exists such an open set W in X, that $x_0 \in W \subset f^{-1}V$ and $[W] \cap (F \cap f^{-1}U) = \varnothing$. Since $[W] \subset f^{-1}[V] \subset f^{-1}U$, then $[W] \cap F = \varnothing$ is regular. If the spaces Y and Z are $T_{3,5} - spaces$, then the proved is similarly. The case for $i = 2$ is obviously.

2. The assertion 2 is obvious.

3. Let $Y \times Z$ be a paracompact space and $\omega = \{U : U$ an open set in $Y, f^{-1}U$ and $U \times Z$ are homomorphic$\}$. In the cover ω we refine some locally finite open cover $\{V_\alpha : \alpha \in A\}$. We can assume that V_α is an F_σ-sets for each $\alpha \in A$. Then $V_\alpha \times Z$ are open F_σ-sets, paracompact in $Y \times Z$. Hence $V_\alpha \times Z$ and $f^{-1}V_\alpha$ are paracompact spaces for all $\alpha \in A$. So, $\{f^{-1}V_\alpha : \alpha \in A\}$ is a locally finit open cover of the space X and $\{f^{-1}V_\alpha$ is paracompact for any $\alpha \in A$. Therefore, X is a paracompact space.

4. The spaces X and $Y \times Z$ are the common locally properties. Suppose that Y and Z are metrizable. Then X is a paracompact locally metrizable space. Thus X is metrizable. The proof is complete.

Theorem 3.16.2. *Let $f : X \to Y$ be a locally trivial fibering with regular bases space Y and the fiber Z. Then:*

1. If the space $Y \times Z$ is weakly paracompact, then X is weakly paracompact too;

2. If the space $Y \times Z$ is meta-Lindelöf, then X is meta-Lindelöf too.

Proof. Let $\omega = \{U | U$-open set in Y; $f^{-1}(U)$ and $U \times Z$ are homeomorphic$\}$. In ω we refine the open pointwise finite cover $\{V_\alpha : \alpha \in A\}$. Then $f^{-1}[V_\alpha]$ is homeomorphic with the closed subspace of the space $Y \times Z$. Hence $f^{-1}[V_\alpha]$ is weakly paracompact for all $\alpha \in A$. Let γ be an open cover of the space X. Then $f^{-1}[V_\alpha]$ contains an open point-wise finite cover $\{W_\beta : \beta \in B_\alpha\}$, which can be refined in γ. Let ξ_α=$\{W_\beta \cap (f^{-1}V_\alpha) : \beta \in B_\alpha\}$. Hence $\bigcup\{\xi_\alpha : \alpha \in A\}$ is an open point-wise cover, refined in γ.

2. Assertion 2 is proved similarly.

The proof is complete.

Corollary 3.16.3. *Let $f : X \to Y$ be a covering mapping. Then:*

1. If Y is a T_i-space, then X is also a T_i-space, where $i \in \{2, 3, 3.5\}$.

2. If the space Y is locally compact, then X is also locally compact.

3. If the space Y is paracompact, then X is also paracompact.

4. If the space Y is metrizable, then X is also metrizable.

5. If the space Y is regular and weakly paracompact, then X is weakly paracompact too.

6. If the space Y is regular and meta-Lindelöf, then X is meta-Lindelöf too.

Proof. It follows from Theorem 3.16.1 and 3.16.2.

Corollary 3.16.4. *Let $f : X \to Y$ be a covering mapping. If the space Y is paracompact and locally compact, then the space X is Lindelöf and the fiber $f^{-1}(y)$ is countable.*

Corollary 3.16.3 allows us to study the topological properties of universal covering spaces.

Corollary 3.16.5. *Let $f : X \to Y$ be a covering mapping. If the space Y is paracompact and locally simple connected, connected and locally compact, then the fundamental group $\pi(Y)$ of the space Y is countable.*

Proof. According to Corollary 3.16.3 the spaces Y and X are Lindelöf, because every connected paracompact and locally compact space is Lindelöf. By the construction of the space X we have $|\pi(Y)| = |\omega^{-1}(a)|$, for every $a \in Y$. The set $\omega^{-1}(a)$ is discrete, but in any Lindelöf space every discrete set is countable or finite. The proof is complete.

Example 3.16.6. In the space $Z = [0, 1] \times \{0, 1\}$ we identify the pairs of points $\{(0, 0), (0, 1)\}$, $\{(1, 0), (1, 1)\}$, $\{(1 - \frac{1}{n}, 0), (1 - \frac{1}{n}, 1)\}$, where $n = 1, 2, ...$. The received quotient space is denoted by X, whereas $p : Z \to X$ is the natural projection. The space X has a countable number of circles $\{\omega_n : n = 1, 2, ...\}$, where $\omega_n = p([\frac{n-1}{n}, \frac{n}{n+1}]) \times \{0, 1\}$. Let $I = [0, 1]$, $I_n = [\frac{n-1}{n}, \frac{n}{n+1}]$, $a_\infty = p((1, 0))$, $a_{n-1} = p((\frac{n-1}{n}, 0))$, where $n = 1, 2, ...$. For every $m \geq 0$ we denote by $\varphi_{mn} : I_n \to \omega_n$ such mapping of I_n onto ω_n that for $m \neq k$ the mappings φ_{mn} and φ_{kn} are not homotopic, $\varphi_{mn}(\frac{n-1}{n}) = a_{n-1}$ and $\varphi_{mn}(\frac{n}{n+1}) = a_n$.

Such mappings exists, because $\pi(\omega_n)$ is countable and isomorphic to Z. We can assume that φ_{mn} contains precisely m full counter-clockwise rotations.

Consider the mapping $h : N \to N$. The mapping h generates the path $r_h : I \to X$, where $r_h(0) = a_0$, $r_h(1) = a_\infty$ and $r_h|I_h = \varphi_{h(n)n}$. If the mapping $h_1, h_2 : N \to N$ are different, then r_{h_1} and r_{h_2} are not homotopic equivalent. Therefore $\pi(X, a_0)$ contains at least N^N elements. The space X is metrizable, compact and not locally simple connected in the point a_∞. Clearly, X is locally simple connected in all points except for point a_∞. Hence, for not locally simple connected spaces the Corollary 3.16.5 is not true.

3.17. Universal Covering Algebras

We consider the complete variety V of topological E-algebras, for which the following condition hold:

A. For every algebra $G \in V$ there is a neutral element 1_G, such that $e_{0G}(E \times G^0) \subseteq \{1_G\}$ and $\omega(1_G, ..., 1_G) = 1_G$ for each $\omega \in E_n$ and $n \geq 1$.

Consider a linearly connected, locally connected and locally simple connected algebra $G \in V$. It is well known that there exists a universal covering $p : G^* \to G$ and a point $1_G^* \in G$, such that $p(1_G^*) = 1_G$ (see[178]). The objects G^* and p are constructed in the following way.

Consider all the paths in G coming from point 1_G. Two paths $f, g : [0, 1] \to G$ are equivalent, if $f(1) = g(1)$ and f, g are homotopic equivalent. For every path $f : [0, 1] \to G$ denote by $[f]$ the class of equivalence contained f. Denote by G^* the totality of all classes of equivalence. If $a^* \in G^*$, then it is uniquely determined by the points $p(a^*)$, where $p(a^*) = f(1)$ for all $f \in a^*$. The topology in G^* is introduced as follows.

Let U be an open simple connected set in G, $a^* \in G^*$ and $p(a^*) \in U$. We fix $f \in a^*$ and set $(U, a^*) = \{[fg] \mid g : [0, 1] \to U$ - is the path and $g(0) = f(1)\}$. The totality of the sets (U, a^*) forms an basis in G^*.

Theorem 3.17.1. *Let G be a topological E-algebra, $G \in V$. Then there exists on G^* an algebraical structure such that $G^* \in V$ and $p : G^* \to G$ is a homomorphism.*

Proof. Fix $1_G^* \in p^{-1}(1_G)$. If $E_0 \neq \varnothing$, then we set $e_{0G}(E_0 \times (G^*)^0) = \{1_G^*\}$. Let $n \geq 1$ and $a_1^*, a_2^*, ..., a_n^* \in G^*$. We fix $f_1 \in a_1^*, f_2 \in a_2^*, ..., f_n \in a_n^*$ and $\omega \in E_n$.

Then we determine the path

$$f(t) = e_{nG}(f_1(t), f_2(t), ..., f_n(t), \omega)$$

for which

$$f(0) = e_{nG}(f_1(0), f_2(0), ..., f_n(0), \omega) = e_{nG}(1_G, 1_G, ..., 1_G, \omega)$$

and

$$f(1) = e_{nG}(f_1(1), f_2(1), ..., f_n(1), \omega).$$

There exists a point $a^* \in G^*$ for which $f \in a^*$.

We put $e_{nG^*}(a_1^*, a_2^*, ..., a_n^*, \omega)= a^*$. We prove that a^* does not depend on the choice of the functions $f_1, f_2, ..., f_n$.

Let $f_1, f_1' \in a_1^*$, $f_2, f_2' \in a_2^*$,...,$f_n, f_n' \in a_n^*$. Then there exists such a continuous function $\varphi_i : [0,1] \times [0,1] \to G$, that $\varphi_{i[0,t]} = f_i(t)$ and $\varphi_{i[1,t]} = f_i^{'}(t)$ for all $i = 1, 2, ..., n$. We set $\varphi^{'}(t) = e_{nG}(f_1'(t), f_2'(t), ..., f_n'(t), \omega)$.

Consider the continuous function

$$\varphi(1, t) = e_{nG}(\varphi_1(s, t), \varphi_2(s, t), ..., \varphi_n(s, t), \omega).$$

Then $\varphi(0, t) = f(t)$ and $\varphi(1, t) = f'(t)$. Hence, f and f' are homotopic equivalent, $f \sim f'$. So, the mapping e_{nG^*} is correctly determined and

$$p(e_{nG^*}(a_1^*, a_2^*, ..., a_n^*, \omega)) = e_{nG^*}(p(a_1^*), p(a_2^*), ..., p(a_n^*), \omega).$$

We now establish the continuity of the mapping e_{nG^*}. Fix $n \geq 1$, $\omega \in E_n$; $a_1^*, a_2^*, ..., a_n^* \in G^*$ and the neighborhood $U^* = (U, a^*)$, where $a^* = e_{nG^*}(a_1^*, a_2^*, ..., a_n^*, \omega)$. There exist a simple connected open sets $U_1, U_2, ..., U_n$ in G and an open set W in E_n such that $p(a_1^*) \in U_1, p(a_2^*) \in U_2, ..., p(a_n^*) \in U_n$, $\omega \in W$ and $e_{nG}(W \times U_1 \times U_2 \times ... \times U_n) \subset U$.

We set $U_1^* = (U_1, a_1^*), U_2^* = (U_2, a_2^*), ..., U_n^* = (U_n, a_n^*)$. Then $e_{nG^*}(W \times U_1^* \times U_2^* \times ... \times U_n^*) \subset U^*$.

Therefore, the mapping e_{nG^*} is continuous and G^* is a topological E-algebra and $p : G^* \to G$ is a continuous homomorphism.

If q is a m-ary operation, then for every $a_1^*, a_2^*, ..., a_m^* \in G^*$:

$$p(q(a_1^*, a_2^*, ..., a_m^*)) = q(p(a_1^*), p(a_2^*), ..., p(a_m^*)).$$

If $f_1 \in a_1^*, f_2 \in a_2^*, ..., f_n \in a_m^*$ and $a^* = q(a_1^*, a_2^*, ..., a_m^*)$, then

$$q(f_1(t), f_2(t), ..., f_m(t)) \in a^*.$$

Now, let ω be an m-ary operation, q be an k-ary operation and

$$\omega(x_1, x_2, ..., x_m) = q(y_1, y_2, ..., y_k)$$

be an identity in V.

Assume $a_1^*, a_2^*, ..., a_m^*, b_1^*, b_2^*, ..., b_k^* \in G^*$. We fix $f_1 \in a_1^*$, $f_2 \in a_2^*$,..., $f_m \in a_m^*$, $g_1 \in b_1^*$, $g_2 \in b_2^*$,..., $g_k \in b_k^*$.

We suppose that:

- $a_i = a_j$, then $f_i = f_j$;
- $a_i = b_j$, then $f_i = g_i$.

Then for any t we have

$$\omega(f_1(t), f_2(t), ..., f_m(t)) = q(g_1(t), g_2(t), ..., g_k(t)), \text{ or}$$
$$\omega(a_1^*, a_2^*, ..., a_m^*)) = q(b_1^*, b_2^*, ..., b_k^*).$$

Hence, the algebra G^* satisfies all the identities of the algebras from V. Therefore, $G^* \in V$ because $e_{nG^*}(1_G^*, ..., 1_G^*, \omega) = 1_G^*$ for all $\omega \in E_n$, $n \geq 1$. The proof is complete.

3.18. Examples of Covering Algebras

Example 3.18.1. Every variety V of topological groups, quasigroups with left or right units, loops, semigroups with unit, the rings without units, satisfies condition (A).

Example 3.18.2. An E-algebra G is a biternary E-algebra if there exist two ternary polynomials p, q such that $p(y, y, x) = q(p(x, y, z), y, z) = p(q(x, y, z), y, z) = x$, for all $x, y, z \in G$. The biternary E-algebras are introduced in [151] and satisfy condition (A). In this case $E = E_3 = \{p, q\}$. We can fix as 1_G any element of algebra, since $p(x, x, x) = x$ and $q(p(x, x, x), x, x) = q(x, x, x) = x$.

Example 3.18.3. An E-algebra G is a biquaternary E-algebra if there exist two quaternary polynomials such that $p(x, x, y, z) = z$; $q(x, y, z, x) = x$; $q(p(x, y, x, z), y, x, z) = x$, for all $x, y, z, \in G$ [151]. We can fix as 1_G any beforehand given element. In this case $E = E_4 = \{p, q\}$.

Example 3.18.4. An E-algebra G is called an homogeneous E-algebra if there exist two binary operations $\{+,\cdot\} = E_2 = E$ such that $x \cdot x = y \cdot y$, $x + x \cdot y = y$, $x \cdot (x + y) = y$, for all $x, y \in G$ [151]. Let the E-algebra G be an homogeneous E-algebra. Then there exists an element $1_G \in G$, such that $x \cdot x = 1_G$ for all $x \in G$. Hence, $1_G \cdot 1_G = 1_G$, $1_G + 1_G = 1_G + 1_G \cdot 1_G = 1_G$.

Example 3.18.5. Ternary algebras of Mal'tev satisfy condition (A). In this case $E = E_3 = \{p\}$ and $p(x, y, y) = p(y, y, x) = x$. We can take any fixed elements as 1_G .

3.19. Conclusions for Chapter 3

In this Chapter there have been studied: topological quasigroups which are obtained by using the isotopies of topological groups, proprieties of medial and paramedial topological groupoids, continuous homomorphisms of topological groupoids with a continuous division and universal covers algebras.

We can mention the following conclusions:

1). there have been introduced some new concepts: (n, m)-identities,(n, m)-homogeneous isotope, (n, m)-homogeneous quasigroup.

2). there have been elaborated methods of construction of Haar measure on medial topological quasigroups.

In this way, using the new concepts and methods we are able:

- to describe the topological quasigroups with (n, m)-homogeneous quasigroup.
- to establish conditions for which there exist right invariant (or left invariant) Haar measures on medial grupoids.
- to construct and demonstrate the uniqueness of Haar measure on medial quasigroups.
- to extend some affirmations of the theory of topological groups on the class of topological (n, m)-homogeneous quasigroups.

3). there have been studied the paramedial topological groupoids and established the correlation between paramediality and associativity.

4). there have been elaborated methods of embedding every topological n-groupoid A into topological n-groupoid with division B.

5). there have been given conditions when continuous homomorphisms of topological groupoids with a continuous division are open.

6). there have been proved that if P is an open compact set of a left identity from a left medial topological loop G, then P contains an open compact left medial subloop Q.

7). there have been constructed a universal covering on topological E-algebras with continuous signature. In particular, the result holds for the topological R-modules, where R is a topological ring. Therefore, the result from this work is stronger then Mal'cev's Theorem for algebras with discrete signature.

The methodology proposed for research in Chapter 3 can be used:

- to introduce the concept of multiple identities for n-medial quasigroups;
- to develop methods of construction of Haar measure on n-quasigroups with multiple identities;
- to study in greater detail the relationship between parameđiality and associativity, mediality and associativity;
- to study properties of a class of (n, m)-homogeneos topological paramedial quasigroups;
- to identify the conditions when continuous homomorphisms of topological n-groupoids with a continuous division are open;
- to give a general solution of the homomorphism problem for fuzzy universal algebras.
- to identify the conditions for which if P is open compact set of paramedial topological quasigroup G, then P contains an open compact paramedial subquasigroup Q;
- to examine the methods of constructions the universal covers on topological groupoid with division;
- to study the topological and algebraical properties of universal covers on topological groupoid with division;
- to study compact subsets of free algebras with topologies.
- to elaborate special methods to investigate of k-algebras.

4. COMPACT SUBSETS OF FREE ALGEBRAS WITH TOPOLOGIES AND EQUIVALENCE OF SPACES

In this Chapter we investigate universal algebras with topologies. On algebras we consider topologies relatively to which operations are continuous on compact subsets. These algebras are called k-algebras. Some properties of compact subsets of free k-algebras and some facts about M_V-equivalence of spaces are established.

The present chapter is connected with results of A.A. Markov [155], M.I. Graev [103], A.I. Mal'cev [153], J. Milnor [160, 161], P.J. Huber [115], H.-E. Porst [177], V.M. Valov and B.A. Pasynkov [203], E.T. Ordman [174] and M.M. Choban [38, 41].

Our objective is to study the category of universal topological algebras and the notions of k-continuous mappings and k-algebras.

The category of k-algebras was studied in [41, 174, 177]. The existence of the free k-algebras follows from the general result from [41]. Our attention is focused on the problem of the description of compact subsets of free topological algebras and free k-algebras.

The notion of a k-group was first considered by J. Milnor [160] and P. J. Huber [115]. Let G be a commutative group. According to J. Milnor, there exists the Eilenberg-MacLane semi-simplicial complex $K(G, n)$, where $n \in N$, which admits the structure of a commutative group. J. Milnor [160] stated that the groups $K(G, n)$ are topological for every countable group G. As was observed by P. J. Huber [115], a closer inspection of Milnor's proof shows that $K(G, n)$ are k-groups for every group G. This deep fact was widely studied and applied (see [22, 115, 157, 161, 203]).

4.0.1 Notations and remarks:

1. $cl_X A$ or clA denotes the closure of a set A in a space X.

2. $|Y|$ denotes the cardinality of a set Y.

3. The Cartesian product of spaces is equipped with the Tychonoff topology.

4. βX denotes the Stone–Čeeh compactification of a completely regular space X.

5. R denotes the spaces of reals, $N = \{0, 1, 2, \dots\}$ is the subspace of natural numbers and $I = [0, 1] \subseteq R$.

6. The word "space" will refer to Hausdorff spaces.

4.1 Spaces and mappings

All spaces under consideration are Hausdorff. A mapping $f : X \to Y$ of a space X into a space Y is called a k-continuous mapping if for every compact subset $\Phi \subseteq X$ the restriction $f|\Phi : \Phi \to Y$ is continuous.

Denote by Tyh the category of all completely regular spaces, by Reg the category of all regular spaces and by Hsd the category of all Hausdorff spaces.

Let $\mathcal{L}$ be a class of spaces and $X \in \mathcal{L}$ be a topological space with a topology $\mathcal{T}$. We put $k(\mathcal{T}) = \{U \subseteq X : U \cap \Phi$ is open in Φ for every compact subset Φ of $X\}$ and $k_{\mathcal{L}}(\mathcal{T})$ is the topology on X generated by the family of all k-continuous mappings $\{f : X \to Y : Y \in \mathcal{L}$ and $f(X) = Y\}$. Clearly $\mathcal{T} \subseteq k_{\mathcal{L}}(\mathcal{T}) \subseteq k(\mathcal{T})$ and $k(\mathcal{T}) = k_{Hsd}(\mathcal{T})$ (see[5, 89, 159]). By $k_{\mathcal{L}}X$ we denote the set X with the topology $k_{\mathcal{L}}(\mathcal{T})$.

If $(X, \mathcal{T})$ is a regular space, then we put $k_\rho(\mathcal{T}) = k_{Reg}(\mathcal{T})$ and $k_\rho X$ or $k_\rho(X)$ is the set X with the topology $k_\rho(\mathcal{T})$.

If $(X, \mathcal{T})$ is a Hausdorff space, then by kX or $k(X)$ we denote the set X with the topology $k(\mathcal{T})$.

If $(X, \mathcal{T})$ is a completely regular space, then we put $k_R(\mathcal{T}) = k_{Tyh}(\mathcal{T})$ and $k_R X$ or $k_R(X)$ is the set X with the topology $k_R(\mathcal{T})$.

If $\mathcal{T}$ is a completely regular topology on X, then $\mathcal{T} \subseteq k_R(\mathcal{T}) \subseteq k_\rho(\mathcal{T}) \subseteq k(X)$.

Definition 4.1.1. *A space X is called:*

– a k-space if $kX = X$;

– a k_ρ-space if X is regular and $k_\rho X = X$;

– a k_R-space if X is completely regular and $k_R X = X$.

Let $kHsd$ be the category of all k-spaces, ρReg be the category of all k_ρ-spaces and $RTyh$ be the category of all k_R-spaces. Then $k : Hsd \to kHsd, eg \to \rho Reg$ and $k_R : Tyh \to RTyh$ are covariant functors.

If X and kX are regular spaces, then $k_\rho X = kX$. Hence every regular k-space is a k_ρ-space.

If X and kX are completely regular spaces, then $k_R X = k_\rho X = kX$. Hence every completely regular k-space (respectively, k_ρ-space) is a k_R-space. The following two statements are obvious.

Proposition 4.1.2. *Let $\mathcal{L} \in \{Tyh, Reg, Hsd\}$ and $(X, \mathcal{T}) \in \mathcal{L}$. The following assertions are equivalent:*

1. $\mathcal{T} = k_{\mathcal{L}}(\mathcal{T})$.

2. Every k*-continuous mapping* $f : X \to Y$*, where* $Y \in \mathcal{L}$*, is continuous.*

Proposition 4.1.3. *For each completely regular space* X *the following assertions are equivalent:*

1. X *is a* k_R*-space.*

2. Every k*-continuous mapping* $f : X \to Y$ *into a completely regular space* Y *is continuous.*

3. Every k*-continuous function* $f : X \to R$ *is continuous.*

4. Every bounded k*-continuous function* $f : X \to R$ *is continuous.*

Let $\mathcal{L} \in \{Hsd, Reg, Tyh\}$, $\{X_\alpha \in \mathcal{L} : \alpha \in A\}$ be a family of spaces and $X = \Pi\{X_\alpha : \alpha \in A\}$. Denote by $\Pi_{\mathcal{L}}\{X_\alpha : \alpha \in A\}$ the space $k_{\mathcal{L}}X$. In particular, $\Pi_k\{X_\alpha : \alpha \in A\} = \Pi_{Hsd}\{X_\alpha : \alpha \in A\}$, $\Pi_\rho\{X_\alpha : \alpha \in A\} = \Pi_{Reg}\{X_\alpha : \alpha \in A\}$ and $\Pi_R\{X_\alpha : \alpha \in A\} = \Pi_{Tyh}\{X_\alpha : \alpha \in A\}$. The operation $\Pi_{\mathcal{L}}$ is a product in the category $\{X \in \mathcal{L} : k_{\mathcal{L}}X = X\}$.

A space X is called a functionally Hausdorff space or an FH-space if for every pair of distinct points $a, b \in X$ there exists a continuous function $f : X \to R$ such that $f(a) \neq f(b)$.

A subset H of a space X is said to be bounded if $f(H)$ is a bounded subset of the reals R for every continuous function $f : X \to R$.

A space X is called a μ-complete space if the closure $cl_X H$ of every bounded subset $H \subseteq X$ is compact.

Proposition 4.1.4. *Let* $(X, \mathcal{T})$ *be a* μ*-complete space and* $\mathcal{T} \subseteq \mathcal{T}' \subseteq k(\mathcal{T})$*. Then the space* $(X, \mathcal{T}')$ *is* μ*-complete.*

Proof. Denote by $comp(Z)$ the family of all compact subsets of a space Z. Then $comp(X, \mathcal{T}') = comp(X, \mathcal{T})$. If H is a bounded subset of $(X, \mathcal{T}')$, then H is bounded in $(X, \mathcal{T})$ too. Therefore $\Phi = cl_{(X,\mathcal{T})}H$ is a compact subset and $\Phi \in comp(X, \mathcal{T}')$. Hence $cl_{(X,\mathcal{T}')}H = \Phi$. The proof is complete.

Corollary 4.1.5. *If* X *is a* μ*-complete space, then* kX *is* μ*-complete.*

Corollary 4.1.6. *If* X *is a regular* μ*-complete space, then* $k_\rho X$ *is* μ*-complete.*

Corollary 4.1.7. *If* X *is a completely regular* μ*-complete space, then* $k_R X$ *is* μ*-complete.*

Example 4.1.8. Let $N = \{0, 1, 2, \dots\}$ be the discrete space of the natural numbers. For every infinite subset A of N we fix a point $r(A) \in cl_{\beta N}A \setminus A$. Consider the set $X = N \cup \{r(A) : A$ is an infinite subset of $N\}$ as a subspace of the Stone–Čech compactification βN of N. Then the space X is pseudocompact (i.e. X is bounded in X) and every compact subset of X is finite. Hence $kX = k_R X = k_\rho X$ is a discrete space, kX is μ-complete and X is not μ-complete.

Example 4.1.9. For every infinite countable subset A of βN we fix some point $p(A) \in cl_{\beta N}A \setminus A$. We put $r(A) = A \cup \{p(B) : B$ is an infinite countable subset of $A\}$. There exists a minimal subspace X of βN such that $N \subseteq X$ and $r(X) = X$. The space X is countably compact and every compact subset of X is finite. Hence the space X is not μ-complete and kX is a discrete μ-complete space. In particular $k_R X = k_\rho X = kX$.

The sequence $\{X_n : n \in N\}$ of subspaces of a space X is called a k_ω-sequence or a k_ω-decomposition of X if:

– $X_n \subseteq X_{n+1}$ and X_n is a compact subset of X for every $n \in N$;

– the subset H of X is closed if and only if $H \bigcap X_n$ is closed in X for every $n \in N$.

A Hausdorff space with a k_ω-sequence is called a k_ω-space.

The following assertions are well known (see [94, 158, 192]):

A1. Every k_ω-space is a normal k-space and a k_R-space.

A2. Every closed subspace of a k_ω-space is a k_ω-space.

A3. Every open Lindelöf subspace of a k_ω-space is a k_ω-space.

A4. The topological product of a finite family of k_ω-spaces is a k_ω-space.

A5. Every k_ω-space is μ-complete.

A6. The discrete sum of a countable family of k_ω-spaces is a k_ω-space.

4.2 Algebras with Topologies

The discrete sum $E = \oplus\{E_n : n \in N\}$ of pairwise disjoint spaces $\{E_n : n \in N\}$ is called a continuous signature.

If E is a discrete space, then the signature E is said to be discrete.

If E is a k_ω-space, then E is called a k_ω-signature.

Definition 4.2.1. *An E-algebra or a universal algebra of the signature E is a family $\{G, e_{nG} : n \in N\}$ for which:*

– *G is a non-empty set;*

– *$e_{nG} : E_n \times G^n \to G$ is a mapping for every $n \in N$.*

The set G is called the support of the E-algebra and the mappings $e_G = \{e_{nG} : n \in N\}$ are called an algebraic structure on G.

The signature E is the space of symbols of operations.

Let G be an E-algebra. Then G^0 is a singleton. If $u \in E_0$, then $u_G = u(G^0) = e_{0G}(u, G^0)$ is a point of G. If $n \geqslant 1$ and $u \in E_n$, then $u(x_1, \ldots, x_n) = e_{nG}(u, x_1, \ldots, x_n)$ is a mapping of G^n into G.

Consider an E-algebra G, a non-empty subset A of G and a non-empty subset H of E. We put $a_0(H, A) = A$, $a_{n+1}(H, A) = a_n(H, A) \cup \cup\{e_{mG}((H \cap E_m) \times a_n(H, A)^m)) : m \in N\}$ for every $n \in N$ and $a(H, A) = \cup\{a_n(H, A) : n \in N\}$.

If $A \neq \emptyset$ and $a(E, A) \subseteq A$, i.e. $e_{nG}(E_n \times A^n) \subseteq A$ for every $n \in N$, then A is called a subalgebra of the E-algebra G. For every non-empty subset A of G, the set $a(E, A)$ is a subalgebra of G.

A mapping $f : A \to B$ of an E-algebra A into an E-algebra B is called a homomorphism if $f(e_{0A}(\{a\} \times A^0)) = e_{0B}(\{a\} \times B^0)$ and $f(e_{nA}(\omega, x_1, \ldots, x_n)) = e_{nB}(\omega, f(x_1), \ldots, f(x_n))$ for every $a \in E_0$, $n \geq 1$, $\omega \in E_n$ and $x_1, \ldots x_n \in A$. If f is a one-to-one homomorphism, then f is called an isomorphism.

Consider a non-empty family $\{B_\alpha : \alpha \in A\}$ of E-algebras. Let $B = \Pi\{B_\alpha : \alpha \in A\}$ and $B^n = \Pi\{B_\alpha^n : \alpha \in A\}$ for every $n \in N$. We consider the mappings $e_{nB} : E_n \times B^n \to B$ with $e_{nB}(b, x) = \{e_{nB_\alpha}(b, x_\alpha) : \alpha \in A\}$ for every $n \in N$, $b \in E_n$ and $x = (x_\alpha \in B_\alpha^n : \alpha \in A) \in B^n$. The set B with the mappings $e_B = \{e_{nB} : n \in N\}$ is called a Cartesian product of the E-algebras $\{B_\alpha : \alpha \in A\}$. The natural projections $\pi_\alpha : B \to B_\alpha$ are homeomorphisms.

If $\{B_\alpha : \alpha \in A\}$ is an empty family of E-algebras, then $\Pi\{B_\alpha : \alpha \in A\}$ is the singleton E-algebra.

Definition 4.2.2. *An E-algebra G together with a given topology on it is called a topological E-algebra if the mappings $e_{nG} : E_n \times G^n \to G$, $n \in N$, are continuous.*

The Tychonoff product of topological E-algebras is a topological E-algebra.

Definition 4.2.3. *An E-algebra G together with a given topology on it is called a k-E-algebra if the mappings $e_{nG} : E_n \times G^n \to G$, $n \in N$, are k-continuous.*

Every topological E-algebra is a k-E-algebra. The Tychonoff product of k-E-algebras is a k-E-algebra. If $\{G_\alpha : \alpha \in A\}$ is a family of k-E-algebras, then $\Pi_k\{G_\alpha : \alpha \in A\}$, $\Pi_\rho\{G_\alpha : \alpha \in A\}$ and $\Pi_R\{G_\alpha : \alpha \in A\}$ are k-E-algebras.

In the category of k-algebras we consider homomorphisms, continuous homomorphisms, isomorphisms, continuous isomorphisms, topological isomorphisms.

Proposition 4.2.4. *Let $\mathcal{T}$ be a topology on an E-algebra G. The following assertions are equivalent:*

1. $(G, \mathcal{T})$ is a k-E-algebra.

2. If $\mathcal{T} \subseteq \mathcal{T}' \subseteq k(\mathcal{T})$, then $(G, \mathcal{T}')$ is a k-E-algebra.

3. For some topology $\mathcal{T}'$, where $\mathcal{T} \subseteq \mathcal{T}' \subseteq k(\mathcal{T})$, $(G, \mathcal{T}')$ is a k-E-algebra.

Proof. It is obvious.

Proposition 4.2.5. *Let E be a k_ω-signature, G be an E-algebra, $\mathcal{T}$ be a topology on G and $(G, \mathcal{T})$ be a k_ω-space. Then G is a topological E-algebra if and only if G is a k-E-algebra.*

Proof. It is obvious.

Example 4.2.6. Let $E_0 = \{0\}$, $E_1 = \{-\}$, $E_2 = \{+\}$ and $E = E_0 \cup E_1 \cup E_2$ be the group signature. Let G be a group with a topology and every compact subset of G be finite. Then G is a k-group, i.e. a k-E-algebra.

Example 4.2.7. Let $E = \{0, -, +\}$ be the group signature and $Z = \{0, \pm 1, \dots\}$ be the additive groups of integers. If $A \subseteq Z$, then $|A|_n$ is the cardinality of the set $\{x \in A : |x| \leq n\}$. The set A is of density 1 if $\lim \frac{|A|_n}{2n} = 1$. On Z consider the topology for which the points $\{n \in Z : n \neq 0\}$ are isolated and $U \subseteq Z$ is a neighbourhood of 0 iff $0 \in U$ and it is of density 1. One can easily see that Z is a normal space and only the finite subsets are compact. The space Z is not a topological group. The space Z is a k-group.

Example 4.2.8. Let G be an uncountable group. Every point $x \neq 0$ is considered to be isolated and the neighbourhoods of 0 are of the form $G \setminus A$, where $0 \notin A$ and A is countable. The space G is Lindelöf, G is a k-group and G is not a topological group. In G every compact subset is finite.

Remark 4.2.9. If $\cup\{E_n : n \geq 1\} = \emptyset$ and E_0 is discrete, then every space admits some structure of a topological E-algebra. In particular, every k-E-algebra is a topological E-algebra.

Remark 4.2.10. If $\cup\{E_n : n \geq 1\} \neq \emptyset$, then there exists a Lindelöf k-E-algebra which is not a topological E-algebra.

Remark 4.2.11. If G is a k-E-algebra and $E_n \times G^n$ is a k-space for every $n \in N$, then G is a topological E-algebra. If G is a regular k-E-algebra and $E_n \times G^n$ is a k_ρ-space for all $n \in N$, then G is a topological E-algebra. If G is a completely regular k-E-algebra and $E_n \times G^n$ is a k_R-space for every $n \in N$, then G is a topological E-algebra.

Let $i \in \{2, 3, 3\frac{1}{2}\}$. Fix a continuous signature $E = \oplus\{E_n : n \in N\}$. Consider that E is a T_i-space.

By $V_i(E)$ we denote the class of all topological E-algebras which are T_i-spaces and by $W_i(E)$ we denote the class of all k-E-algebras which are T_i-spaces. Clearly $V_i(E) \subseteq W_i(E)$. If $U(E)$ is the class of all E-algebras, then there exist the forgetful functors $v : V_i(E) \to U(E)$ and $w : W_i(E) \to U(E)$, where $v(G)$ or $w(G)$ is the k-E-algebra G without any topology on it. Since $V_i(E) \subseteq W_i(E)$ we have $v = w|V_i(E)$.

If V is any class of k-E-algebras and $V \subseteq W_i(E)$, then we denote:

– $S(V)$: the class of all k-E-algebras topologically isomorphic to subalgebras of members of V;

– $\Pi(V)$: the class of all Tychonoff products of families of members of V;

– $\Gamma(V)$: the class of all isomorphic images of members of the class V;

– $\Gamma_i(V) = \Gamma(V) \cap V_i(V)$;

– $\Gamma'_i(V) = \Gamma(V) \cap W_i(V)$.

Definition 4.2.12. *A class V of k-E-algebras is called:*

– a T_i-quasivariety if $V = \Pi(V) = S(V)$ and $V \subseteq W_i(E)$;

– a complete T_i-quasivariety if $V = \Pi(V) = S(V) = \Gamma'_i(V) \subseteq W_i(V)$ and $\Gamma_i(V') \neq \emptyset$ for every non-empty subclass $V' \subseteq V$;

– a t-complete T_i-quasivariety if $V = \Pi(V) = S(V) = \Gamma_i(V) \subseteq V_i(V)$.

Lemma 4.2.13. *If V is a T_i-quasivariety of topological E-algebras, then $\Gamma_i(V)$ is a t-complete T_i-quasivariety and $\Gamma'_i(V)$ is a complete T_i-quasivariety.*

Proof. It is obvious.

Let G be an E-algebra and $H \subseteq G$. If $a(E, H) = G$, then we say that the set H algebraically generates the E-algebra G.

Definition 4.2.14. *Let V be a T_i-quasivariety of k-E-algebras and X be a non-empty space.*

(A). A couple $(F^a(X,V), a_X)$ is called a free algebra of the space X in the class V if the following conditions hold:

A1. $F^a(X,V) \in V$ and $a_X : X \to F^a(X,V)$ is a mapping.

A2. The set $a_X(X)$ algebraically generates $F^a(X,V)$.

A3. For each mapping $f : X \to G \in V$ there exists a continuous homomorphism $af : F^a(X,V) \to G$ such that $f = a_X \circ af$.

(B). A couple $(F^k(X,V), q_X)$ is called a k-free algebra of the space X in the class V if the following conditions hold:

B1. $(F^k(X,V) \in V$ and $q_X : X \to F^k(X,V)$ is a k-continuous mapping.

B2. The set $q_X(X)$ algebraically generates $F^k(X,V)$.

B3. For each k-continuous mapping $f : X \to G \in V$ there exists a continuous homomorphism $qf : F^k(X,V) \to G$ such that $f = q_X \circ qf$.

(C). A couple $(F(X,V), t_X)$ is called a t-free algebra of the space X in the class V if the following conditions hold:

C1. $F(X,V) \in V$ and $t_X : X \to F(X,V)$ is a continuous mapping.

C2. The set $t_X(X)$ algebraically generates $F(X,V)$.

C3. For each continuous mapping $f : X \to G \in V$ there exists a continuous homomorphism $tf : F(X,V) \to G$ such that $f = t_X \circ tf$.

Theorem 4.2.15. *Let V be a T_i-quasivariety of k-E-algebras. Then for every non-empty space X there exist:*

– the unique free E-algebra $(F^a(X,V), a_X)$;

– the unique k-free E-algebra $(F^k(X,V), q_X)$;

– the unique t-free E-algebra $(F(X,V), t_X)$;

– the unique continuous homomorphisms $b_X : F^a(X,V) \to F^k(X,V)$, $c_X : F^a(X,V) \to F(X,V)$ and $l_X : F^k(X,V) \to F(X,V)$ such that $q_X = b_X \circ a_X$ and $t_X = c_X \circ a_X = l_X \circ q_X$.

Proof. Let τ be an infinite cardinal and $\tau \geqslant |X| + |E|$. Consider that $\{f_\alpha : X \to G_\alpha \in V : \alpha \in A\}$ is the set of all mappings for which $|G_\alpha| \leqslant \tau$. Let

$C = \{\alpha \in A : f_\alpha : X \to G_\alpha$ is continuous $\}$ and $D = \{\alpha \in A : f_\alpha : X \to G_\alpha$ is k-continuous $\}$. Fix a subset $B \subseteq A$. Consider the diagonal product $f_B : X \to G_B = \Pi\{G_\alpha : \alpha \in B\}$, where $f_B(x) = (f_\alpha(x) : \alpha \in B)$. Let $F(X, B)$ be the subalgebra of G_B generated by the set $g_B(X)$. Consider the projection $g_\alpha : F(X, B) \to G_\alpha$ for every $\alpha \in B$.

If $\alpha \in B$, then $f_\alpha = g_\alpha \circ f_B$.

Fix a mapping $f : X \to G \in V$.

Put $G' = a(E, f(X))$. Then $|G'| \leqslant \tau$ and $f = f_\alpha$, $G' = G_\alpha$ for some $\alpha \in A$.

If $\alpha \in B$, then $f = g_\alpha \circ f_B$ and $g_\alpha : F(X, B) \to G$ is a continuous homomorphism. Hence $(F^a(X, V), a_X) = (F(X, A), f_A)$, $(F^k(X, V), q_X) = (F(X, D), f_D)$ and $(F(X, V), t_X) = (F(X, C), f_C)$. The assertions are proved.

Remark 4.2.16. If a space X is discrete, then the mappings $b_X : F^a(X, V) \to F^k(X, V)$ and $c_X : F^a(X, V) \to F(X, V)$ are topological isomorphisms. If X is a k-space, then $l_X : F^k(X, V) \longrightarrow F(X, V)$ is a topological isomorphism.

Definition 4.2.17. *Let V be a T_i-quasivariety of k-E-algebras. If $b_X : F^a(X, V) \to F^k(X, V)$ is an isomorphism, then we say that the algebra $F^k(X, V)$ is algebraically free in V. If $c_X : F^a(X, V) \to F(X, V)$ is an isomorphism, then we say that the algebra $F(X, V)$ is algebraically free in V.*

Remark 4.2.18. If the algebra $F(X, V)$ is algebraically free in V, then the algebra $F^k(X, V)$ is algebraically free in V, too.

Remark 4.2.19. Let $i \in \{2, 3, 3\frac{1}{2}\}$ and V be a complete T_i-quasivariety. Then:

1. If $i = 2$, then for every space X the algebra $F^k(X, V)$ is a k-space.
2. If $i = 3$, then for every space X the algebra $F^k(X, V)$ is a k_ρ-space.
3. If $i = 3\frac{1}{2}$, then for every space X the algebra $F^k(X, V)$ is a k_R-space.

Theorem 4.2.20. *Let $i \in \{2, 3, 3\frac{1}{2}\}$ and V be a complete T_i-quasivariety. Then:*

1. For every completely regular space X the algebras $F^k(X, V)$ and $F(X, V)$ are algebraically free in V.

2. If the class V is not trivial, i.e. $|G| \geqslant 2$ for some $G \in V$, then for every completely regular space X the mapping $t_X : X \to F(X, V)$ is an embedding and $t_X(X)$ is a closed subset of $F(X, V)$.

Proof. Let $tV = \{G \in V : G$ is a topological E-algebra $\}$. By definition, the class tV is a t-complete T_i-quasivariety of topological E-algebras. Moreover, there exists a unique continuous isomorphism $h_X : F^a(X, V) \to F^a(X, tV)$ such that $h_X(a_X(x)) = a_X(x)$ for every $x \in X$. The algebra $F(X, tV)$ is algebraically free in tV (see [38] Theorem 2.5). Hence the algebra $F(X, V)$ is algebraically free in V. By Remark 8.3.7 the algebra $F^k(X, V)$ is algebraically free in V, too. The assertion 1 is proved. The assertion 2 follows from the assertion (2) of Theorem 2.5 in [38]. The proof is complete.

Corollary 4.2.21. *Let V be a non-trivial complete T_i-quasivariety of k-E-algebras. Then for every FH-space X the algebras $F^k(X, V)$ and $F(X, V)$ are topologically free in V and $t_X : X \to F(X, V)$ is a continuous injection.*

Remark 4.2.22. If E is a discrete signature and $V = \Pi(V) = S(V) = \Gamma'_i(V) \subseteq W_i(E)$, then V is a complete T_i-quasivariety of k-E-algebras.

This fact is not true for a non-discrete signature E.

Example 4.2.23. Let B be a countable group for which $(B, \mathcal{T})$ is a Hausdorff topological group if and only if $\mathcal{T}$ is a discrete topology. On B consider some non-discrete completely regular topology $\mathcal{T}_0$ relative to which every compact subset of $(B, \mathcal{T}_0)$ is finite. Let $E_0 = \{0\}$, $E_1 = \{-\} + (B, \mathcal{T}_0)$, $E_2 = \{+\}, \{0, -, +\}$ be the group signature and $E = E_0 + E_1 + E_2$. We put $e_{1B}(u, x) = u + x$ for all $u \in B \subseteq E_1$ and $x \in B$. Then B is a k-E-algebra.

Let A be an infinite commutative group and $e_{1A}(u, x) = 0$ for all $u \in B$ and $x \in A$. Then A is an E-algebra, too.

Denote by V the complete T_2-quasivariety of E-algebras generated by the family $\{A, B\}$. If $G \in V$ and G is a topological E-algebra, then G as a group is commutative. The class $tV = \{G \in V : G$ is a topological algebra$\}$ is the t-complete T_2-quasivariety of E-algebras generated by the class $\{A\}$. In particular, $(B, \mathcal{T}_0) \in V$ and $\Gamma_i(\{(B, \mathcal{T}_0)\}) = \emptyset$.

4.3 Free Algebras of μ-Complete Spaces

Fix a k_ω-signature E, $i \in \{2, 3, 3\frac{1}{2}\}$ and a non-trivial complete T_i-quasivariety V of k-E-algebras. Let $tV = \{G \in V : G$ is a topological E-algebra$\}$. Then tV is a non-trivial t-complete T_i-quasivariety of topological E-algebras and for every completely regular space X the mappings $t_X : X \to F(X, V)$ and t_X :

$X \to F(X, tV)$ are embeddings. Moreover, there exists a continuous isomorphism $h_X : F(X, V) \to F(X, tV)$ such that $h_X(t_X(x)) = t_X(x)$ for all $x \in X$.

Theorem 4.3.1. *If X is a completely regular space, then:*

1. For every bounded subset B of $F(X, V)$ there exist a bounded subset L of X, a compact subset H of E and $n \in N$ such that $B \subseteq a_n(H, i_X(L))$.

2. For every bounded subset B of $F^k(X, V)$ there exist a bounded subset L of X, a compact subset H of F and $n \in N$ such that $B \subseteq a_n(H, k_X(L))$.

3. If X is μ-complete, then the spaces $F(X, V)$, $F^k(X, V)$ are μ-complete.

4. If X is μ-complete and $i = 2$, then $F^k(X, V) = k(F(X, V))$.

5. If X is μ-complete and $i = 3$, then $F^k(X, V) = k_\rho(F(X, V))$.

6. If X is μ-complete and $i = 3\frac{1}{2}$, then $F^k(X, V) = k_R(F(X, V))$.

7. If X is a k_ω-space, then $F^k(X, V) = F(X, V)$ is a k_ω-space and a topological E-algebra.

8. The spaces $F(X, V)$ and $F^k(X, V)$ are FH-spaces.

9. If $F(X, V)$ is μ-complete, then X is μ-complete, too.

Proof. Let B be a bounded subset of $F(X, V)$. Then $B_1 = h_X(B)$ is a bounded subset of $F(X, tV)$. By virtue of Theorem 3.4 from [38], $B_1 \subseteq a_n(H, t_X(L))$ for some $m \in N$, a compact-subset $H \subseteq E$ and a bounded subset L of X. Since h_X is an isomorphysm, then $B_1 \subseteq a_n(H, i_X(L))$. The assertion 1 is proved.

Let B be a bounded subset of $F^k(X, V)$. The mapping $l_X : F^k(X, V) \to F(X, V)$ is a continuous isomorphism and the set $l_X(B)$ is bounded in $F(X, V)$. If $l_X(B) \subseteq a_n(H, t_X(L))$, then $B \subseteq a_n(H, k_X(L))$. Hence the assertion 2 follows from the assertion 1. The assertion 3 follows from the assertions 1 and 2. From the assertions 1 and 2 it follows that

$$comp(F(Y, V)) = \{l_X(\Phi) : \Phi \in comp(F^k(Y,\ V))\}$$

for every μ-completely regular space Y. Hence the assertions 4, 5 and 6 follow from the assertions 1 and 2. The assertion 7 follows from [82].

If $\{X_n : n \in N\}$ is a k_ω-sequence in X and $\{H_n : n \in N\}$ is a k_ω-sequence in E, then $\{a_n(H_n, t_X(X_n) : n \in N\}$ is a k_ω-sequence in $F(X, V)$. Hence $F(X, V)$ is a k_ω-space and $F^k(X, V) = F(X, V)$. The proof is complete.

4.4 On M-equivalence of spaces

Fix a k_ω-signature E, $i \in \{2, 3, 3\frac{1}{2}\}$ and a non-trivial complete T_i-quasivariety V of k-E-algebras. Let $tV = \{G \in V : G \text{ is a topological } E\text{-algebra}\} = V \cap V_i(E)$.

Definition 4.4.1. *The spaces X and Y are called:*

1. M_V-equivalent if the algebras $F(X, V)$ and $F(Y, V)$ are topologically isomorphic.

2. wM_V-equivalent if the algebras $F^k(X, V)$ and $F^k(Y, V)$ are topologically isomorphic.

3. tM_V-equivalent if the algebras $F(X, tV)$ and $F(Y, tV)$ are topologically isomorphic.

Definition 4.4.2. *A space X is called V-perfect if the mapping $q_X : X \to F^k(X, V)$ is an embedding.*

Remark 4.4.3. Let X be a completely regular space. Then:

1. If X is a k-space, then X is V-perfect.

2. If $i \geqslant 3$ and X is a k_ρ-space, then X is V-perfect.

3. If $i = 3\frac{1}{2}$ and X is a k_R-space, then X is V-perfect.

Lemma 4.4.4. *Let G be a k-E-algebra. Then there exist a topological E-algebra $G/t_i \in V_i(E)$ and a continuous homomorphism $p_{iG} : G \to G/t_i$ such that for every continuous homomorphism $\varphi : G \to A \in V_i(E)$ there exists a continuous homomorphism $\varphi_{iG} : G/t_i \to A$ such that $\varphi = \varphi_{iG} \circ p_{iG}$.*

Proof. Let $\{f_\mu : G \to A_\mu : \mu \in M\}$ be the family of all continuous homomorphisms $f : G \to A$ with $A \in V_i(E)$ and $|A| \leqslant |G|$. Consider the diagonal product $\varphi_{iG} : G \to \Pi\{A_\mu : \mu \in M\}$, where $\varphi_{iG}(x) = \{f_\mu(x) : \mu \in M\}$. Setting $G/t_i = \varphi_{iG}(G)$ we complete the proof.

The following two statements are obvious.

Lemma 4.4.5. *$F(X, tV) = F(X, V)/t_i$ for every space X.*

Lemma 4.4.6. *If $f : A \to B$ is a continuous homomorphism of a k-E-algebra A into k-E-algebra B, then there exists a continuous homomorphism $if : A/t_i \to B/t_i$ such that $\varphi_{iB} \circ f = if \circ \varphi_{iA}$.*

Corollary 4.4.7. *If the spaces X and Y are M_V-equivalent, then they are tM_V-equivalent too.*

Theorem 4.4.8. *Let X and Y be completely regular μ-complete V-perfect spaces. The following assertions are equivalent:*

1. *The spaces X and Y are M_V-equivalent.*
2. *The spaces X and Y are tM_V-equivalent.*
3. *The spaces X and Y are wM_V-equivalent.*

Proof. The theorem is obvious if V is trivial. Suppose that the class V is not trivial. The implication $1 \to 2$ follows from Corollary 4.4.7. For every V-perfect space X we have $F(X, tV) = F^k(X, V)/t_i$ and $F^k(X, K) = F(X, K)$. This proves the implications $3 \to 2$ and $1 \to 3 \to 1$.

Let Z be a completely regular μ-complete space. Consider the continuous isomorphism $m_Z : F^k(Z, V) \to F(Z, tV)$, where $m_Z(q_Z(z)) = t_Z(z)$ for every $z \in Z$.

Since $comp(F(Z, tV) = \{m_Z(\Phi) : \Phi \in comp(F^k(Z, V))\}$ we have $F^k(Z, V) = k_V(F(Z, tV)$. This proves the implication $3 \to 2$. The proof is complete.

Corollary 4.4.9. *Let X and Y be completely regular μ-complete k-spaces. The following assertions are equivalent:*

1. *The spaces X and Y are M_V-equivalent.*
2. *The spaces X and Y are wM_V-equivalent.*
3. *The spaces X and Y are tM_V-equivalent.*

Corollary 4.4.10. *Let $i \geq 3$ and X, Y be completely regular μ-complete k_ρ-spaces. The following assertions are equivalent:*

1. *The spaces X and Y are M_V-equivalent.*
2. *The spaces X and Y are wM_V-equivalent.*
3. *The spaces X and Y are tM_V-equivalent.*

Corollary 4.4.11. *Let $i = 3\frac{1}{2}$ and X, Y be completely regular μ-complete k_R-spaces. The following assertions are equivalent:*

1. *The spaces X and Y are M_V-equivalent.*
2. *The spaces X and Y are wM_V-equivalent.*
3. *The spaces X and Y are tM_V-equivalent.*

Remark 4.4.12. Every paracompact space is μ-complete. Every normal metacompact space is μ-complete.

Remark 4.4.13. Let X and Y be completely regular wM_V-equivalent V-perfect spaces. Then the following assertions are equivalent:

1. X is μ-complete.
2. $F^k(X,V)$ is μ-complete.
3. $F(X,V)$ is μ-complete.
4. $F(X,tV)$ is μ-complete.
5. Y is μ-complete.

4.5 One Special Construction

Fix a k_ω-signature E, a non-trivial complete T_2-quasivariety V of k-E-algebras and a non-empty locally compact space P. Then $Z \times P$ is a k-space for every k-space Z (see [89], Theorem 3.3.27).

For every space X the space $F^k(X,V)$ is a k-space.

If Z is a completely regular k-space, then $q_Z : Z \to F^k(Z,V)$ is an embedding and we identify Z with $q_Z(Z)$. In this condition $Z \subseteq F^k(Z,V)$ and for every k-continuous mapping $f : Z \to G \in V$ the mapping f is continuous and there exists a unique continuous homomorphism $\bar{f} : F^k(Z,V) \longrightarrow G$ such that $f = \bar{f}|Z$.

Let X be a completely regular k-space. Then $X \times P$ is a k-space too. For every $z \in P$ there exists a continuous homomorphism $\Psi_z : F^k(X,V) \to F^k(X \times P, V)$ such that Ψ_z is an embedding and $\Psi_z(x) = (x,z)$ for every $x \in X$. Now we consider the mapping

$$\Psi : F^k(X,V) \times P \to F^k(X \times P, V),$$

where $\Psi(x,z) = \Psi_z(x)$ for every $(x,z) \in F^k(X,K) \times P$.

The set of the polynomials or of the derived operations is the smallest set $P(E)$ of operations on E-algebras such that:

1. $E \subseteq P(E)$ and $e \in P(E)$, where $e(x) = x$ for every $x \in G$ and an E-algebra G.

2. If $n \geqslant 1$, $\omega \in E_n$ and $u_1, \dots, u_n \in P(E)$, then $\omega(u_1, \dots, u_n) \in P(E)$. The type of $\omega(u_1, \dots, u_n)$ is equal to the sum of types of polynomials $u_1, \dots, u_n$.

If G is a k-E-algebra and u is a polynomial of the type m, then $u : G^m \to G$ is a k-continuous mapping. For $n \in N$ and $u \in E_n$ we have $u(x) = e_{nG}(u,x)$ for all $x \in G^n$.

If $y \in F^k(X,V)$ and $X \subseteq F^k(X,V)$, then there exist $n \in N$, a polynomial u of type n and $x \in X^n$ such that $u(x) = y$.

Denote by $C_nP_0(E)$ the family of all compact subsets of the space E_n. Let $m \geqslant 1$ and the families $\{C_nP_i(E) : i < m, n \in N\}$ of compact spaces be defined such that every compact space $H \in C_nP_i(E)$ is a set of polynomials of type n and for every k-E-algebra G the mapping $u_H : H \times G^n \to G$, where $u_H(h,x) = h(x)$ for every $h \in H$ and $x \in G^n$, is k-continuous. Fix $n \geqslant 1$, $H_0 \in C_nT_0(E)$, $H_1 \in C_{m_1}P_{i_1}(E), \ldots, H_n \in C_{m_n}P_{i_n}(E)$. If $p = m_1 + \cdots + m_n$, then $H = H_0 \times H_1 \times \cdots \times H_n$ is a compact space from the family $C_pP_m(E)$ and

$$\omega_H((h_0, h_1, \ldots, h_n), (x_1, \ldots, x_p)) =$$

$$= h_0(h_1(x_1, \ldots, x_{m_1}), \ldots, h_n(x_{m_1+\cdots+m_{n-1}+1}, \ldots, x_p)).$$

By induction, the families $\{C_nP_m(E) : n, m \in N\}$ of compact spaces are defined.

Theorem 4.5.1. *Let X be a completely regular μ-complete k-space. Then the mapping $\Psi : F^k(X,V) \times P \to F^k(X \times P, V)$ is continuous.*

Proof. Consider the family $\mathcal{L} = \{\omega_H(H \times \Phi^n) : H \in C_nP_m(E), \Phi$ is a compact subset of X and $n, m \in N\}$ of compact subsets of $F(X,V)$. From Theorem 8.4.1 it follows that for every compact subset L of $F^k(X,V)$ there exists a finite family $L_1, \ldots, L_m \in \mathcal{L}$ such that $L \subseteq L_1 \cup \cdots \cup L_m$. Hence it is sufficient to prove that for every $L \in \mathcal{L}$ the mapping $\Psi|L \times P$ is continuous.

Fix $n, m \in N$, a compact space $H \in C_nP_m(E)$, a compact subset Φ of X and a compact subset Γ of P. Then $L = u_H(H \times \Phi^n)$ is a compact subset of $F^k(X, L)$. The mappings

$g_1 : H \times \Phi^n \times \Gamma \to F^k(X,K) \times P$, $g_2 : H \times \Phi^n \times \Gamma \to H \times (\Phi \times \Gamma)^n$ and
$g_3 : H \times (\Phi \times \Gamma)^n \to F^k(X \times P, K)$, where
$g_1(h, x_1, \ldots, x_n, y) = (h(x_1, \ldots, x_n), y)$,
$g_2(h, x_1, \ldots, x_n, y) = (h, (x_1, y), \ldots, (x_n, y))$ and
$g_3(h, (x_1, y_1), \ldots, (x_n, y_n)) = h((x_1, y_1), \ldots, (x_n, y_n))$, are continuous.

By construction, $\Psi(g_1(h, x_1, \ldots, x_n, y)) = h((x_1, y), \ldots, (x_n, y))$ for every $h \in H$, $y \in \Gamma$ and $(x_1, \ldots, x_n) \in \Phi^n$. Hence for $g_4 = \Psi|L \times \Gamma$ we have $g_3 \cdot g_2 = g_4 \cdot g_1$.

Since the mapping g_1 is perfect and the mappings g_2, g_3 are continuous, then for every closed subset A of $F^k(X \times P, V)$ the set $g_n^{-1}(A) = g_1(g_2^{-1}(g_3^{-1}(A)))$ is closed in $L \times \Gamma$. Hence the mapping g_4 is continuous. Therefore the mapping Ψ is k-continuous. The proof is complete.

Corollary 4.5.2. *For every completely regular μ-complete space X the mapping*

$$\Psi : F^k(X, V) \times P \longrightarrow F^k(X \times P, V)$$

is continuous and the mappings

$$\Psi : F(X, V) \times P \longrightarrow F(X \times P, V),$$

$$\Psi : F(X, tV) \times P \longrightarrow F(X \times P, tV)$$

are k-continuous.

Corollary 4.5.3. *If X and P are k_ω-spaces, then the mapping*

$$\Psi : F(X, tV) \times P \longrightarrow F(X \times P, tV)$$

is continuous.

4.6 Homotopy Classes of Mappings

Fix a signature E and a topological E-algebra G. Then the family $C(X, G)$ of all continuous mappings $f : X \longrightarrow G$ of a space X in G is an E-algebra. For every $n \in N$, $x \in X$ and f_1, ..., $f_n \in C(X, G)$ we have $\omega(f_1, ..., f_n)(x) = \omega(f_1(x), ..., f_n(x))$.

The mappings $f, g \in C(X, G)$ are called homotopic if there exists a continuous mapping $\varphi : X \times I \longrightarrow G$ where $\varphi(x, 0) = f(x)$ and $\varphi(x, 1) = g(x)$ for all $x \in X$. Consider on $C(X, G)$ the homotopy equivalence $f \overset{h}{\sim} g$.

Theorem 4.6.1. *If G is a topological E-algebra, then the homotopy equivalence $\overset{h}{\sim}$ is a congruence relation on $C(X, G)$.*

Proof. If $n \geq 1$, $\omega \in E_n$ and $\varphi_1, ..., \varphi_n : X \times I \longrightarrow G$ are continuous mappings, then $\omega(\varphi_1, ..., \varphi_n) : X \times I \longrightarrow G$ is a continuous mapping. The proof is complete.

Denote by $\Gamma(X,G)$ the quotient algebra of homotopy classes in $C(X,G)$. Then there exists a homomorphism $h_X : C(X,G) \longrightarrow \Gamma(X,G)$ onto $\Gamma(X,G)$ such that $h_X(f) = h_X(g)$ if and only if $f \overset{h}{\sim} g$.

Theorem 4.6.2. *Let G be a k-E-algebra and X be a k-space. Then $C(X,G)$ is an E-algebra and the homotopy equivalence $\overset{h}{\sim}$ is a congruence relation on $C(X,G)$.*

Proof. Let $f_1,...,f_n : X \longrightarrow G$ be continuous mappings, $n \geq 1$ and $\omega \in E_n$. Consider the mapping $f = \omega(f_1,...,f_n)$. Let Φ be a compact subset of X. Then $\Phi_i = f_i(\Phi)$ is a compact subset of G for every $i \leq n$. We put $H = \Phi_1 \times ... \times \Phi_n$ and $g = \omega|H$. The mapping g and the mapping $\varphi : \Phi \to H$, $\varphi(x) = (f_1(x),...,f_n(x))$, are continuous. By construction, $f|\Phi = g \circ \varphi$ is a continuous mapping. Thus f is a continuous mapping and $C(X,G)$ is an E-algebra.

Let $\varphi_1,...,\varphi_n : X \times I \longrightarrow G$ be continuous mappings, $n \geq 1$ and $\omega \in E_n$.

Since $X \times I$ is a k-space, $\omega(\varphi_1,...,\varphi_n) : X \times I \longrightarrow G$ is a continuous mapping. Hence $\overset{h}{\sim}$ is a congruence relation on $C(X,G)$.

Hence for every k-E-algebra G and every k-space X there exists a homomorphism $h_X : C(X,G) \longrightarrow \Gamma(X,G)$. The proof is complete.

4.7 Homotopy Classes of Homomorphisms

Fix a signature E. Let A and B be k-E-algebras. Denote by $Hom(A,B)$ the family of all continuous homomorphisms $f : A \longrightarrow B$. By construction, $Hom(A,B) \subseteq C(A,B)$. The homomorphisms $f,g \in Hom(A,B)$ are called a-homotopic if there exists a continuous mapping $\varphi : A \times I \longrightarrow B$ such that:

– $\varphi(x,0) = f(x)$ and $\varphi(x,1) = g(x)$ for every $x \in A$;

– for every $t \in I$ the mapping $\varphi_t : A \longrightarrow B$, where $\varphi_t(x) = \varphi(x,t)$, is a continuous homomorphism.

If $f,g \in Hom(A,B)$ and f,g are a-homotopic, then we put $f \overset{a}{\sim} g$.

Definition 4.7.1. *The E-algebra G is an algebra with permutable operations if the following conditions hold:*

– if $E_0 \neq 0$, then $e_{0G}(E_0 \times G^0)$ is a singleton subalgebra of G;

– if $u \in E_0$, $n \geq 1$ and $v \in E_n$, then $v(u_G,...,u_G) = u_G$;

– *if* $n, m \geq 1$, $u \in E_n$ *and* $v \in E_m$, *then* $u(v(x_{11}, ..., x_{1m}), ..., v(x_{n1}, ..., x_{nm}) = v(u(x_{11}, ..., x_{n1}), ..., u(x_{1m}, ..., x_{nm}))$ *for all* $\{x_{ij} : i \leq n, j \leq m\} \subseteq G$.

If G is an E-algebra, then for every E-algebra A the set G^A is the E-algebra of all mappings $f : A \longrightarrow G$ and $hom(A, G)$ is the set of all homomorphisms $g : A \longrightarrow G$.

Proposition 4.7.2. *Let* G *be an* E*-algebra with permutable operations. Then for every* E*-algebra* A *the set* $hom(A, G)$ *is a subalgebra of the* E*-algebra* G^A.

Proof. Let $n \geq 1$, $n \in E_n$, $f_1, ..., f_n \in hom(A, G)$ and $f = u(f_1, ..., f_n)$, i. e. $f(x) = u(f_1(x), ..., f_n(x))$ for each $x \in A$. If $m \geq 1$, $v \in E_m$ and $x_1, ..., x_m \in A$, then

$$f(v(x_1, ..., x_m)) = u(f_1(v(x_1, ..., x_n)), ..., f_n(v(x_1, ..., x_m))) =$$

$$= u(v(f_1(x_1), ..., f_1(x_m)), ..., v(f_n(x_1), ..., f_n(x_m))) =$$

$$= v(f(x_1), ..., f(x_n)), ..., u(f_1(x_m), ..., f_n(x_m)) = v(f(x_1), ..., f(x_m))$$

and $f \in hom(A, B)$. The proof is complete.

Since $Hom(A, G) = hom(A, G) \bigcap C(A, G)$, then from Proposition 4.7.2 we obtain:

Corollary 4.7.3. *Let* G *be a* k*-*E*-algebra with permutable operations. Then:*

1. If G *and* A *are topological* E*-algebras, then* $Hom(A, G)$ *is a subalgebra of the* E*-algebra* $C(A, G)$.

2. If a k*-*E*-algebra* A *is a* k*-space, then* $Hom(A, G)$ *is a subalgebra of the* E*-algebra* $C(A, G)$.

Remark 4.7.4. Let G be an E-algebra and $Hom(G, G)$ be a subalgebra of the E-algebra G^G. Then:

- if $E_0 \neq 0$, then $e_{0G}(E_0 \times G^0)$ is a singleton subalgebra of G;
- if $u \in E_0$, $n \geq 1$ and $v \in E_n$, then $v(u_G, ..., u_G) = u_G$;
- if $n, m \geq 1$, $u \in E_n$ and $v \in E_m$, then

$$u(v(x_1, ..., x_m), ..., v(x_1, ..., x_m)) = v(u(x_1, ..., x_1), ..., u(x_m, ..., x_m))$$

for all $x_1, ..., x_n \in G$.

Suppose that $Hom(A,B)$ is a subalgebra of the E-algebra $C(A,B)$. Then the a-homotopy equivalence is a congruence relation on $Hom(X,G)$.

Let $\Gamma Hom(A,B)$ be the set of classes of a-homotopy equivalence on $Hom(A,B)$ and

$$d_X : Hom(A,B) \longrightarrow \Gamma Hom(A,B)$$

be the natural homomorphism, where $d_X(f) = d_X(g)$ iff $f \overset{a}{\sim} g$.

If $f \overset{a}{\sim} g$, then $f \overset{h}{\sim} g$. The converse is false.

Example 4.7.5. Let $E_0 = \{\omega_0\}$, $E_1 = (0,1]$ and $E = E_0 \bigcup E_1$. Consider the space $G = [0,1]$. We put $e_{0G}(E_0 \times G^0) = \{0\}$ and $e_{1g}(u,x) = \min\{1, \frac{x}{u}\}$ for all $u \in E_1$ and $x \in G$. Then E, G are k_ω-spaces and G is a topological E-algebra. The mappings $f, g : G \longrightarrow G$, where $f(x) = 0$ and $g(x) = x$, are continuous homomorphisms. It is obvious that the homomorphisms f, g are homotopic mappings and are not a-homotopic homomorphisms.

Theorem 4.7.6. *Let V be a non-trivial complete T_2-quasivariety of k-E-algebras and X be a completely regular k-space. Then $X \subseteq F^k(X,V)$ and for every $G \in V$ there exists a unique mapping $\pi_X : C(X,G) \longrightarrow Hom(F^k(X,V),G)$ such that $f = \pi_X(f)|X$ for any $f \in C(X,G)$.*

Proof. By conditions of the theorem, $F^k(X,V)$ is a k-space and $q_X : X \longrightarrow F^k(X,V)$ is an embedding. By Definition 8.3.3 (B), for every $f \in C(X,G)$ there exists a unique continuous homomorphism $\pi_X(f) = qf : F^k(X,V) \rightarrow G$ such that $\pi_X(f)|X = f$. The proof is complete.

From Theorem 4.7.6 and Proposition 4.7.2 the next assertion follows:

Corollary 4.7.7. *Let V be a non-trivial complete T_2-quasivariety of k-E-algebras and X be a completely regular k-space. If $G \in K$ is a k-E-algebra with permutable operations, then the mapping $\pi_X : C(X,G) \longrightarrow Hom(F^k(X,V),G)$ is an isomorphism.*

Theorem 4.7.8. *Let E be a k_ω-signature, V be a non-trivial complete T_2-quasivariety of k-E-algebras, $G \in V$ and X be a completely regular μ-complete k-space. Then:*

1. There exists a unique one-to-one mapping

$$\varphi_X : \Gamma(X,G) \longrightarrow \Gamma Hom(F^k(X,V),G)$$

such that $d_X \cdot \pi_X = \varphi_X \cdot h_X$.

2. If G is a k-E-algebra with permutable operations, then φ_X is an isomorphism.

Proof. Let $f, g \in C(X,G)$ and $f_1 = \pi_X(f)$, $g_1 = \pi_X(g)$. If the homomorphisms f_1, g_1 are a-homotopic, then f, g are homotopic, too. Assume that f, g are homotopic. There exists a continuous mapping $\varphi : X \times I \longrightarrow G$ such that $\varphi(x, 0) = f(x)$ and $\varphi(x, 1) = g(x)$ for all $x \in X$. Since $X \times I$ is a completely regular k-space, we consider that $X \times I = q_{X \times I}(X \times I) \subseteq F^k(X \times I, V))$. By Definition 3.3 (B), there exists a continuous homomorphism $\varphi_1 : F^k(X \times I, V) \longrightarrow G$ such that $\varphi_1 | X \times I = \varphi$.

By virtue of the Theorem 4.6.1, there exists a continuous mapping

$$\psi : F^k(X, V) \times I \longrightarrow F^k(X \times I, V),$$

where for every $t \in I$ the mapping

$$\psi_t : F^k(X, V) \longrightarrow F^k(X \times I, V)$$

with $\psi_t(x) = (x, t)$ for all $x \in X$ is a continuous homomorphism.

Consider the continuous mapping

$$\varphi_2 = \varphi_1 \circ \psi : F^k(X, V) \times I \longrightarrow G.$$

By construction for every $t \in I$ the mapping $\varphi_t : F^k(X, V) \longrightarrow G$, where $\varphi_t(x) = \varphi_2(x, t)$, is a continuous homomorphism. By construction $\varphi_2(x, 0) = f_1(x)$ and $\varphi_2(x, 1) = g_1(x)$ for every $x \in F^k(X, V)$. Hence $f_1 \overset{a}{\sim} g_1$. Therefore $f \overset{h}{\sim} g$ if and only if $f_1 \overset{a}{\sim} g_1$. Thus

$$\varphi_X : \Gamma(X, G) \longrightarrow \Gamma Hom(F^k(X, V), G),$$

where $\varphi_X(\xi) = d_X(\pi_X(h_X^{-1}(\xi))$ for each $\xi \in \Gamma(X, G)$, is a one-to-one mapping.

If π_X is a homomorphism, then φ_X is a isomorphism too. Corollary 4.8.7 completes the proof.

Corollary 4.7.9. *Let E be a k_ω-signature, V be a non-trivial complete T_2-quasivariety of k-E-algebras, $G \in V$ be an E-algebra with permutable operations and X, Y be completely regular μ-complete k-spaces. Then:*

1. If the spaces X, Y are M_V-equivalent, then the E-algebras $\Gamma(X, G)$, $\Gamma(Y, G)$ are isomorphic.

2. If the spaces X, Y are wM_V-equivalent, then the E-algebras $\Gamma(X, G)$, $\Gamma(Y, G)$ are isomorphic.

3. If the spaces X, Y are tM_V-equivalent, then the E-algebras $\Gamma(X, G)$, $\Gamma(Y, G)$ are isomorphic.

4.8 On Homotopical Cohomology

Let G be a commutative group and for every $n \in N$ by $K(G, n)$ denote an Eilenberg-MacLane complex, i. e. a CW-complex having a single non-vanishing homotopy group G in dimension n. The *n-th* homotopical cohomology group $H^n(X, G)$ of a space X is defined as the group of homotopy classes of continuous maps of X into $K(G, n)$, the group structure being induced by any of the H-structures of $K(G, n)$.

The space $K(G, n)$ can be realized as a commutative k-group (see [115, 160]). Moreover, if G is a countable group, then $K(G, n)$ can be realized as a topological commutative group [160]. Consider that $\{K(G, n) : n \in N\}$ are commutative k-groups. Then the *n-th* homotopical cohomology group $H^n(X, G)$ of a k-space X can be defined as the group $\Gamma(X, K(G, n))$ of homotopy classes of continuous mappings $C(X, K(G, n))$ (see [115, 160]). For a paracompact k-space X the homotopical cohomology group $H^n(X, G)$ is isomorphic with the Čech cohomology group $\tilde{H}^n(X, G)$ (see [115]).

Fix a complete T_2-quasivariety V of k-groups and consider that $\{K(G, n) : n \in N\} \subseteq V$. Every commutative group is an algebra with permutable operations. Thus from Corollary 8.8.9 we obtain

Corollary 4.8.1. *Let X, Y be completely regular μ-complete k-spaces. If the spaces X, Y are M_V-equivalent, or wM_V-equivalet, or tM_V-equivalent, then the groups $H^n(X, G)$, $H^n(Y, G)$ are isomorphic.*

Corollary 4.8.2. *Let X, Y be paracompact k-spaces. If the spaces X, Y are M_V-equivalent, or wM_V-equivalent, or tM_V-equivalent, then the groups $\tilde{H}^n(X, G)$, $\tilde{H}^n(Y, G)$ are isomorphic.*

Let V_g be the variety of all topological groups and V_{ag} be the variety of all topological commutative groups. Then $F(X) = F(X, V_g)$ is Markov's free group of a space X and $A(X) = F(X, V_{ag})$ is Markov's Abelian free group of a space X.

Corollary 4.8.3.(B. A. Pasynkov [203]. *Let X and Y be compact spaces. If the topological groups $F(X)$ and $F(Y)$, or the topological groups $A(X)$ and $A(Y)$ are topologically isomorphic, then the groups $\tilde{H}^n(X, G)$, $\tilde{H}^n(Y, G)$ are isomorphic, too.*

Remark 4.8.4. Let Λ be a discrete ring and G be a Λ-module. Then $K(G, n)$ can be considered as Λ-modules (see [160]). Every Λ-module is an algebra with permutable operations. In this case the natural mapping

$$h_X : C(X, K(G, n)) \longrightarrow \Gamma(X, K(G, n))$$

is a homomorphism of the Λ-module $C(X, K(G, n))$ onto the Λ-module $\Gamma(X, K(G, n))$.

For every group A and every Λ-module B the group $hom(A, B)$ of group homomorphisms of A into B is a Λ-module. Hence Corollary 4.7.9 yields.

Corollary 4.8.5. *Let X, Y be completely regular μ-complete k-spaces, Λ be a discrete ring and G be a Λ-module. If $\{K(G, n) : n \in N\}$ as the k-groups are from quasivariety V and the spaces X, Y are M_V-equivalent, or wM_V-equivalent, or tM_V-equivalent, then the Λ-modules $H^n(X, G)$, $H^n(Y, G)$ are isomorphic. Moreover, if X, Y are paracompact k-spaces, then the Λ-modules $\tilde{H}^n(X, G)$, $\tilde{H}^n(Y, G)$ are isomorphic.*

4.9 Spaces of Mappings

Fix a k_ω-signature E, a non-trivial complete T_2-quasivariety V of k-E-algebras and a non-trivial topological E-algebra $G \in V$.

Let X be a space. On the space $C(X, G)$ consider the topology of pointwise convergence. If $A, B \in V$, then we consider $Hom(A, B)$ as a subspace of the space $C(A, B)$.

Theorem 4.9.1. *For every space X there exists a homeomorphism $\varphi : C(X, G) \to Hom(F(X, V), G)$ such that:*

1. $f = i_X \circ \varphi(f)$ for every $f \in C(X, G)$.

2. If G is a topological E-algebra with permutable operations, then φ is a topological isomorphism of the topological E-algebra $C(X,G)$ onto $Hom(F(X,V),G)$.
3. If X is a μ-complete k-space and $f,g \in C(X,G)$, then $f \overset{h}{\sim} g$ if and only if $\varphi(f) \overset{a}{\sim} \varphi(g)$.

Proof. We consider that $X = i_X(X) \subseteq F(X,V)$. For every $f \in C(X,G)$ there exists a unique continuous homomorphism $\varphi(f) : F(X,V) \longrightarrow G$ such that $f = t_X \circ \varphi(f)$, i. e. $f = \varphi(f)|X$.

Let $f \in C(X,G)$ and $\{f_\mu : \mu \in M\}$ be a net in $C(X,G)$, where M is a directed set. Then $\{\varphi(f_\mu) : \mu \in M\}$ is a net in $Hom(F(X,V),G)$. If $\varphi(f) = \lim \varphi(f_\mu)$, then $f = \lim f_\mu$. Suppose that $f = \lim f_\mu$. Fix $z \in F(X,V)$. Then there exists $n \in N$, a polynomial t and $x_1,...,x_n \in X$ such that $z = t(x_1,...,x_n)$. If $n = 0$, then $\varphi(f)(z) = \varphi(f_\mu)(z)$ for all $\mu \in M$ and $\lim \varphi(f_\mu)(z) = \varphi(f)(z)$. Let $n \geq 1$. Fix a neighborhood U of the point $\varphi(f)(z)$ in G. Then there exist some neighborhoods $U_1,...,U_n$ of the points $f(x_1) = \varphi(f)(x_1),...,f(x_n) = \varphi(f)(x_n)$ in G such that $t(U_1 \times ... \times U_n) \subseteq U$. Since $f(x_i) = \lim f_\mu(x_i)$, there exists $\mu_0 \in M$ such that $f_\mu(x_i) \in U_i$ for all $\mu \geq \mu_0$ and $i \leq n$. Hence

$$\varphi(f_\mu)(z) = \varphi(f_\mu)(t(x_1,...,x_n)) = t(\varphi(f_\mu)(x_1),...,\varphi(f_\mu)(x_n)) =$$

$$= t(f_\mu(x_1),...,f_\mu(x_n)) \in t(U_1 \times ... \times U_n) \subseteq U$$

for all $\mu \geq \mu_0$ and $\varphi(f)(z) = \lim \varphi(f_\mu)(z)$. Therefore

$$\varphi : C(X,G) \longrightarrow Hom(F(X,V),G)$$

is a homeomorphism. The assertions 1 and 2 are proved. The assertion 3 follows from Theorem 4.7.8. The proof is complete.

Corollary 4.9.2. *Let X and Y be μ-complete k-spaces and the algebras $F(X,V)$ and $F(Y,V)$ be topologically isomorphic. Then there exists a homeomorphism $u : C(X,G) \longrightarrow C(Y,G)$ such that:*

1. If $f,g \in C(X,G)$, then $f \overset{h}{\sim} g$ if and only if $u(f) \overset{h}{\sim} u(g)$.

2. If G is an algebra with permutable operations, then u is an isomorphism.

Remark 4.9.3.The assertion 1 of the Corollary 4.10.2 for the varieties V_g, V_{ag} of topological groups and compact spaces X,Y was proved by L. S. Pontrjagin ([103], §9).

4.10. Conclusions for Chapter 4

In this Chapter compact subsets of free algebras with topologies have been studied. On algebras we have been considered topologies relatively to which operations are continuous on compact subsets. Studying methods have been elaborated and some properties of k-free algebra and M_K-equivalence have been established.

The scientific innovation of this Chapter is determined by the following:

1). there have been introduced the concepts of: k-E-topological algebra, T_i-quasivariety of k-E-algebras, k-E-free algebra and relation of M_V-equivalence, wM_V-equivalence, tM_V-equivalence.

2). there have been elaborated studying methods of k-algebras.

3). the implementation of new concepts and methods of free algebras and k-algebras allowed obtaining new important results, with large applications in topological algebra:

- we have identified significant properties regarding the relation of M_K-equivalence. In particular, we obtained that homological groups are preserved under the relation of M_K-equivalence. Some similar results for the varieties of topological groups and compact spaces was proved by L. S. Pontrjagin and B. A. Pasynkov.

- we have described compact subsets of free topological algebras and of k-algebras.

The methodology proposed for research in this Chapter can be used:

- to introduce the concept of k-free topological groupoid with a continuous division;
- to describe compact subsets of k-free topological groupoid with a continuous division;
- to elaborate studying methods of k-free topological groupoid with a continuous division;
- to improve the methods of free topological algebras for the study of equivalence of spaces.
- to elaborate a general methods of decomposition of algebras with invertibility properties.

5. RESOLVABILITY OF SOME SPECIAL ALGEBRAS WITH TOPOLOGIES

A space X is called resolvable if in X there exist two disjoint dense subsets. In [64] M. Choban and L. Chiriac has proved the following assertion.

Theorem. *Let G be an infinite group of cardinality τ. Then there exists a disjoint family $\{B_\mu : \mu \in M\}$ of subsets of G such that:*
1. $|M| = |G|$.
2. $G = \cup\{B_\mu : \mu \in M\}$.
3. $(G \setminus B\mu) \cdot K \neq G$ for all $\mu \in M$ and every finite subset K of G.
4. The sets $\{B_\mu : \mu \in M\}$ are dense in all totally bounded topologies on G.

This fact is a generalization of one Protasov's result [180]. In this Chapter the assertions of Theorem are proved for the special algebras - I_nP_k-n-groupoids.

5.1 Introductory notions

We shall use the notation and terminology from [17, 26, 106]. In particular, $|X|$ is the cardinality of a set $|X|$, $N = 0, 1, 2, ...$, R is the space of reals. By ω_0 we denote the first infinite cardinal. If τ is an infinite cardinal, then τ^+ is the first cardinal larger than τ. If $\tau \geq 1$ is a cardinal, then the space X is called τ-resolvable if there exists a family of pairwise disjoint dense subsets $\{B_\alpha : \alpha \in A\}$ of X such that $|A| = \tau$. Every space is 1-resolvable. If the space X is 2-resolvable, then we say that X is resolvable.

Denote by a_1^m a sequence $a_1, a_2, ..., a_m$. If $a_1 = a_2 = ... = a_m$, then we denote this sequence by a^m. For every space X we put

$$m(X) = min\{|U| : U \neq \emptyset, U \subseteq X, U \in \tau\}.$$

A space X is maximal resolvable if it is $m(X)$-resolvable. It is clear that if X is τ-resolvable then $\tau \leq m(X)$. If $m(X) = |X| > 1$ and X is maximal resolvable, then we say that X is superresolvable.

For every mapping $f : X \to X$ we put $f' = f$ and $f^{n+1} = f \circ f^n$ for any $n \in N$. We can consider that $f^0 : X \to X$ is the identity mapping.

The problem of resolvability of totally bounded topological groups was solved by V.I. Malykhin, W.W. Comfort, S. Van Mill [37], I.V.Protasov [180] and M.M. Choban, L.L. Chiriac [64].

5.2 Groupoids with invertibility properties

Fix a sequence $\{E_n : n \in N\}$ of pairwise disjoint spaces. The discrete sum $E = \cup\{E_n : n \in N\}$ is called a signature or a set of fundamental operations. A universal algebra of signature E, or briefly, an E-algebra is a non-empty set G and a sequence of mappings $e_G = \{e_{nG} : E_n \times G^n \longrightarrow G : n \in N\}$. The set G is called a support of the E-algebra G and the mappings e_G are called the algebraical structure on G. Let G be an E-algebra. If $u \in E_0$, then the element $u_G = e_{0G}(\{u\} \times G^0)$ is called a constant of G and we put $u(x) = u_G$ for all $x \in G$. If $n \geq 1, u \in E_n$ and $x_1, ..., x_n \in G$, then we put $u(x_1, ..., x_n) = e_{nG}(u, x_1, ..., x_n)$. A pair (G, ω) is said to be a n-groupoid if G is a non-empty set and $\omega : G^n \to G$ is a mapping.

Definition 5.2.1 *Let $k \leq n$. An n-groupoid (G, ω) is called:*
1. an I_nP_k-n-groupoid if there exist the mappings
$r_1, ..., r_{k-1}, r_{k+1}, ..., r_n : G \to G$ such that
$\omega(r_1(x_1), ..., r_{k-1}(x_{k-1}), \omega(x_1, ..., x_{k-1}, y, x_{k+1}, ..., x_n), r_{k+1}(x_{k+1}), ...,$
$r_n(x_n)) = y$ or $\omega(r_1^{k-1}(x_1^{k-1}), \omega(x_1^{k-1}, y, x_{k+1}^n), r_{k+1}^n(x_{k+1}^n)) = y$
for all $x_1, ..., x_{k-1}, x_{k+1}, ..., x_n, y \in G$.
The mapping $r_i(x)$ is called k-involution, $i \in \{1, ..., k-1, k+1, ..., n\}$.
2. an I_nP-n-groupoid in the large sense if it is I_nP_k-n-groupoid for all $k = \overline{1, n}$. In this case the mapping $r_i(x)$ is called involution, $i \in \{1, ..., n\}$.
3. an I_nP-n-groupoid, or I_nP-n-groupoid in strong sense, if there exist the mappings $\{r_i : G \to G$: $i = \overline{1, n}\}$ such that $\{r_i : i \leqslant n, i \neq k\}$ is a family of k-involutions for any $k = \overline{1, n}$.
4. an I_0P_k-n-groupoid if there exist the mappings $r_1, ..., r_{k-1}, r_{k+1}, ..., r_n : G \to G$ such that $\omega(r_1(x), ..., r_{k-1}(x), \omega(x^{k-1}, y, x^{n-k}), r_{k-1}(x), ..., r_n(x)) =$
$= y$ for all $x, y \in G$.
5. an I_0P-n-groupoid if it is I_0P_k-n-groupoid for all $k = \overline{1, n}$.

Example 5.2.2 Let $(G, \cdot)$ be a topological non commutative group with the identity e. If we put $\omega(x, y, z, u) = y \cdot x \cdot u \cdot z$, then (G, ω) is an I_0P-4-quasigroup. Indeed:

1. (G, ω) is an I_0P_1-4-quasigroup for $r_2(y) = y^{-1}$, $r_3(z) = z^{-1}$, $r_4(u) = u^{-1}$. We have $\omega(\omega(x, t, t, t), r_2(t), r_3(t), r_4(t)) = r_2(t) \cdot t \cdot x \cdot t \cdot t \cdot r_4(t) \cdot r_3(t) = t^{-1} \cdot t \cdot x \cdot t \cdot t \cdot t^{-1} \cdot t^{-1} = e \cdot x \cdot t \cdot e \cdot t^{-1} = x \cdot t \cdot t^{-1} = x$.
2. (G, ω) is an I_0P_2-4-quasigroup for $r_1(x) = x^{-1}, r_3(z) = z^{-1}$, $r_4(u) = u^{-1}$. Really, $\omega(r_1(t), \omega(t, y, t, t), r_3(t), r_4(t)) = y \cdot t \cdot t \cdot t \cdot r_1(t) \cdot r_4(t) \cdot r_3(t) = y$.
3. (G, ω) is an I_0P_3-4-quasigroup for $r_1(x) = x^{-1}$, $r_2(y) = y^{-1}$, $r_4(u) = u^{-1}$. Really, $\omega(r_1(t), r_2(t), \omega(t, t, z, t), r_4(t)) = r_2(t) \cdot r_1(t) \cdot r_4(t) \cdot t \cdot t \cdot t \cdot z = z$.
4. (G, ω) is an I_0P_4-4-quasigroup for $r_1(x) = x^{-1}$, $r_2(y) = y^{-1}$, $r_3(z) = z^{-1}$. Really, $\omega(r_1(t), r_2(t), r_3(t), \omega(t, t, t, u)) = r_2(t) \cdot r_1(t) \cdot t \cdot t \cdot u \cdot t \cdot r_3(t) = u$.

In this case (G, ω) is an I_0P_i-4-quasigroup for every $i \in \{1, 2, 3, 4\}$. Hence, (G, ω) is an I_0P-4-quasigroup.

Example 5.2.3 Let $(G, \cdot)$ be a topological group with the identity e. We put $\omega(x, y, z) = x \cdot y \cdot z$. In this case:
1. (G, ω) is a 3-groupoid;
2. (G, ω) is an I_0P_i-3-groupoid for every $i \in \{1, 2, 3\}$ and for
$r_1(x) = r_2(x) = r_3(x) = x^{-1}$;
3. (G, ω) is an I_3P_2-3-groupoid for $r_1(x) = x^{-1}, r_3(x) = z^{-1}$.
Indeed, $\omega(r_1(x), \omega(x, y, z), r_3(z)) = x^{-1} \cdot x \cdot y \cdot z \cdot z^{-1} = e \cdot y \cdot e = y$;
4. If the group G is non commutative, then (G, ω) is not an I_3P_i-3-groupoid for $i = \{1, 3\}$.

Example 5.2.4 Let C be the field of the complex numbers, R be the field of the reals numbers. Let $A = C \setminus \{0\}$, $B = R \setminus \{0\}$ and $G = \{r \in R : r > 0\}$. Then $(A, \cdot)$, $(B, \cdot)$ and $(G, \cdot)$ are commutative multiplicative groups. We put $\omega(x, y, z) = x \cdot y^n \cdot z$, $n \geq 1$.
1. If $n = 1$, then (A, ω), (B, ω) and (G, ω) are I_3P-3-quasigroups.
2. If $n \geq 2$, then (A, ω), is a 3-groupoid with divisions. The equation
$\omega(a, y, c) = d$ has n solutions.
3. If $n > 1$ and n is odd, then (B, ω) and (G, ω) are 3-quasigroups.
4. If $n \geq 2$ and n is even, then (B, ω) is not a 3-groupoid with divisions and

(G, ω) is a 3-quasigroup.
5. (A, ω), (B, ω), (G, ω) are I_3P_1-3-groupoids and I_3P_3-3-groupoids.
If $n \geq 2$, then (A, ω), (B, ω) and (G, ω) are not I_3P_2-3-groupoids.

Example 5.2.5 Let C be the field of the complex numbers and $A = C \setminus \{0\}$. We fix $k \in A$ and put $\omega_n(x_1, x_2, ..., x_n) = k \cdot x_1 \cdot x_2 \cdot ... \cdot x_n$, $(n \geq 2)$. In this case:
1. (A, ω_n) is a commutative quasigroup.
2. (A, ω_n) is an I_nP-n-groupoid in strong sense. Denote $r_i(x_i) = \sqrt[n-1]{\frac{1}{k^2}} \cdot x_i^{-1}$. Hence,

$$\omega_n(r_1(x_1), ..., r_{i-1}(x_{i-1}), \omega_n(x_1, ..., x_{i-1}, x_i, x_{i+1}, ..., x_n),$$
$$r_{i+1}(x_{i+1}), ..., r_n(x_n)) = k \cdot (\sqrt[n-1]{\frac{1}{k^2}})^{i-1} \cdot x_1^{-1} \cdot x_2^{-1} \cdot ... \cdot x_{i-1}^{-1} \cdot k \cdot x_1 \cdot x_2 \cdot ... \cdot x_{i-1} \cdot x_i \cdot$$
$$x_{i+1} \cdot ... \cdot x_n \cdot (\sqrt[n-1]{\frac{1}{k^2}})^{n-i} \cdot \cdot x_{i+1}^{-1} \cdot ... \cdot x_n^{-1} = k^2 (\sqrt[n-1]{\frac{1}{k^2}})^{n-1} \cdot x_i = k^2 \cdot \frac{1}{k^2} \cdot x_i = x_i.$$

In strong sense there are $n-1$ complete involutions.
3. Let $n \geq 2$ and $m = 2 + (n-1)$. There is $k \in A$ such that $k^m = 1$ and $k^i \neq 1$ for $i < m$. If $r_i(x_i) = k \cdot x_i^{-1}$ then $\{r_1, r_2, ..., r_n\}$ are involutions in strong sense. Hence,
$\omega_n(r_1(x_1), ..., r_{i-1}(x_i), \omega_n(x_1, x_2, ..., x_n), r_{i+1}(x_{i+1}), ..., r_n(x_n)) = k^{n-1} \cdot k^2 \cdot x_1^{-1} \cdot ... \cdot x_{i-1}^{-1} \cdot x_1 \cdot ...x_{i-1} \cdot x_i \cdot x_{i+1} \cdot ... \cdot x_n \cdot x_{i+1}^{-1} \cdot ... \cdot x_n^{-1} = k^{2+n-1} \cdot x_i = k^m \cdot x_i = x_i$.
4. Let $n = 2$, $m \geq 3$, $k^m = 1$ and $k^i \neq 1$ for $i < m$. We put $\omega(x, y) = k \cdot x \cdot y$, $r_1(x) = k^{m-2}x^{-1}$, $r_2(y) = k^{m-2}y^{-1}$. In this case $\{r_1(x), r_2(x)\}$ are unique involutions in strong sense and
$r_i^2(x_i) = k^{m-2}(r_i(x_i))^{-1} = k^{m-2}((k^{m-2} \cdot x^{-1})^{-1}) = k^{m-2} \cdot \frac{1}{k^{m-2}} \cdot x_i = x_i$.

Example 5.2.6 Let $(G, \cdot)$ be a topological group with the identity. If we put $\omega(x, y) = x \cdot y$, then:
1. (G, ω) is a 2-groupoid or, briefly, groupoid;
2. (G, ω) is an RIP-groupoid for $r_2(x) = x^{-1}$.
Indeed, $\omega(\omega(y, x), r_2(x)) = (y \cdot x) \cdot x^{-1} = y$;
3. (G, ω) is an LIP-groupoid for $r_1(x) = x^{-1}$.
Indeed, $\omega(r_1(x_1), \omega(x, y)) = x^{-1}(x \cdot y) = y$;
4. (G, ω) is an IP-groupoid if it is both an RIP-groupoid and an LIP-groupoid. The notions LIP, RIP in the class of groupoids were introduced by R. H. Bruck [20].

Proposition 5.2.7 *Let (G, ω) be an I_nP_1-n-groupoid and* $r_2, r_3, ..., r_n : G \to G$ *be* 1*-involutions. Then the following assertions are equivalent:*

1. $\omega(\omega(y, x_2, ..., x_n), r_2(x_2), ..., r_n(x_n)) = y$*;*

2. $\omega(\omega(y, r_2(x_2), ..., r_n(x_n)), x_2, ..., x_n) = y$ *for all* $x_2^n \in G$*.*

Proof. Suppose that

$$\omega(\omega(y, x_2, ..., x_n), r_2(x_2), ..., r_n(x_n)) = y \tag{5.1}$$

for all $x_2^n, y \in G$. From (5.1) we have

$$\omega(\omega(\omega(y, x_2, ..., x_n), r_2(x_2), ..., r_n(x_n)), r_2^2(x_2), ..., r_n^2(x_n)) = \omega(y, x_2, ..., x_n) \tag{5.2}$$

and

$$\omega(\omega(\omega(y, x_2, ..., x_n), r_2(x_2), ..., r_n(x_n)), r_2^2(x_2), ..., r_n^2(x_n)) = \tag{5.3}$$

$$= \omega(y, r_2^2(x_2), ..., r_n^2(x_n)).$$

Using (5.2) and (5.3) we obtain

$$\omega(y, x_2, ..., x_n) = \omega(y, r_2^2(x_2), ..., r_n^2(x_n)). \tag{5.4}$$

Let in (5.4) $y = \omega(y, r_2(x_2), ..., r_n(x_n))$. Therefore from (5.4)

$$\omega(\omega(y, r_2(x_2), ..., r_n(x_n)), x_2, ..., x_n)) =$$

$$= \omega(\omega(y, r_2(x_2), ..., r_n(x_n)), r_2^2(x_2), ..., r_n^2(x_n)).$$

The implication $1 \to 2$ is proved.
Suppose that

$$\omega(\omega(y, r_2(x_2), ..., r_n(x_n)), x_2, ..., x_n) = y. \tag{5.5}$$

From (5.5) it follows that

$$\omega(\omega[y, r_2^2(x_2), ..., r_n^2(x_n)], r_2(x_2), ..., r_n(x_n)) = y. \tag{5.6}$$

It is clear that

$$\omega(\omega[\omega[y, r_2^2(x_2), ..., r_n^2(x_n)], r_2(x_2), ..., r_n(x_n)], x_2, ..., x_n) = \tag{5.7}$$

$$= \omega(y, r_2^2(x_2), ..., r_n^2(x_n)).$$

From (5.6) we obtain

$$\omega(\omega[\omega[y, r_2^2(x_2), ..., r_n^2(x_n)], r_2(x_2), ..., r_n(x_n)], x_2, ..., x_n) = \tag{5.8}$$

$$= \omega(y, x_2, ..., x_n).$$

Using (5.7) and (5.8) we have

$$\omega(y, r_2^2(x_2), ..., r_n^2(x_n)) = \omega(y, x_2, ..., x_n). \tag{5.9}$$

Therefore

$$\omega(\omega(y, x_2, ..., x_n), r_2(x_2), ..., r_n(x_n)) =$$

$$= \omega(\omega[y, r_2^2(x_2), ..., r_n^2(x_n)], r_2(x_2), ..., r_n(x_n)) = y.$$

The implication $2 \to 1$ is proved. The proof is complete.

Definition 5.2.8 *An n-groupoid (G, ω) is called:*
1. a k-cancellative n-groupoid if for every $a, b, x_1, ..., x_{k-1}, x_{k+1}, ..., x_n \in G$ we have $\omega(x_1, ..., x_{k-1}, a, x_{k+1}, ..., x_n) = \omega(x_1, ..., x_{k-1}, b, x_{k+1}, ..., x_n)$ if and only if $a = b$.
2. a cancellative n-groupoid if it is k-cancellative groupoid for all $k = \overline{1, n}$
3. an n-quasigroup if the equation $\omega(a_1^{i-1}, x, a_{i+1}^n) = b$ has unique solution for every a_i^n, b and each $i = \overline{1, n}$.

Definition 5.2.9 *An element e from (G, ω) is called:*
1. a k-identity of n-groupoid (G, ω) if $\omega(e^{k-1}, x, e^{n-k}) = x$ for every $x \in G$.
2. an identity of n-groupoid (G, ω) if $\omega(e^{i-1}, x, e^{n-i}) = x$ for every $x \in G$ and each $i = \overline{1, n}$.
If n-quasigroup (G, ω) contains at least one identity, then (G, ω) is called n-loop.

Proposition 5.2.10 *Let (G, ω) be an I_nP_1-n-groupoid and $r_2, r_3, ..., r_n : G \to G$ be 1-involutions. Then:*
1. $\omega(x_1, x_2, ..., x_n) = \omega(x_1, r_2^2(x_2), ..., r_n^2(x_n))$ for all $x_1^n \in G$.
2. $\omega(\omega(y, r_2(x_2), ..., r_n(x_n)), x_2, ..., x_n) = y$ for all $x_2^n, y \in G$.
3. (G, ω) is 1-cancellative .
4. For every $b, a_2^n \in G$, the equation $\omega(y, a_2, ..., a_n) = b$ has a unique solution.

Proof. The proof of the assertion 1 is contained in the proof of Proposition 5.2.7. The assertion 2 follows from Proposition 5.2.7. Let $a, b, x_2^n \in G$ and $\omega(a, x_2, ..., x_n) = \omega(b, x_2, ..., x_n)$. Then

$$a = \omega(\omega(a, x_2, ..., x_n), r_2(x_2), ..., r_n(x_n)) =$$
$$= \omega(\omega(b, x_2, ..., x_n), r_2(x_2), ..., r_n(x_n)) = b.$$

The assertion 3 is proved. We consider the equation $\omega(y, a_2, ..., a_n) = b$. Then from Proposition 10.2.7 we have $y = \omega(b, r_2(x_2), ..., r_n(x_n))$. Hence the equation $\omega(y, a_2, ..., a_n) = b$ has a unique solution. The proof is complete.

Corollary 5.2.11 *Let (G, ω) be an I_nP-n-groupoid in the large sense and $r_i : G \to G, i = \overline{1, n}$, are the involutions on G. Then (G, ω) is cancellative.*

Proof. The assertion follows from Proposition 5.2.10.

Academician M.M. Choban observed the following interesting fact.

Proposition 5.2.12 *Let (G, ω) be an I_nP-n-groupoid in the large sense and $r_i : G \to G, i = \overline{1, n}$, are the involutions on G. Then $x_i = r_i^{2(n-1)}(x_i)$, for every $i = \overline{1, n}$ and $n \geq 2$.*

Proof. It is sufficient to prove that $x_1 = r_1^{2(n-1)}(x_1)$ for any $x_1 \in G$. Fix $x_1, x_2, ..., x_n \in G$. From Proposition 10.2.10 we have

$$\omega(x_1, x_2, ..., x_n) = \omega(x_1, r_2^2(x_2), ..., r_n^2(x_n)) =$$
$$= \omega(r_1^2(x_1), r_2^2(x_2), r_3^4(x_3), ..., r_n^4(x_n)) = ...$$
$$= \omega(r_1^{2i}(x_1), ..., r_{i+1}^{2i}(x_{i+1}), r_{i+2}^{2(i+1)}(x_{i+2}), ..., r_n^{2(i+1)}(x_n)) =$$
$$... = \omega(r_1^{2(n-1)}(x_1), r_2^{2(n-1)}(x_2), ..., r_n^{2(n-1)}(x_n)), \text{ i.e.}$$

It is obvious that $\omega(x_1, x_2, ..., x_n) = \omega(x_1, r_2^{2m}(x_2), ..., r_n^{2m}(x_n))$ for any $m \geq 1$. Hence for $m = n - 1$, we have
$\omega(x_1, r_2^{2(n-1}(x_2), ..., r_n^{2(n-1)}(x_n)) = \omega(r_1^{2(n-1)}(x_1), r_2^{2(n-1)}(x_2), ..., r_n^{2(n-1)}(x_n))$.
Therefore $x_1 = r_1^{2(n-1)}(x_1)$ for any $x_1 \in G$ and $x_i = r_i^{2(n-1)}(x_i)$, for every $i = \overline{1, n}$ and $n \geq 2$. The proof is complete.

Proposition 5.2.13 *Let (G, ω) be an I_nP-n-groupoid in the large sense and $r_i : G \to G, i = \overline{2, n}$, are the involutions on G. If $e_1, e_2, ..., e_n \in G$, $e_i = r_i^{2m}(e_i)$, for all $i = \overline{2, n}$, then $x_i = r_i^{2m}(x_i)$, for every $x_i \in G$ and $n \geq 2$.*

Proof. From Proposition 5.2.10 it follows that
$\omega(x_1, x_2, ..., x_n) = \omega(x_1, r_2^{2m}(x_2), ..., r_n^{2m}(x_n))$. Fix $i = \overline{2, n}$. Then

$\omega(e_1, e_2, ..., e_{i-1}, x_i, e_{i+1}, ..., e_n) = \omega(e_1, e_2, ..., e_{i-1}, r_i^{2m}(x_i), e_{i+1}, ..., e_n)$. Hence, $x_i = r_i^{2m}(x_i)$, for every $x_i \in G$, $i = \overline{2, n}$ and $n \geq 2$. The proof is complete.

5.3 Topologies on algebras

We consider arbitrary topologies on universal algebras.
There are a lot of types of bounded topology. We fix $n \geq 2$ and $k \leq n$. Consider a mapping $\varphi : \{1, 2, ..., n\} \to \{1, 2, ..., n\}$. We will use Choban's bounded topology.

Definition 5.3.1 *Let (G, ω) be an n-groupoid and $L_1, L_2, ..., L_n$ be a family of subsets of G. Then:*
1. The sets $L_1, L_2, ..., L_n$ are k-α-associated with the mapping φ and denote $(L_1, L_2, ..., L_n)\alpha(k)\varphi$ if $L_i = L_j$ provided $\varphi(i) = \varphi(j)$ and $i \neq k$, $j \neq k$.
2. If $x_1, x_2, ..., x_n \in G$ and $(\{x_1\}, \{x_2\}, ..., \{x_n\})\alpha(k)\varphi$, then we put $(x_1, x_2, ..., x_n)\alpha(k)\varphi$.
3. We put $\Delta_{\varphi(k)}\omega(L_1, L_2, ..., L_n) = \{\omega(x_1, x_2, ..., x_n) : x_1 \in L_1, x_2 \in L_2, ..., x_n \in L_n$ and $(x_1, x_2, ..., x_n)\alpha(k)\varphi\}$.

Remark 5.3.2 *Let $L_1, L_2, ..., L_n$ be subsets of G, and $L_k^{'}=L_k$ and $L_i^{'}=\bigcap\{L_j : j \leq n, \varphi(j) = \varphi(i)\}$ for any $i \neq k$. Then $(L_1^{'}, L_2^{'}, ..., L_n^{'})\alpha(k)\varphi$ and $\Delta_{\varphi(k)}\omega(L_1^{'}, L_2^{'}, ..., L_n^{'})=\Delta_{\varphi(k)}\omega(L_1, L_2, ..., L_n)$.*

Definition 5.3.3 *Let $k \leq n$. An n-groupoid (G, ω) is called an $I_\varphi P_k$-n-groupoid if there exist the mappings $r_i : G \to G$, $i \in \{1, ..., k-1, k+1, ..., n\}$ such that $\omega(r_1(x_1), ..., r_{k-1}(x_{k-1}), \omega(x_1, ..., x_{k-1}, y, x_{k+1}, ..., x_n), r_{k+1}(x_{k+1}), ..., r_n(x_n)) = y$ provided $(x_1, ..., x_{k-1}, x_{k+1}, ..., x_n)\alpha(k)\varphi$ for all $x_1, ..., x_{k-1}, x_{k+1}, ..., x_n, y \in G$.*

We say that the mapping $r_i : G \to G$, $i \in \{1, ..., k-1, k+1, ..., n\}$ is called k-φ-involution.

If $\varphi(i) = \varphi(j)$ for all $i, j \leq n$, then $I_\varphi P_k$-n-groupoid is an $I_0 P_k$-n-groupoid.

Definition 5.3.4 *Let (G, ω) be an n-groupoid and λ be an infinite cardinal. A topology $\mathcal{T}$ on G is called:*
- a λ-k-φ-bounded topology if for every non-empty open set $U \in \mathcal{T}$ there exists a subset $K \subseteq G$ such that $|K| < \lambda$ and $\Delta_{\varphi(k)}\omega(K^{k-1}, U, K^{n-k}) = = G$.

- *a* λ-φ*-bounded topology if it is* λ-k-φ*-bounded topology for every* $k = \overline{1, n}$.

An ω_0-k-φ*-bounded topology is called a* k-φ*-totally bounded topology. The topology is said to be* φ*-totally bounded if it is a* k-φ*-totally bounded topology for every* $k = \overline{1, n}$.

Remark 5.3.5 *If in Definition 5.3.4 the mapping* φ *is one-to-one, then a topology* $\mathcal{T}$ *on* G *is called respectively: a* λ-k*-bounded topology, a* λ*-bounded topology, a* ω_0-k*-bounded topology, an* k*-totally bounded topology and totally bounded topology, for every* $k = \overline{1, n}$.

Proposition 5.3.6 *Let* $\varphi : \{1, 2, ..., n\} \to \{1, 2, ..., n\}$ *be a mapping,* (G, ω) *be an* n*-groupoid with the properties:*

1. The equation $\omega(a^{k-1}, x, a^{n-k}) = b$ *is solvable for every* $a, b \in G$.

2. For every $a, b \in G$ *there exist* $a_1, a_2, ..., a_n \in G$ *such that* $a_k = a$, $(a_1, a_2, ..., a_n)\alpha(k)\varphi$ *and* $\omega(a_1, a_2, ..., a_n) = b$.

Then the minimal compact T_1*-topology* $\mathcal{T} = \{\varnothing\} \cup \{G \setminus F : F$ *is a finite subset of* $G\}$ *is a* k-φ*-totally bounded topology on* G.

Proof. Let $U \in \mathcal{T}$ and $U \neq \varnothing$. Then the set $F = G \setminus U$ is finite. Fix $a \in U$. Then $h_a : G \to G$, where $h_a(x) = \omega(a^{k-1}, x, a^{n-k})$ for any $x \in G$ is a mapping of G onto G. Thus $F' = G \setminus h_a(U) \subseteq h_a(F)$ is a finite set. For any $x \in G$ there exist $y_1(x), y_2(x), ..., y_n(x) \in G$ such that $y_k(x) = a$, $(y_1(x), y_2(x), ..., y_n(x))\alpha(k)\varphi$ and $\omega(y_1(x), y_2(x), ..., y_n(x)) = x$. We put $\Phi = \{a\} \cup \{\{y_1(x), y_2(x), ..., y_n(x)\} : x \in F'\}$. The set Φ is finite. By construction, $\Delta_{\varphi(k)}\omega(\Phi^{k-1}, U, \Phi^{n-k}) = G$. The proof is complete.

Proposition 5.3.7 *Let* $\varphi : \{1, 2, ..., n\} \to \{1, 2, ..., n\}$ *be a mapping,* (G, ω) *be an* n*-groupoid with the properties:*

1. For every $a, b \in G$ *there exist* $a_1, a_2, ..., a_n \in G$ *such that* $a_k = a$, $(a_1, a_2, ..., a_n)\alpha(k)\varphi$ *and* $\omega(a_1, a_2, ..., a_n) = b$.

2. There exists $e \in G$ *such that* $G \setminus \omega(e^{k-1}, G, e^{n-k})$ *is a finite set (in particular,* $\omega(e^{k-1}, x, e^{n-k}) = x$ *for every* $x \in G$*).*

Then the minimal compact T_1*-topology* $\mathcal{T} = \{\varnothing\} \cup \{G \setminus F : F$ *is a finite subset of* $G\}$ *is a* k-φ*-totally bounded topology on* G.

Proof. Let $U \in \mathcal{T}$ and $U \neq \varnothing$. Then the set $F = G \setminus U$ is finite. Fix $a \in U$.

Consider the mapping $h_e : G \to G$, where $h_e(x) = \omega(e^{k-1}, x, e^{n-k})$ for any $x \in G$. The set $G \setminus h_e(G)$ is finite. Thus the set $F^{'} = G \setminus h_e(U) \subseteq (G \setminus h_e(G)) \bigcup h_e(F)$ is a finite set. For any $x \in F^{'}$ fix $\{y_1(x), y_2(x), ..., y_n(x)\} \subseteq G$ such that $y_k(x) = a$, $(y_1(x), y_2(x), ..., y_n(x))\alpha(k)\varphi$ and $\omega(y_1(x), y_2(x), ..., y_n(x)) = x$. Let $\Phi = \{e\} \cup$ $\cup\{\{y_1(x), y_2(x), ..., y_n(x)\} : x \in F^{'}\}$. The set Φ is finite. By construction, $\Delta_{\varphi(k)}\omega(\Phi^{k-1}, U, \Phi^{n-k}) = G$. The proof is complete.

Proposition 5.3.8 *Let $\varphi : \{1, 2, ..., n\} \to \{1, 2, ..., n\}$ be a mapping, (G, ω) be an infinite I_nP_k-n-groupoid, $B \subseteq G$, m be an infinite cardinal and $\Delta_{\varphi(k)}\omega(K^{k-1}, G\setminus B, K^{n-k}) \neq G$ for every subset K of cardinality $|K| < m$. Then the set B is dense in every m-k-φ-bounded topology $\mathcal{T}$ on G.*

Proof. Suppose that $\mathcal{T}$ is an m-k-φ-bounded topology on G and $U = G \setminus cl_G B \neq \varnothing$. Then $U \in \mathcal{T}$ and $U \subseteq G \setminus B$. By assumption there exists a subset K of G such that $\Delta_{\varphi(k)}\omega(K^{k-1}, U, K^{n-k}) = G$ and $|K| < m$. Since $U \subseteq G \setminus B$, we have $G \supseteq \Delta_{\varphi(k)}\omega(K^{k-1}, G \setminus B, K^{n-k}) \supseteq \Delta_{\varphi(k)}\omega(K^{k-1}, U, K^{n-k}) = G$, a contradiction. The proof is complete.

5.4 Decomposition of I_nP_k-n-groupoids

We fix $n \geq 2$ and $k \leq n$. Consider a mapping $\varphi : \{1, 2, ..., n\} \to \{1, 2, ...n\}$.

Lemma 5.4.1 *Let G be an infinite I_nP_k-n-groupoid, $r_1, ..., r_{k-1}, r_{k+1}, ..., r_n : G \to G$ be k-involutions, L and M be subsets of G and $|L \cup M| < |G|$. Then there exists an element $a \in G$ such that $\omega(L^{k-1}, a, L^{n-k}) \cap M = \varnothing$ and $\Delta_{\varphi(k)}\omega(L^{k-1}, a, L^{n-k}) \cap M = \varnothing$.*

Proof. Let $H = \{\omega(r_1(y_1), ..., r_{k-1}(y_{k-1}), x, r_{k+1}(y_{k+1}), ..., r_n(y_n)) :$ $x \in M, y_1, ..., y_{k-1}, y_{k+1}, ..., y_n \in L\}$. Thus $|H| < |G|$ and there exists an element $a \in G \setminus H$. Suppose that $\omega(L^{k-1}, a, L^{n-k}) \cap M \neq \varnothing$. Fix $\omega(L^{k-1}, a, L^{n-k}) \cap M$. Then $x = \omega(y_1, ..., y_{k-1}, a, y_{k+1}, ..., y_n)$ for some $y_1, ..., y_{k-1}, y_{k+1}, ..., y_n \in L$. Hence

$$a = \omega(r_1(y_1), ..., r_{k-1}(y_{k-1}), \omega(y_1^{k-1}, a, y_{k+1}^n), r_{k+1}(y_{k+1}), ..., r_n(y_n)) =$$
$$\omega(r_1(y_1), ..., r_{k-1}(y_{k-1}), x, r_{k+1}(y_{k+1}), ..., r_n(y_n)) \in$$
$$\omega(r_1(y_1), ..., r_{k-1}(y_{k-1}), M, r_{k+1}(y_{k+1}), ..., r_n(y_n)) \subseteq H,$$

a contradiction.

By construction, $\Delta_{\varphi(k)}\omega(L^{k-1}, M, L^{n-k}) \subseteq \omega(L^{k-1}, M, L^{n-k})$. Hence, $\Delta_{\varphi(k)}\omega(L^{k-1}, a, L^{n-k}) \cap M = \varnothing$. The proof is complete.

Theorem 5.4.2 *Let G be an infinite I_nP_k-n-groupoid, $\mathcal{L}$ be a a non-empty family of non-empty subsets of G, $|\mathcal{L}| \leq |G|$ and for every set A and mapping $\Psi : A \to \mathcal{L}$ we have $| \cup\{\Psi(\alpha) : \alpha \in A\} |<| G |$ provided $| A |<| G |$. Then there exists a family $\{B_\mu : \mu \in M\}$ of non-empty subsets of G such that:*

1. $| M |=| G |$.

2. $B_\mu \cap B_\eta = \varnothing$ for all $\alpha, \beta \in M$ and $\alpha \neq \beta$.

3. $G = \cup\{B_\mu : \mu \in M\}$.

4. $\omega(K^{k-1}, G \setminus B_\mu, K^{n-k}) \neq G$ for all $\mu \in M$ and $K \in \mathcal{L}$.

5. $\Delta_{\varphi(k)}\omega(K^{k-1}, G \setminus B_\mu, K^{n-k}) \neq G$ for all $\mu \in M$ and $K \in \mathcal{L}$.

Proof. Consider on G some k-involutions, $r_1, ..., r_{k-1}, r_{k+1}, ..., r_n : G \to G$. Let $\tau = |G|$. Denote by $|\alpha|$ the cardinality of the ordinal number α. We put $\Omega_\tau = \{\alpha : 1 \leq |\alpha| < \tau\}$. If $K \subseteq G$, then $K_i^{-1} = \{r_i(x_i) : x_i \in K\}, i = 1, ..., k-1, k+1, ..., n$, and $K^{-1} = \cup\{K_i^{-1} : i = 1, 2, ..., k-1, k+1, ...n\}$. Let $\mathcal{L}_\infty = \{K^{-1} : K \in \mathcal{L}\} \cup \mathcal{L}$. It is clear that $|\mathcal{L}_1| \leq \tau$. Moreover, if A is a set, $|A| < \tau$ and $\Psi : A \to \mathcal{L}_1$ is a mapping, then $| \cup \{\Psi(\alpha) : \alpha \in A\}| < \tau$. Fix a set M of the cardinality τ. Since $|\Omega_\tau| = |M \times \mathcal{L}_1| = \tau$ then there exists a bijection $h : \Omega_\tau \to M \times \mathcal{L}_1$. If $\alpha \in \Omega_\tau$, then we consider that $h(\alpha) = (\mu_\alpha, K_\alpha) \in M \times \mathcal{L}_1$. If $\mu \in M$, then we put $A_\mu = h^{-1}(\{\mu\} \times \mathcal{L}_1)$. It is obvious that
$A_\mu = \{\alpha \in \Omega_\tau : \mu_\alpha = \mu\}$ and $\{K_\alpha : \alpha \in A_\mu\} = \mathcal{L}_1$. Now we affirm that there exists a transfinite sequence $\{a_\alpha : \alpha \in \Omega_\tau\} \subseteq G$ such that
$\omega(K_\alpha^{k-1}, a_\alpha, K_\alpha^{n-k}) \cap \omega(K_\beta^{k-1}, a_\beta, K_\beta^{n-k}) = \varnothing$ for all $\alpha, \beta \in \Omega_\tau$ and $\alpha \neq \beta$. We fix $a_1 \in G$. Let $1 < \beta, \beta \in \Omega_\tau$ and the elements $\{a_\alpha : \alpha < \beta\}$ are constructed. We put now $H_\beta = \cup\{\omega(K_\alpha^{k-1}, a_\alpha, K_\alpha^{n-k}) : \alpha < \beta\}$.
Since $|\alpha \in \Omega_\tau : \alpha < \beta| \leq |\beta| < |G|$, then $|H_\beta| < |G|$. From Lema 10.4.1 it follows that there exists $a_\beta \in G$ such that $\omega(K_\beta^{k-1}, a_\beta, K_\beta^{n-k}) \cap H_\beta = \varnothing$. By the transfinite induction if follows that the set $\{a_\alpha : \alpha \in \Omega_\tau\}$ is constructed. We put $P_\mu = \cup\{\omega(K_\alpha^{k-1}, a_\alpha, K_\alpha^{n-k}) : \alpha \in A_\mu\}$ for every $\mu \in H$. Fix $\mu, \eta \in M$ and $\mu \neq \eta$. Then $A_\mu \cap A_\eta = \varnothing$. Since $\omega(K_\alpha^{k-1}, a_\alpha, K_\alpha^{n-k}) \cap \omega(K_\beta^{k-1}, a_\beta, K_\beta^{n-k}) = \varnothing$ for all $\alpha \in A_\alpha$ and $\beta \in A_\eta$, then $P_\mu \cap P_\eta = \varnothing$. Fix $\mu \in M$ and $K \in \mathcal{L}$. Then $K^{-1} \in \mathcal{L}_1$ and $(\mu, K^{-1}) = (\mu_\alpha, K_\alpha)$ for some $\alpha \in A_\mu$. Suppose that $\omega(K^{k-1}, G \setminus P_\mu, K^{n-k}) = G$. Then $a_\alpha \in \omega(K^{k-1}, G \setminus P_\mu, K^{n-k})$, i.e. $a_\alpha = \omega(y_1^{k-1}, x, y_{k+1}^n)$ for some $x \in G \setminus P_\mu$ and $y_1, ..., y_{k-1}, y_{k+1}, ..., y_n \in K$. By construction, we have $r_1(y_1), ..., r_{k-1}(y_{k-1}), r_{k+1}(y_{k+1}), ..., r_n(x_n) \in K_\alpha$ and

$$\omega((r_1(y_1), ..., r_{k-1}(y_{k-1}), a_\alpha, r_{k+1}(y_{k+1}), ..., r_n(x_n)) \in$$
$$\in \omega(K_\alpha^{k-1}, a_\alpha, K_\alpha^{n-k}) \subseteq P_\mu.$$

By assumption, we have that

$$\omega(r_1(y_1), ..., r_{k-1}(y_{k-1}), a_\alpha, r_{k+1}(y_{k+1}), ..., r_n(x_n)) =$$
$$= \omega(r_1(y_1), ..., r_{k-1}(y_{k-1}), \omega(y_1^{k-1}, x, y_{k+1}^{n-k}), r_{k+1}(y_{k+1}), .., r_n(y_n)) =$$
$$= x \in G \setminus P_\mu,$$

a contradiction. Hence $\omega(K^{k-1}, G \setminus P_\mu, K^{n-k}) \neq G$ for all $\mu \in M$ and $K \in \mathcal{L}$. Now we fix $\mu_0 \in M$. We put $B_\mu = P_\mu$ for all $\mu \in M \setminus \{\mu_0\}$ and $B_{\mu_0} = G \setminus \cup\{P_\mu : \mu \in M \setminus \{\mu_0\}\}$. By construction, we have $P_\mu \subseteq B_\mu$ for all $\mu \in M$ and $G = \cup\{B_\mu : \mu \in H\}$. If $\mu \in M$, then $G \setminus B_\mu \subseteq G \setminus P_\mu$ and $\omega(K^{k-1}, G \setminus B_\mu, K^{n-k}) \neq G$ for all $K \in \mathcal{L}$. The proof is complete.

Theorem 5.4.3 *Let (G, ω) be an infinite I_nP_k-n-groupoid, $\tau = |G|$, m be an infinite cardinal, $\tau = \sum\{\tau^q : q < m\}$ and either $m < \tau$, or τ be a regular cardinal. If $\mathcal{L}_m = \{K \subseteq G : |K| < m\}$, then there exists a family $\{B_\mu : \mu \in M\}$ of non-empty subsets of G such that:*
1. $|M| = \tau$.
2. $B_\mu \cap B_\eta = \varnothing$ for all $\mu, \eta \in M$ and $\mu \neq \eta$.
3. $G = \cup\{B_\mu : \mu \in M\}$.
4. $\omega(K^{k-1}, G \setminus B_\mu, K^{n-k}) \neq G$ for all $\mu \in M$ and $K \in \mathcal{L}_m$.
5. $\Delta_{\varphi(k)}\omega(K^{k-1}, G \setminus B_\mu, K^{n-k}) \neq G$ for all $\mu \in M$ and $K \in \mathcal{L}_m$
6. The sets B_μ are dense in every m-k-φ-bounded topology on G.
7. Relative to every m-k-φ-bounded topology G is super-resolvable.
8. The sets B_μ are dense in every m-k-bounded topology on G.
9. Relative to every m-k-bounded topology G is super-resolvable.

Proof. Since $\tau = \sum\{\tau^q : q < m\}$, we have $m \leq \tau$. Let A be a set, $|A| < \tau, \Psi : A \to L_m$ be a mapping and $H = \cup\{\Psi(\alpha) : \alpha \in A\}$. If $m < \tau$, then $|H| \leq \omega(m, ...m, |A|, m, ..., m) = \omega(m^{k-1}, |A|, m^{n-k}) < \tau$. If $m = \tau$ and $|H| = \tau$, then $cf(\tau) \leq |A| < \tau$ and the cardinal τ is not regular. Hence $|H| < \tau$. Theorem 5.4.2 and Proposition 5.3.8 complete the proof.

Corallary 5.4.4 *Let G be an infinite I_nP_k-n-groupoid. Then there exists a family $\{B_\mu : \mu \in M\}$ of non-empty subsets of G such that:*

1. $|M| = |G|$.
2. $B_\mu \cap B_\eta = \varnothing$ *for all* $\mu, \eta \in M$ *and* $\mu \neq \eta$.
3. $G = \cup\{B_\mu : \mu \in M\}$.
4. $\omega(K^{k-1}, G \setminus B_\mu, K^{n-k}) \neq G$ *for all* $\mu \in M$ *and every finte subset* K *of* G.
5. $\Delta_{\varphi(k)}\omega(K^{k-1}, G \setminus B_\mu, K^{n-k}) \neq G$ *for all* $\mu \in M$ *and every finte subset* K *of* G.
6. The sets $\{B_\mu : \mu \in M\}$ *are dense in every* k-φ*-totally bounded topology on* G.
7. Relative to every k-φ*-totally bounded topology* G *is super-resolvable.*
8. The sets $\{B_\mu : \mu \in M\}$ *are dense in every* k*-totally bounded topology on* G.
9. Relative to every k*-totally bounded topology* G *is super-resolvable.*

Corollary 5.4.5 *Let* G *be an infinite* I_nP_k*-n-groupoid,* $\tau = |G|$, m *be an infinite cardinal and* $\tau^m = \tau$. *Then there exists a family* $\{B_\mu : \mu \in M\}$ *of non-empty subsets of* G *such that:*
1. $|M| = |G|$.
2. $B_\mu \cap B_\eta = \varnothing$ *for all* $\mu, \eta \in M$ *and* $\mu \neq \eta$.
3. $G = \cup\{B_\mu : \mu \in M\}$.
4. If $\mu \in M$, $K \subseteq G$ *and* $|K| < m$ *then* $\omega(K^{k-1}, G \setminus B_\mu, K^{n-k}) \neq G$.
5. If $\mu \in M$, $K \subseteq G$ *and* $|K| < m$ *then* $\Delta_{\varphi(k)}\omega(K^{k-1}, G \setminus B_\mu, K^{n-k}) \neq G$.
6. The sets $\{B_\mu : \mu \in M\}$ *are dense in every* m^+-k-φ*-bounded topology on* G.
7. Relative to every m^+-k-φ*-bounded topology* G *is super-resolvable.*
8. The sets $\{B_\mu : \mu \in M\}$ *are dense in every* m^+-k*-bounded topology on* G.
9. Relative to every m^+-k*-bounded topology* G *is super-resolvable.*

5.5 Decomposition of I_nP-n-groupoids

We fix $n \geq 2$ and $k \leq n$. Consider a mapping $\varphi : \{1, 2, ..., n\} \to \{1, 2, ...n\}$.

Lemma 5.5.1 *Let* G *be an infinite* I_nP*-gruopoid,* $r_1, ..., r_n : G \to G$ *be involutions,* L *and* M *be subsets of* G *and* $|L \cup M| < |G|$. *Then there exists an element* $a \in G$ *such that:*
1. $\bigcup_{k=1}^{n} \omega(L^{k-1}, a, L^{n-k}) \cap M = \varnothing$, *where* $\bigcup_{k=1}^{n} \omega(L^{k-1}, a, L^{n-k}) = = \omega(a, L^{n-1}) \cup \omega(L^1, a, L^{n-2}) \cup ... \cup \omega(L^{n-1}, a)$.
2. $\bigcup_{k=1}^{n} \Delta_{\varphi(k)}\omega(L^{k-1}, a, L^{n-k}) \cap M = \varnothing$, *where* $\bigcup_{k=1}^{n} \Delta_{\varphi(k)}\omega(L^{k-1}, a, L^{n-k}) =$

$= \Delta_{\varphi(k)}\omega(a, L^{n-1}) \cup \Delta_{\varphi(k)}\omega(L^1, a, L^{n-2}) \cup ... \cup \Delta_{\varphi(k)}\omega(L^{n-1}, a)$.

Proof. Let $H = \{\omega(x, r_2(y_2), ..., r_n(y_n)) : x \in M, y_2, ...y_n \in L\} \cup \cup \{\omega(r_1(y_1), x, r_3(y_3), ..., r_n(y_n)) : x \in M, y_1, y_3, ..., y_n \in L\} \cup ... \cup \{\omega(r_1(y_1), ..., r_{n-1}(y_{n-1}), x) : x \in M, y_1, ...y_{n-1} \in L\}$. Since $|H| < |G|$, then there exists an element $a \in G \setminus H$. Let $\omega(a, L, ...L) \cap M \neq \varnothing$. Fix $x \in \omega(a, L, ...L) \cap M$. Then $x = \omega(a, y_2, ..., y_n)$ for some $y_2, ..., y_n \in L$. Hence $a = \omega(\omega(a, y_2, ..., y_n), r_2(y_2), ..., r_n(y_n)) = \omega(x, r_2(y_2), ..., r_n(y_n)) \in \omega(M, r_2(y_2), ..., r_n(y_n)) \leq H$, a contradiction. In similar way we prove that $\omega(L^{k-1}, a, L^{n-k}) \cap M$ for all $k = \overline{1, n}$.
Hence $\bigcup_{k=1}^{n} \omega(L^{k-1}, a, L^{n-k}) \cap M = \varnothing$. By construction, $\Delta_{\varphi(k)}\omega(L^{k-1}, a, L^{n-k}) \subseteq \omega(L^{k-1}, a, L^{n-k})$. Hence, $\bigcup_{k=1}^{n} \Delta_{\varphi(k)}\omega(L^{k-1}, a, L^{n-k}) \cap M = \varnothing$. The proof is complete.

Theorem 5.5.2 *Let G be an infinite I_nP-n-groupoid, $\mathcal{L}$ be a non-empty family of non-empty subsets of G, $|\mathcal{L}| \leq |G|$ and for every set A and mapping $\Psi : A \to \mathcal{L}$ we have $| \cup\{\Psi(\alpha) : \alpha \in A\} |<| G |$ provided $| A |<| G |$. Then there exists a family $\{B_\mu : \mu \in M\}$ of non-empty subsets of G such that:*
1. $| M |=| G |$.
2. $B_\mu \cap B_\eta = \varnothing$ for all $\alpha, \beta \in M$ and $\alpha \neq \beta$.
3. $G = \cup\{B_\mu : \mu \in M\}$.
4. $\bigcup_{k=1}^{n} \omega(K^{k-1}, G \setminus B_\mu, K^{n-k}) \neq G$ for all $\mu \in M$ and $K \in \mathcal{L}$
5. $\bigcup_{k=1}^{n} \Delta_{\varphi(k)}\omega(K^{k-1}, G \setminus B_\mu, K^{n-k}) \neq G$ for all $\mu \in M$ and $K \in \mathcal{L}$.

Proof. Consider on G involutions, $r_1, ..., r_n : G \to G$. Let $\tau = |G|$. Denote by $|\alpha|$ the cardinality of the ordinal number α. We put $\Omega_\tau = \{\alpha : 1 \leq |\alpha| < \tau\}$. If $K \subseteq G$, then $K_i^{-1} = \{r_i(x_i) : i = \overline{1, n}, x_i \in K\}$. We put $K^{-1} = \cup K_i^{-1}$ and $\mathcal{L}_1 = \{K^{-1} : K \in \mathcal{L}\} \cup \mathcal{L}$. It is clear that $|\mathcal{L}_1| \leq \tau$. Moreover, if A is a set, $|A| < \tau$ and $\Psi : A \to \mathcal{L}_1$ is a mapping, then $| \cup \{\Psi(\alpha) : \alpha \in A\}| < \tau$. Fix a set M of the cardinality τ. Since $|\Omega_\tau| = |M \times \mathcal{L}_1| = \tau$, then there exists a bijection $h : \Omega_\tau \to M \times \mathcal{L}_1$. Let $A_\mu = h^{-1}(\{\mu\} \times \mathcal{L}_1) = \alpha \in \Omega_\tau : \mu_\alpha = \mu\}$. If $\alpha \in \Omega_\tau$, then we consider that $h(\alpha) = (\mu_\alpha, K_\alpha) \in M \times \mathcal{L}_1$. It is obvious that
$A_\mu = \{\alpha \in \Omega_\tau : \mu_\alpha = \mu\}$ and $\{K_\alpha : \alpha \in A_\mu\} = \mathcal{L}_1$. As in the proof of Theorem 5.4.2 from Lemma 5.5.1 it follows that there exists a transfinite sequence $\{a_\alpha \in G : \alpha \in \Omega_\tau\}$ such that
$(\bigcup_{k=1}^{n} \omega(K_\alpha^{k-1}, a_\alpha, K_\alpha^{n-k})) \cap (\bigcup_{k=1}^{n} \omega(K_\beta^{k-1}, a_\beta, K_\beta^{n-k})) = \varnothing$ for all $\alpha, \beta \in \Omega_\tau$ and

$\alpha \neq \beta$. Now we put $P_\mu = \cup\{\bigcup_{k=1}^{n} \omega(K_\alpha^{k-1}, a_\alpha, K_\alpha^{n-k}) : \alpha \in A_\mu\}$ for every $\mu \in M$. If $P_\mu^k = \bigcup_{k=1}^{n} \omega\{(K_\alpha^{k-1}, a_\alpha, K_\alpha^{n-k}) : \alpha \in A_\mu\}$ for all $k = \overline{1,n}$, then $P_\mu = \bigcup_{k=1}^{n} P_\mu^k$ and $\omega(K^{k-1}, G \setminus P_\mu^k, K^{n-k}) \neq G$ for every $K \in \mathcal{L}$. Suppose that $K \in \mathcal{L}$, $\mu \in M$ and $G = \bigcup_{k=1}^{n} \omega(K^{k-1}, G \setminus P_\mu^k, K^{n-k})$. For some $\alpha \in A_\mu$ we have $K_\alpha = \bigcup_{i=1}^{n} K_i^{-1} = K^{-1}$. Then $\bigcup_{k=1}^{n} \omega(K_\alpha^{k-1}, a_\alpha, K_\alpha^{n-k}) \subseteq P_\mu$ and $a_\alpha \in G$. Suppose that $a_\alpha \in \omega(K^{k-1}, G \setminus P_\mu^k, K^{n-k})$. Then $a_\alpha = \omega(y_1, ..., y_{k-1}, x, y_{k+1}, ..., y_n)$ for some $y_1, ..., y_{k-1}, y_{k+1}, ..., y_n \in K$ and $x \in G \setminus P_\mu$. Therefore

$$\omega(r_1(y_1), ..., r_{k-1}(y_{k-1}), a_\alpha, r_{k+1}(y_{k+1}), ..., r_n(y_n)) =$$
$$= \omega(r_1(y_1), ..., r_{k-1}(y_{k-1}), \omega(y_1^{k-1}, x, y_{k+1}^{n-k}), r_{k+1}(y_{k+1}), .., r_n(y_n)) =$$
$$= x \in G \setminus P_\mu.$$

Since $r_i(y_i \in K_\alpha)$, $i = \overline{1,n}$, we have

$$x = \omega(r_1(y_1), ..., r_{k-1}(y_{k-1}), a_\alpha, r_{k+1}(y_{k+1}), ..., r_n(y_n)) \in \omega(K_\alpha^{k-1}, a_\alpha, K_\alpha^{n-k}) \subseteq P_\mu,$$

a contradiction. Hence $\bigcup_{k=1}^{n} \omega(K^{k-1}, G \setminus P_\mu, K^{n-k}) \neq G$ for all $\mu \in M$ and $K \in \mathcal{L}$. Now we fix $\mu_0 \in M$. We put $B_\mu = P_\mu$ for all $\mu \in M \setminus \{\mu_0\}$ and $B_{\mu_0} = G \setminus \cup\{P_\mu : \mu \in M \setminus \{\mu_0\}\}$. By construction, we have $P_\mu \subseteq B_\mu$ for all $\mu \in M$ and $G = \cup\{B_\mu : \mu \in H\}$. If $\mu \in M$, then $G \setminus B_\mu \subseteq G \setminus P_\mu$ and $\bigcup_{k-1}^{n} \omega(K^{k-1}, G \setminus B_\mu, K^{n-k}) \neq G$ for all $K \in \mathcal{L}$. The proof is complete.

Theorem 5.5.3 *Let (G) be an infinite I_nP-n-groupoid, $\tau = |G|$, m be an infinite cardinal, $\tau = \sum\{\tau^q : q < m\}$ and either $m < \tau$, or τ be a regular cardinal. If $\mathcal{L}_m = \{K \subseteq G : |K| < m\}$, then there exists a family $\{B_\mu : \mu \in M\}$ of non-empty subsets of G such that:*

1. $|M| = \tau$.

2. $B_\mu \cap B_\eta = \varnothing$ for all $\mu, \eta \in M$ and $\mu \neq \eta$.

3. $G = \cup\{B_\mu : \mu \in M\}$.

4. $\bigcup_{k=1}^{n} \omega(K^{k-1}, G \setminus B_\mu, K^{n-k}) \neq G$ for all $\mu \in M$ and $K \in \mathcal{L}_m$.

5. $\bigcup_{k=1}^{n} \Delta_{\varphi(k)}\omega(K^{k-1}, G \setminus B_\mu, K^{n-k}) \neq G$ for all $\mu \in M$ and $K \in \mathcal{L}_m$.

6. The sets B_μ are dense in every m-φ-bounded topology on G.

7. Relative to every m-φ-bounded topology T on G the space (G, T) is super-resolvable.

8. The sets B_μ are dense in every m-bounded topology on G.

9. Relative to every m-bounded topology T on G the space (G, T) is super-resolvable.

Proof. Is similar to the proof of Theorem 5.4.3.

Corollary 5.5.4 *Let G be an infinite I_nP-n-groupoid. Then there exists a family $\{B_\mu : \mu \in M\}$ of non-empty subsets of G such that:*
1. $|M| = |G|$.
2. $B_\mu \cap B_\eta = \varnothing$ for all $\mu, \eta \in M$ and $\mu \neq \eta$.
3. $G = \cup\{B_\mu : \mu \in M\}$.
4. $\bigcup_{k=1}^{n} \omega(K^{k-1}, G \setminus B_\mu, K^{n-k}) \neq G$ for all $\mu \in M$ and every finte subset K of G.
5. $\bigcup_{k=1}^{n} \Delta_{\varphi(k)}\omega(K^{k-1}, G \setminus B_\mu, K^{n-k}) \neq G$ for all $\mu \in M$ and every finte subset K of G.
6. The sets $\{B_\mu : \mu \in M\}$ are dense in every φ-totally bounded topology on G.
7. Relative to every φ-totally bounded topology G is super-resolvable.
8. The sets $\{B_\mu : \mu \in M\}$ are dense in every totally bounded topology on G.
9. Relative to every totally bounded topology G is super-resolvable.

Corollary 5.5.5 *Let G be an infinite I_nP-n-groupoid, $\tau = |G|$, m be an infinite cardinal and $\tau^m = \tau$. Then there exists a family $\{B_\mu : \mu \in M\}$ of non-empty subsets of G such that:*
1. $|M| = |G|$.
2. $B_\mu \cap B_\eta = \varnothing$ for all $\mu, \eta \in M$ and $\mu \neq \eta$.
3. $G = \cup\{B_\mu : \mu \in M\}$.
4. If $\mu \in M$, $K \subseteq G$ and $|K| \leq m$ then $\bigcup_{k=1}^{n} \omega(K^{k-1}, G \setminus B_\mu, K^{n-k}) \neq G$.
5. If $\mu \in M$, $K \subseteq G$ and $|K| \leq m$ then $\bigcup_{k=1}^{n} \Delta_{\varphi(k)}\omega(K^{k-1}, G \setminus B_\mu, K^{n-k}) \neq G$.
6. The sets $\{B_\mu : \mu \in M\}$ are dense in every m^+-k-φ-bounded topology on G.
7. Relative to every m^+-k-φ-bounded topology G is super-resolvable.
8. The sets $\{B_\mu : \mu \in M\}$ are dense in every m^+-k-bounded topology on G.
9. Relative to every m^+-k-bounded topology G is super-resolvable.

5.6. Conclusions for Chapter 5

In this Chapter we elaborated general methods of decomposition of universal algebras with invertibility properties. The scientific innovation of this Chapter is determined by the following:

1). there have been introduced the concepts of: I_nP_k-n-groupoid, λ-k-φ-bounded topology, k-φ-total bounded topology, bounded topology of Choban.

2). there has been elaborated a general theory of decomposition of topological groupoids with invertibility properties.

3). the implementation of new concepts and general theory of decomposition allowed:

- to generalize some results and constructions proposed, in the case of totally bounded topological groups, by V.I. Malykhin, W.W. Comfort, S. Van Mill [37], I.V.Protasov [180] and M.M. Choban, L.L. Chiriac [64].
- to obtain that the result holds for the some classes of n-topological quasigroups and n-loops with invertibility properties.

The mathematically developed device facilitates the research of various classes of groupoid and n-groupoids with invertibility properties.

The methodology proposed for research in this Chapter can be used:

- to examine the decomposition of various classes of n-gruopoids with invertibility properties which are not included in the class I_nP_k - n-groupoids;
- to investigate the topologo-algebraic properties in decomposition classes of algebras considered.
- to investigate a homogeneous basis of fuzzy groupoid with division.

6. ON FUZZY ALGEBRAS

Certain problems about fuzzy universal algebras were considered in [15, 42, 52, 58, 101]. In papers [169, 182] the problem of the homomorphism for fuzzy algebras was formulated and solved for some homomorphisms of the fuzzy groupoids, groups and rings. This section gives a general solution of the homomorphism problem for fuzzy universal algebras. The results of this section were published in [61].

6.1 The Lattice of L-fuzzy Algebras

Fix a complete lattice L.

If $B \subseteq L$, then $\wedge B$ (respectively, $\vee B$) denotes the infimum (respectively, supremum) of B. Let $0 = \wedge L$, $1 = \vee L$ and $0 \neq 1$.

Let A be a non-empty set. A mapping $\mu : A \to L$ is called an L-fuzzy subset of A. The lattice L^A is called the lattice of all L-fuzzy subsets of A.

Let $f : A \to B$ be a mapping of a set A into a set B and μ be an L-fuzzy subset of A.The image of μ under f, denoted by $f(\mu)$, is an L-fuzzy subset of B defined by $f(\mu)(y) = \vee f^{-1}(y)$ if $y \in f(A)$, and $f(\mu)(y) = 0$ if $y \notin f(A)$.

Fix a sequence $\{E_n : n \in N = \{0, 1, 2, \dots\}\}$ of pairwise disjoint sets. The sum $E = \cup\{E_n : n \in N\}$ is called *a signature* or a set of *fundamental operations.*

An E-algebra or a universal algebra of the signature E is a pair (G, e_G) for which:

– G is a non-empty set;

– $e_G = \{e_{nG} : E_n \times G^n : n \in N\}$ is a sequence of mappings.

The set G is called the *support* of the E-algebra and the mappings e_G are called *the algebraical structure* on G.

Let G be an E-algebra. If $\omega \in E_0$, then the element $\omega_G = e_{0G}(\{\omega\} \times G^0)$ is called a constant of G. If $n \geqslant 1$, $\omega \in E_n$ and $x_1, \dots, x_n \in G$, then we put $\omega(x_1, \dots, x_n) = e_{nG}(\omega, x_1, \dots, x_n)$.

A mapping $f : A \to B$ of an E-algebra A into an E-algebra B is said to be *a homomorphism* if:

1. $f(\omega_A) = \omega_B$ for every $\omega \in E_0$;

2. $f(\omega(x_1, \dots, x_n)) = \omega(f(x_1), \dots, f(x_n))$ for every $n \geqslant 1$, $\omega \in E_n$ and $x_1, \dots, x_n \in A$.

A triple (G, e_G, μ) is called *an L-fuzzy E-algebra* if the following conditions hold:

(A) (G, e_G) is an E-algebra;

(F) $\mu : G \to L$ is an L-fuzzy subset of G;

(AF) the set $\{x \in G : \mu(x) \geqslant l\}$ is empty or is an E-subalgebra of G for every $l \in L$.

Theorem 6.1.1. ([101], Proposition 2.1). *Let (A, e_A) be an E-algebra and $\mu : A \to L$ be a mapping. The triple (A, e_A, μ) is an L-fuzzy E-algebra if and only if the following assertions are true:*

(AF1). *If $\omega \in E_0$, then $\mu(\omega_A) = \vee\mu(A)$;*

(AF2). *If $n \geqslant 1$, $\omega \in E_n$ and $x_1, \dots, x_n \in A$, then $\mu(\omega(x_1, \dots, , x_n)) \geqslant$ $\geqslant \mu(x_1) \wedge \dots \wedge \mu(x_n)$.*

Proof. The assertions (AF1) and (AF2) are equivalent with the condition that the set $\{x \in A : \mu(x) \geqslant l\}$ is non-empty or it is an E-subalgebra of the E-algebra A. Let A be an E-algebra. We put $LF(A) = \{\mu \in L^A : (A, e_A, \mu) \text{ is an } L - \text{fuzzy} E - \text{algebra}\}$. The proof is complete.

Lemma 6.1.2. *Let $H \subseteq LF(A)$ be a non-empty set, where A is an E-algebra, and $\lambda(x) = \wedge\{\mu(x) : \mu \in H\}$ for all $x \in A$. Then $\lambda \in LF(A)$.* **Proof.** If $l \in L$, then $\{x \in A : \lambda(x) \geqslant l\} = \cap\{\{x \in A : \mu(x) \geqslant l\} : \mu \in H\}$. The proof is complete.

Corollary 6.1.3. *Let A be an E-algebra. Then $LF(A)$ is a complete lattice of L-fuzzy E-subalgebras of the E-algebra A.*

Let B be a subalgebra of an E-algebra $B, m, n \in L$ and $m \leqslant n$. We put $(m, n)_B(x) = n$ if $x \in B$, and $(m, n)_B(x) = m$ if $x \notin B$. Let $n_B(x) = (0, n)_B(x)$ for every $x \in A$. Then $(m, n)_B \in LF(A)$ and $n_B \in LF(A)$. By construction, $1_A = \vee LF(A)$ and $0_A = \wedge LF(A)$. It is obvious that 0_A is the minimal element in L^A and 1_A is the maximal element in L^A.

Lemma 6.1.4. *The mapping $i : L \to LF(A)$, where $i(l) = l_A$ for every $l \in L$, is an embedding of the lattice L in the lattice $LF(A)$.*

By virtue of the Corollary 6.1.3, the lattice $LF(A)$ borrowed many "bad" properties from the lattice L.

Remark 6.1.5. *The concept of fuzzy group was introduced by A. Rosenfeld [182]. Fuzzy groupoids, rings, linear spaces and modules have been studied by G.V. Negoiţă and D.A. Ralescu [171], G. Gerla and K. Tortora [101], S. Nanda [169].Every L-fuzzy algebra is an L-algebraic system [42]. Further results on Cartesian products and on lattices $LF(A)$ can be found in [15, 52, 58, 167, 171]. Concrete general problems about fuzzy E-algebras were raised in ([171], p. 59–63).*

6.2 The Fuzzy Homomorphisms

A homomorphism $f : A \to B$ of an L-fuzzy E-algebra (A, e_A, μ) into an L-fuzzy E-algebra (B, e_B, η) is called *a fuzzy homomorphism* if $\eta(f(x)) \geqslant \mu(x)$ for every $x \in A$.

Theorem 6.2.1. (the fuzzy homomorphism theorem). *Let $f : A \to B$ be a homomorphism of an L-fuzzy E-algebra A onto an E-algebra B. Then there exists a unique mapping $\lambda(f, \mu) : B \to L$ such that:*

1. *$(B, e_B, \lambda(f, \mu))$ is an L-fuzzy E-algebra.*
2. *If $g : B \to C$ is a homomorphism of the E-algebra into an L-fuzzy E-algebra (C, e_C, η) and the composition $g \cdot f : A \to C$ is a fuzzy homomorphism of A into C, then g is a fuzzy homomorphism of $(B, e_B, \lambda(f, \mu))$ into (C, e_C, η).*
3. *$\lambda(f, \mu)(y) \geqslant f(\mu)(y)$ for every $y \in B$ and f is a fuzzy homomorphism of (A, e_A, μ) onto $(B, e_B, \lambda(f, \mu))$.*

Proof. Consider the set $H = \{\lambda \in LF(B) : f(\mu) \leqslant \lambda\}$. Since $1_B \geqslant f(\mu)$, $1_B \in H$ and the set H is non-empty. We put $\lambda(f, \mu) = \wedge H$. By virtue of the Lemma 13.1.2, $f(\mu) \leqslant \lambda(f, \mu)$ and $\lambda(f, \mu) \in H$. The assertions 1 and 3 are proved. Fix a homomorphism $g : B \to C$ into an L-fuzzy E-algebra (C, e_C, η) and suppose that the composition $g \cdot f : A \to C$ is a fuzzy homomorphism. For every $y \in B$ we put $\lambda(y) = \eta(g(y))$. Then $\lambda \in H$. Hence $g : B \to C$ is a fuzzy homomorphism. The proof is complete.

Definition 6.2.2. *Let $f : A \to B$ be a fuzzy homomorphism of an L-fuzzy L-algebra (A, e_A, μ) onto an L-fuzzy E-algebra (B, e_B, η). Then:*

1. *If $\eta = f(\mu)$, then f is called a factor homomorphism and (B, e_B, η) is called a fuzzy factor-algebra of the algebra (A, e_A, μ).*

2. *If $\eta = \lambda(f, \mu)$, then f is called an s-factor homomorphism and (B, e_B, η) is called a fuzzy s-factor-algebra of the algebra (A, e_A, μ).*

Every factor homomorphism is an s-factor homomorpfism.

Problem 6.2.3. (the fuzzy homomorphism problem). *Let $f : A \to B$ be a homomorphism of an L-fussy E-algebra (A, e_A, μ) onto an E-algebra B. Under which conditions $\lambda(f, \mu) = f(\mu)$, i.e. $(B, e_B, f(\mu))$ is an L-fuzzy E-algebra?*

Our main goal is to solve the Problem 6.2.3 for some important cases. We shall close this section with tree examples which show that the answer to the Problem 6.2.3, in general, is "no".

Example 6.2.4. Let $L = \{0, 1, a, b, c\}$, where $a \wedge b = a \wedge c = b \wedge c = 0$, $a \vee b = a \vee c = b \vee c = 1$, $A = \{0, a, b, c\}$, $B = \{0, a, b\}$. Consider the mapping $\mu : A \to L$, where $\mu(x) = x$ for every $x \in A$, and the mapping $f : A \to B$, where $f(0) = 0$, $f(a) = a$, $f(b) = f(c) = b$. Let $E = E_2 = \{\cdot\}$ and $x \cdot y = x \wedge y$ for all $x, y \in A$. Then A is an L-fuzzy commutative semigroup, B is a subsemigroup of A and $f : A \to B$ is a homomorphism. By construction, $f(\mu)(0) = 0$, $f(\mu)(a) = a$ and $f(\mu)(b) = 1$. It is evident that $\lambda(f, \mu)(0) = \lambda(f, \mu)(a) = a$ and $\lambda(f, \mu)(b) = 1$. Hence $f(\mu) \neq \lambda(f, \mu)$.

Example 6.2.5. Let $L = \{0, 1, b, a_n : n \in N\}$, where $b \wedge a_n = 0$, $b \vee a_n = 1$ for all $n \in N$ and $a_n < a_m$ for $n < m$. The lattice L is complete and distributive. Consider the commutative semigroup $A = \{0, b, a_n : n \in N\}$, where $x \cdot y = x \wedge y$ for all $x, y \in A$, the subsemigroup $B = \{0, b, a_0\}$ of A, the fuzzy structure $\mu : A \to L$, where $\mu(x) = x$ for all $x \in A$, and the mapping $f : A \to B$, where $f(0) = 0$, $f(b) = b$ and $f^{-1}(a_0) = \{a_n : n \in N\}$. Then $f : A \to B$ is a homomorphism and (A, μ) is an L-fuzzy commutative semigroup. By construction, $f(\mu)(0) = 0$,

$f(\mu)(b) = b$ and $f(\mu)(a_0) = 1$. It is obvious that $\lambda(f, \mu)(0) = \lambda(f, \mu)(b) = b$ and $\lambda(f, \mu)(a_0) = 1$. Hence $f(\mu) \neq \lambda(f, \mu)$.

Example 6.2.6. Let L be the lattice from the Example 13.2.5, $E_0 = \{1\}$, $E_2 = \{\cdot\}$, $E = E_0 \cup E_2$, $A = L$, $1_A = 1$, $x \cdot y = x \wedge y$, for all $x, y \in A$, $\mu(x) = x$ for every $x \in A$. Then A is an L-fuzzy semigroup with identity. Consider the subsemigroup $B = \{0, b, a_0, 1\}$ and the mapping $f : A \to B$, where $f(0) = 0$, $f(b) = b$, $f(1) = 1$ and $f^{-1}(a_0) = \{a_n : n \in N\}$. The mapping f is a homomorphism and $f(\mu) \neq \lambda(f, \mu)$.

Remark 6.2.7. Let $E = \emptyset$. Then every L-fuzzy set is an L-fuzzy E-algebra. Therefore in this case $f(\mu) = \lambda(f, \mu)$ for every homomorphism $f : A \to B$ of some fuzzy E-algebra (A, μ) onto an E-algebra B.

6.3 Case of the Proper Homomorphisms

Definition 6.3.1. *A homomorphism $f : A \to B$ of an L-fussy E-algebra (A, e_A, μ) into an E-algebra B is called a proper homomorphism if for every $y \in B$ there exists an element $x(y) \in f^{-1}(y)$ such that $\mu(x(y)) = f(\mu)(y)$.*

Theorem 6.3.2. *Let $f : A \to B$ be a proper homomorphism of an L-fuzzy E-algebra (A, e_A, μ) onto an E-algebra B. Then $f(\mu) = \lambda(f, \mu)$ and $(B, e_B, f(\mu))$ is an L-fussy E-algebra.*

Proof. By virtue of the conditions of the theorem, we have $\{y \in B : f(\mu)(y) \geqslant l\} = f(\{x \in A : \mu(x) \geqslant l\})$. The image of an E-subalgebra of A is an E-subalgebra of B. The proof is complete.

Remark 6.3.3. S. Nanda [169] has proved the similar result for groupoids, groups and modules.

6.4 Case of Distributive Lattices

Let τ be an infinite cardinal. A complete lattice L is called τ-distributive if $a \wedge (\vee H) = \vee\{a \wedge x : x \in H\}$ for every non-empty subset H of L provided $|H| < \tau$, where by $|H|$ we denote the cardinality of the set H. The lattice L is called infinite distributive if it is τ-distributive for every cardinal τ.

If ω_0 is first infinite cardinal, then the lattice L is distributive if and only if it is ω_0-distributive.

Lemma 6.4.1. ([26], Section 5.5). *Let τ be an infinite cardinal, L be a complete τ-distributive lattice, $n \geqslant 2$ and $H_1, H_2, \dots, H_n$ be the non-empty subsets of L of cardinality $< \tau$. Then*

$$(\vee H_1) \wedge (\vee H_2) \wedge \cdots \wedge (\vee H_n) = \vee\{x_1 \wedge x_2 \wedge \cdots \wedge x_n : x_i \in H_i,\ i \leqslant n\}.$$

Theorem 6.4.2. *Let τ be an infinite cardinal, L be a τ-distributive complete lattice, $f : A \to B$ be a homomorphism of an L-fuzzy E-algebra (A, e_A, μ) onto an E-algebra B and $|f^{-1}(y)| < \tau$ for every $y \in B$. Then $f(\mu) = \lambda(f, \mu)$ and $(B, e_B, f(\mu))$ is an L-fuzzy E-algebra.*

Proof. Let $\omega \in E_0$. Then $f(\mu)(\omega_B) = \vee f^{-1}(\omega_B) = \mu(\omega_A) = \vee\mu(A) = \vee f(\mu)(B)$. Let $n \geqslant 1$, $\omega \in E_n$, $y_1, \dots, y_n \in B$ and $y = \omega(y_1, \dots, y_n)$. Then

$$f(\mu)(y) = \vee\mu(f^{-1}(y)) \geqslant \vee\{\mu(x_1, \dots, x_n) : x_i \in f^{-1}(y_i),\ i \leqslant n\} \geqslant$$

$$\geqslant \vee\{\mu(x_1) \wedge \cdots \wedge \mu(x_n) : x_i \in f^{-1}(y_i),\ i \leqslant n\} =$$

$$= (\vee\mu(f^{-1}(y_1))) \wedge (\vee\mu(f^{-1}(y_2))) \wedge \cdots \wedge (\vee\mu(f^{-1}(y_n))) =$$

$$= f(\mu)(y_1) \wedge f(\mu)(y_2) \wedge \cdots \wedge f(\mu)(y_n).$$

The proof is complete.

Corollary 6.4.3. *Let L be a complete distributive lattice, $f : A \to B$ be a homomorphism of an L-fuzzy E-algebra (A, e_A, μ) onto an E-algebra B and the set $f^{-1}(y)$ is finite for every $y \in B$. Then $f(\mu) = \lambda(f, \mu)$.*

Corollary 6.4.4. *Let L be an infinite distributive lattice and $f : A \to B$ be a homomorphism of an L-fuzzy E-algebra (A, e_A, μ) onto an E-algebra B. Then $f(\mu) = \lambda(f, \mu)$ and $(B, e_B, f(\mu))$ is an L-fuzzy E-algebra.*

Remark 6.4.5. Wang-Jin Liu ([**?**] proved the Corollary 6.4.4 for groups and rings.

6.5 Case of Dense Homomorphism

In [[171],p. 26] C.V. Negoiţă and D.A. Ralescu considered complete lattices with the following property:

(NR) **for all** $H \subseteq L$ **and every** $b < \vee H$ **there exists** $c \in H$ **such that** $b \leqslant c$**.**

Every lattice with the property (NR) is infinite distributive.

Definition 6.5.1. *A homomorphism* $f : A \to B$ *of an* L*-fuzzy* E*-algebra* A *onto an* E*-algebra* B *is called a dense homomorphism if for every* $y \in B$ *and every* $t < \vee\mu_A(f^{-1}(y)$ *there exists an element* $t(y) \in f^{-1}(y)$ *for which* $t < \mu(t(y))$.

Theorem 6.5.2. *Let* L *be a complete lattice and* $f : A \to B$ *be a dense homomorphism of an* L*-fuzzy* E*-algebra* (A, e_A, μ) *onto an* E*-algebra* B*. Then* $f(\mu) = \lambda(f, \mu)$.

Proof. Note $\eta = f(\mu)$ and $l_0 = \vee\mu(A) = \vee\eta(B)$. If $\omega \in E_0$, then $\eta(\omega_B) = \mu(\omega_A) = l_0$.

Let $n \geqslant 1$, $\omega \in E_n$, $y_1, \ldots, y_n \in B$, $y = \omega(y_1, \ldots, y_n)$ and $l = \eta(y_1) \wedge \cdots \wedge \eta(y_n)$. We will prove that $\eta(y) \geqslant l$. Let $L^-(l) = \{t \in L : t < l\}$ and $l_1 = \vee L^-(l)$. We consider two possible cases.

C a s e 1. $l_1 < l$.

In this case for every $i \leqslant n$ we have $\vee\mu(f^{-1}(y_i)) \geqslant l$ and there exists an element $x_i \in f^{-1}(y_i)$ for which $\mu(x_i) \geqslant l$. If $x = \omega(x_1, \ldots, x_n)$, then $y = f(x)$ and $\eta(y) = \vee\mu(f^{-1}(y)) \geqslant \mu(x) \geqslant l$.

C a s e 2. $l_1 = l$.

For every $t \in L^-(l)$ and $i \leqslant n$ there exists $t(y_i) \in f^{-1}(y_i)$ such that $\mu(t(y_i)) \geqslant t$. Hence $\eta(y) \geqslant \mu(\omega(t(y_1), \ldots, t(y_n))) \geqslant t$ and $\eta(y) \geqslant \vee L^-(l) = l$. The proof is complete.

Example 6.5.3. Every proper homomorphisms is a dense homomorphism.

If L is a lattice with the property (NR) and $f : A \to B$ is a homomorphism of an L-fuzzy E-algebra B, then the homomorphism f is dense.

Example 6.5.4. Let L be a complete and linearly ordered lattice. Then the lattice L is infinite distributive, has the property (NR) and every homomorphism $f : A \to B$ of an L-fuzzy E-algebra (A, e_A, μ) onto an E-algebra B is dense.

6.6 Algebras with Fuzzy Operations

Fix a non-empty signature $E = \cup\{E_n : n \in N\}$. Let $D \subseteq E$ and $D_n = D \cap E_n$ for every $n \in N$.

If A is an E-algebra, then every E-subalgebra B of A is a D-subalgebra. The converts assertion is not true: there exists D-subalgebras of A which are not E-subalgebras. If $D = \emptyset$, the every non-empty subset of A is a D-subalgebra of A.

Fix an L-fuzzy subset $\theta : E \to L$ of E. For every $t \in L$ consider the level subset $E(\theta, t) = \{x \in E : \theta(x) \geqslant t\}$ of E. It is clear that $E(\theta, 0) = E$ and $E(\theta, t) \subseteq E(\theta, t')$ whenever $t' \leqslant t$.

A triple (G, e_G, μ) is called an (L, θ)-fuzzy E-algebra if the following conditions hold:

(A) (G, e_G) is an E-algebra;

(F) $\mu : G \to L$ is an L-fuzzy subset of G;

(AFF) For every $t \in L$ the set $\{x \in G : \mu(x) \geqslant t\}$ is empty or is an $E(\theta, t)$-subalgebra of G.

The proofs of following assertions are simple.

Theorem 6.6.1. *Let (A, e_A) be an E-algebra and $\mu : A \to L$ be a mapping. The triple (A, e_A, μ) is an (L, θ)-fuzzy E-algebra if and only if the following assertions are true:*

1. *If $\omega \in E_0$, then $\mu(\omega_A) \geqslant \vee\{\theta(\omega) \wedge \mu(x) : x \in A\}$;*
2. *If $n \geqslant 1$, $\omega \in E_n$ and $x_1, \ldots, x_n \in A$, then $\mu(\omega(x_1, \ldots, x_n)) \geqslant$ $\geqslant \theta(\omega) \wedge \mu(x_1) \wedge \cdots \wedge \mu(x_n)$.*

Proposition 6.6.2. *Let (A, e_A) be an E-algebra and $\mu : A \to L$ be a mapping. The triple (A, e_A, μ) is an L-fuzzy E-algebra if and only if it is an $(L, 1_E)$-fuzzy E-algebra.*

Proposition 6.6.3. *Let (A, e_A, μ) be an (L, θ)-fuzzy E-algebra and $\lambda : E \to L$ be a mapping. If $\lambda \leqslant \theta$, then (A, e_A, μ) is an (L, λ)-fuzzy E-algebra.*

Proposition 6.6.4. *Let (A, e_A) be an E-algebra and $\mu : A \to L$ be a mapping. Then (A, e_A, μ) is an $(L, 0_E)$-fuzzy E-algebra.*

Applying 6.6.1 – 6.6.4 we obtain

Theorem 6.6.5. *The assertions of the Theorems* 6.2.1, 6.3.2, 6.4.2, 6.5.2, *of the Corollaries* 6.1.3, 6.1.4, 6.4.3, 6.4.4 *and of the Lemma* 6.1.2 *are true for* (L, θ)*-fuzzy* E*-algebras.*

Example 6.6.6. The set $\{0, -, +\}$ is the signature of commutative groups, $\{\cdot\}$ is the signature of groupoids and $\{0, 1, -, +, \cdot\}$ is the signature of rings with identity. Let R be a ring with the identity 1, $R_1 = \{n \cdot 1 : n = 0, \pm 1, \pm 2, \dots\}$ and $R \neq R_1$. The set R_1 is the subring of the integers of the ring R. We consider that $0 \cdot 1 = 0$, $(n+1) \cdot 1 = n \cdot 1 + 1$ and $(-n) \cdot 1 = -(n \cdot 1)$ for every number n. Every R-module is an E-algebra of the signature $E = E_0 \cup E_1 \cup E_2$, where $E_0 = \{0\}$, $E_1 = R \cup \{-\}$, $E_2 = \{+\}$. Consider the mapping $\theta : E \to L$, where $\theta^{-1}(1) = \{0, -, +\} \cup R_1$ and $\theta^{-1}(0) = R \setminus R_1$, and the mapping $\mu : R \to R$, where $\mu^{-1}(1) = R_1$ and $\mu^{-1}(0) = R \setminus R_1$. Then (R, μ) is an L-fuzzy ring and (R, μ) is an (L, θ)-fuzzy E-algebra, but (R, μ) is not an L-fuzzy E-algebra.

The module R is a fuzzy module over the fuzzy ring R ([171], Definition 16).

6.7 On Fuzzy Finitely Generated Qroupoids

This section provides a study of the category of fuzzy groupoids with division. The results of this section were published in [60]. The category of fuzzy quasigroups was studied in [59].

Let L be a complete lattice, $0 = inf(L) < sup(L) = 1$ and for every pair of different elements $x, y \in L$ we have $x < y$ or $x < y$.

Example 6.7.1. $L = \{0, 1\}$.

Example 6.7.2. $L = [0, 1]$.

The mapping $\mu : A \to I$ is called an L-fuzzy subset of a set A. The mapping $f : X \to Y$ of an L-fuzzy set (X, μ) into an L-fuzzy set (Y, μ) is called a fuzzy mapping if $\eta(f(x)) \geq \mu(x)$ for every $x \in X$.

The disjoint sum of the sets $\{E_n : n \in N = \{0, 1, 2, \dots\}\}$ is denoted by E and is called a signature or a set of fundamental operations.

We say that an E-algebra or an algebra G of a signature E is given if the set G is non-empty and there are the mappings $e_G = \{e_{nG} : E_n \times G^n \to G : n \in N\}$.

The class of E-algebras was studied in [42].

Definition 6.7.3 *An E-algebra G is an:*

1. *E-groupoid with left division if there exist the operations $A, G, W \in E_2$ for which $A(C(y,x),x) = W(C(x,y),x) = y$ for every points $x, y \in G$.*

2. *E-groupoid with right division if there exist the operations $A, B, V \in E_2$ for which $A(x, B(x,y)) = V(x, B(y,x)) = y$ for every $x, y \in G$.*

3. *E-groupoid with division if there exist the operation $A, B, G, V, W \in E_2$ such that $A(x, B(x,y)) = A(G(y,x),x) = V(x, B(y,x)) = W(G(x,y),x) = y$ for every $x, y \in G$.*

4. *E-quasigroup if there exist the operations $A, B, C \in E_2$ for which $A(x, B(x,y)) = B(x, A(x,y)) = A(G(y,x),x) = G(A(y,x),x) = y$.*

Every E-groupoid with division is an E-groupoid with left and right divisions. If G is an E-groupoid, then G is also an E-groupoid with division and $V = G, W = B$.

Let X be a non-emty subset of the E-algebra G. Denote

$$s_0(X,G) = X,\ s_{n+1}(X,G) = s_n(V,G) \cup \cup\{e_{mG}(E_m \times s_n(X,G)^m) : m \in N\},$$
$$s(X,G) = \cup\{s_n(X,G) : n \in N\}.$$

Then $s(X,G)$ is called the subalgebra of G generated by X.

The E-algebra G is finitely generated if $G = s(Y,G)$ for some finite subset Y of G.

Definition 6.7.4. *The triple (G, e_G, μ_G) is called an L-fuzzy E-algebra if it satisfies the following conditions:*

1. *(G, e_G) is an E-algebra;*

2. *(G, μ_G) is a L-fuzzy set;*

3. *If $n \geq 1, a \in E_n$ and $x_1, ..., x_n \in G$, then $\mu_G(e_{nG}(a, x_1, ..., x_n)) \geq inf(\mu_G(x_1), ..., \mu_G(x_n))$.*

The concept of a fuzzy set was introduced by L. Zadeh [210]. The class of fuzzy groups was studied in [182, 190]. The notion of fuzzy modules was introduced by Negoita and Ralescu [171] and studied in [95, 96].

6.8 The Basis of the Fuzzy Algebras

Let (G, e_G, μ_G) be an L-fuzzy E-algebra. If $G = s(X, G)$, then X is called a basis of G. Denote $l(G, e_G, \mu_G) = inf(\mu_G(x) : x \in X)$.

Definition 6.8.1. *The basis X of (G, e_G, μ_G) is homogeneous, if $\mu(x) = l(G, e_G, \mu_G)$ for every $x \in X$.*

Lemma 6.8.2. (see [59]). *Let (G, e_G, μ_G) be an L-fuzzy E-algebra and X be a basis of G. Then $l(G, e_G, \mu_G) = inf\{\mu_G(x) : x \in X\}$.*

Proof. Let $d = inf(\mu_G(x) : x \in X)$, $x \in s_1(X, G)$. Then $x = = e_{nG}(e, x_1, ..., x_n)$ for some $n \geq 1$, $e \in E_n$, $x_1, ..., x_n \in X$. In virtue of condition 3 of Definition 9.7.4, $\mu_G(x) \geq inf(\mu_G(x_1), \mu_G(x_2), ..., \mu_G(x_n)) \geq \geq d$. Hence $\mu_G(x) \geq d$ for every $x \in s_1(X, G)$. By construction $s_{n+1}(X, G) = s_1(s_n(X, G), G)$. Therefore, $\mu_G(x) \geq d$ for every $x \in G$. The proof is complete.

Lemma 6.8.3. *Let $(G, e_G, \mu_G))$ be an L-fuzzy groupoid with right division. If $\mu_G(y) \neq \mu_G(x)$, then $\mu_G(B(x, y)) = inf(\mu_G(x), \mu_G(y))$.*

Proof. If $\mu_G(y) < \mu_G(x)$, then we have

$$\mu_G(y) = \mu_G(A(x, B(x, y))) \geq inf(\mu_G(x), \mu_G(B(x, y)) \geq \\ \geq inf(\mu_G(x), \mu_G(y)) = \mu_G(y)).$$

Hence $\mu_G(B(x, y)) = \mu_G(y)$. If $\mu_G(x) < \mu_G(y)$, then $\mu_G(x) = = \mu_G(V(y, B(x, y))) \geq inf(\mu_G(B(y, x), \mu_G(y)) = \mu_G(x)$. Therefore, $\mu_G(B(x, y) = \mu_G(x)$. The proof is complete.

Corollary 6.8.4. *Let $(G, e_G, \mu_G))$ be an L-fuzzy groupoid with left division. If $\mu_G(y) \neq \mu_G(x)$, then $\mu_G(y, x) = inf\{\mu_G(x), \mu_G(y)\}$.*

Theorem 6.8.5. *Let (G, e_G, μ_G) be an L-fuzzy E-groupoid with left or right division. Then:*

1. *If $a \in G$ and $\mu_G(a) = l(G, e_G, \mu_G)$, then exists a homogeneous basis for G.*

2. *If $d \in G$ and $\mu_G(d) \geq l(G, e_G, \mu_G)$, then there exists a basis X of G such that $\mu_G(x) \leq d$ for every $x \in X$.*

3. *If G is finitely generated, then there exists a homogeneous basis for G.*

Proof. Let G be a groupoid with right division, Y be a basis of G and $d \in Y$. We put $X_1 = Y_1 = \{x \in Y : \mu_G(x) \leq \mu_G(d)\}$, $Y_2 = Y \setminus Y_1$, $X_2 = \{B(x,d) : x \in Y_2\}$ and $X = X_1 \cup X_2$. By construction, $d \in Y_1, Y_2 = \{V(d, B(x,d)) : x \in X_2\}$ and X is a basis of G. In virtue of Lemma 6.8.3, $\mu_G(x) \leq \mu_G(d)$ for every $x \in X$. The assertion 2 is proved. From it the assertions 1 and 3 follow. The proof is complete.

Corollary 6.8.6. *Let $L = [0,1]$). For an arbitrary L-fuzzy finitely generated L-fuzzy module there exists a homogeneous basis.*

Corollary 6.8.7. *For an arbitrary finitely generated L-fuzzy group there exists a homogeneous basis.*

Corollary 6.8.8. (see [59]). *For an arbitrary finitely generated L-fuzzy quasigroup there exists a homogeneous basis.*

6.9. Conclusions for Chapter 6

In this Chapter we solved the homomorphism problem for fuzzy universal algebras and studied a homogeneous basis of fuzzy groupoid with division. Homomorphism problem is a fundamental problem of algebraical structures.

We come to the following conclusions:

1). there have been introduced the concepts of fuzzy universal algebras, fuzzy homomorphism of fuzzy universal algebras, fuzzy groupoids with division.

2). there have been elaborated the studying methods of fuzzy universal algebras, fuzzy groupoids with division, a homogeneous basis of fuzzy groupoid with division and fuzzy homomorphism.

3). in this way, using the new concepts and new methods of research we obtained the following results:

- we have offered a general solution of the homomorphism problem for fuzzy universal algebras. In papers [169, 182] the problem of the homomorphism for fuzzy algebras was formulated and solved for some homomorphisms of the fuzzy groupoids, groups and rings.
- we have investigated the category of fuzzy groupoids with division and, in particular, we have proved that an arbitrary L-fuzzy finitely generated L-fuzzy module, L-fuzzy quasigroup, L-fuzzy group, there exists a homogeneous basis. Result from this work is stronger than Theorems of S. Nanda, Wang-Jin Liu.

Further investigation could focus on:

- introducing the concept of multiple identities for fuzzy algebras;
- developing methods of construction of Haar measure on fuzzy groups.

General Conclusions and Recommendations

The main problem solved in accordance with the objectives of the thesis, consist to determine the influence of the algebraic structures on the topological properties of the universal topological algebras and application of topological algebraic structures in the study of the properties of topological spaces.

This topic is directly related to the celebrated Hilbert's problem V.

The obtained results in the respective piece of work are directly intertwined with the solving of Problems 1-12, which were formulated above. The main results of the work are new. There have been solved concrete problems, or some aspects of the problems formulated by A.I. Mal'cev, L.S. Pontrjagin, A.V. Arhangelskii, M.M.Choban, I.V. Protasov.

The research conducted in this thesis covers objectives of the investigation and allows formulating the following conclusions:

1. In this thesis we developed general theories, concepts and efficient research methods to various classes of topological algebras:

- the method of uniform structures;

- the method of free algebras;

- the method of k-algebras;

- a general theory on the decomposition of the topological algebras;

- the concept of multiple identities;

- methods of the investigation of topological quasigroups with multiple identities;

- the method of fuzzy algebras and fuzzy homomorphism.

2. Applying the method of uniform structures succeeded developing a general construction which allows describing topological structures of free algebras generated by pseudocompact and countable compact spaces. This construction is more general and efficient comparative with methods of A. Arhangel'skii, E. Nummela, V. Pestov, T.H.Fay, B.V. Smith-Thomas and A. Tkacenko, which were successfully used in the research of free topological groups generated by compact and countable compact spaces. Our construction made it possible to studying the case of the pseudocompact spaces and implement new methods of studying topologies on free topological algebras with continuous signature generated by pseudocompact and countable compact spaces.

3. There has been elaborated the method of k-algebras. This allowed obtaining new important results, with large applications in topological algebra. The implementation of the methods of free algebras and k-algebras contributed to the identification of some significant properties regarding the relation of M_K-equivalence. Some properties of compact subsets of free k-algebras and some facts about M_K-equivalence of spaces are established. For instance, we obtained that the homological groups obey a relation of M_K-equivalence. Some similar results for varieties of topological groups and compact spaces were proved by L. S. Pontrjagin, B. A. Pasynkov and V. Valov.

4. There has been elaborated a general method of decomposition of abstract algebra in a maximal number of subsets, which remains dense in any bounded topology of Choban. The implementation of new concepts and general method of decomposition allowed to generalize some results and constructions proposed by V.I. Malykhin, W.W. Comfort, S. Van Mill, I.V.Protasov and M.M. Choban which successfully was used in research the problem of resolvability of totally bounded topological groups. The mathematically developed device allows researching various classes of groupoid and n-groupoids with invertibility properties.

5. We introduced some new concepts: (n, m)-identities,(n, m)-homogeneous isotope, (n, m)-homogeneous quasigroup. Using the new concepts and methods we describe the topological quasigroups with (n, m)-identities, which are obtained by using isotopies of topological groups. We extend some affirmations from the theory of topological groups on the class of topological (n, m)-homogeneous quasigroups. We establish conditions for which there exist right invariant (or left invariant) Haar measures on a medial grupoid.

In this way, using the new investigation methods we are able construct and demonstrate the uniqueness of Haar measure on medial quasigroups. The proposed methodology can be used to investigate n-quasigroups with multiple identities.

6. We introduced the concepts of fuzzy universal algebras and fuzzy homomorphism for these algebras. The problem of homomorphism for fuzzy algebras was formulated and solved by S. Nanda, A. Rozenfeld, A.Wang-Jin Liu for some homomorphisms of the fuzzy groupoids, groups and rings.

This work gives a general solution of the homomorphism problem for fuzzy universal algebras and conditions for which there exists a homogeneous basis for the category of fuzzy groupoids with division. The results from this work are stronger than the results obtained by the authors mentioned above. In particular, we prove that for an arbitrary L-fuzzy finitely generated L-fuzzy module, L-fuzzy quasigroup, L-fuzzy group, there exists a homogeneous basis.

The applied methodology, the developed concepts and methods as well as the results obtained in work can be used:

- in studying the free topological algebras generated by diverse topological spaces.
- in studying the topological-algebraical properties of groupoids with multiple identities.
- in investigating the topological-algebraical structure of various classes of topological algebras.
- in studying some special classes of automata or semi-automata.
- in constructing the free topological algebras with some special topological-algebraic properties.
- in studying algebraic properties of fuzzy universal algebras.
- in elaborating optional courses.

Topological algebraic systems, as a branch of topological algebra, represent an important field of research in modern mathematics. The obtained results and the methods elaborated within this work can be successfully implemented not only in theoretical mathematics (abstract algebra, topology, topological algebra, algebraic topology, harmonic analysis), but also in applied mathematics, physics, computer science, fuzzy algebra, theory of automata and semi-automata, etc.

The prospective purpose lays in:

- in studying the types of algebraic structures that can be considered on the space, which make it a topological algebra.
- in investigating the kinds of topologies, which can be considered on the universal algebra that makes it a topological algebra;
- in a more detailed studying of the concept of multiple identities for various

classes of algebra;

- in studying the role of universal topological algebras in the theory of automata, semi-automata and in diverse informational systems;
- in elaborating special courses in the theory of topological algebra for students and doctorates.

Concluzii Generale şi Recomandări

Problema principală rezolvată conform obiectivelor tezei, constă în determinarea influenţei structurilor algebrice asupra proprietăţilor topologice ale algebrelor universale topologice şi aplicarea acestora la studierea proprietăţilor spaţiilor topologice.

De această direcţie ţine şi vestita Problmă V a lui Hilbert.

Rezultatele obţinute în lucrare respectivă sunt nemijlocit legate de soluţionarea Problemelor 1-12 formulate mai sus. Rezultatele principale ale lucrării sunt noi. Au fost rezolvate probleme concrete, ori unele aspecte ale problemelor formulate de A.I. Malţev, L.S. Pontrjagin, A.V. Arhangelsk'ii, M.M.Cioban, I.V. Protasov.

Cercetările realizate în această lucrare se referă la obiectivele propuse pentru investigaţie şi permit să formulăm următoarele concluzii:

1. În lucrare au fost elaborate teorii generale, concepte şi metode eficiente de cercetare a diverselor clase de algebre topologice:

- metoda structurilor uniforme;
- metoda algebrelor libere;
- metoda k-algebrelor;
- teoria generală de descompunere a algebrelor;
- conceptul unităţilor multiple;
- metode de cercetare a quasigrupurilor topologice cu unităţi multiple;
- metoda algebrelor universale fuzzy şi omomorfismelor fuzzy.

2. Aplicând metoda structurilor uniforme, s-a reuşit de elaborat o construcţie generală care permite descrierea structurilor topologice a algebrelor libere generate de spaţii pseudocompacte şi numărabil compacte. Această construcţie este mai generală şi mai eficientă comparativ cu cele elaborate de A. Arhangelschi, E. Nummela, V. Pestov şi A. Tkacenko folosite pentru examinarea grupurilor topologice libere generate de spaţii compacte şi numărabil compacte. Construcţia propusă permite să studiem şi cazul spaţiilor pseudocompacte şi implementarea acestei metode privind studierea topologiilor pe algebre topologice libere cu signatura continuă.

3. A fost elaborată metoda k-algebrelor topologice care permite să determinăm proprietăţile topologice ce se păstrează la relaţia de M_k-echivalenţă cum ar fi, de exemplu, grupurile omologice, care formează o proprietate fundamentală care ţine de topologia algebrică. Afirmaţiile obţinute generalizează unele din rezultatele demonstrate pentru varietăţi de grupuri topologice şi spaţii compacte de L. S. Pontrjagin, B. A. Pasynkov, V. Valov.

4. A fost elaborată o metodă generală de descompunere a algebrei abstracte într-un număr maximal de submulţimi, care rămân dense în orice topologie mărginită în sensul Cioban. Rezultatele obţinute generalizează unele construcţii propuse de V.I. Malykhin, W.W. Comfort, S. Van Mill, I.V.Protasov şi M.M. Cioban care au examinat diverse aspecte ale descompunerilor grupurilor topologice total mărginite. Afirmaţiile demonstrate sunt juste şi pentru anumite clase de n-quasigrupuri şi n-bucle cu propietăţi de invertibilitate. Aparatul matematic elaborat permite efectuarea cercetărilor pentru diverse clase de grupoizi şi n-grupoizi cu invertibilitate.

5. A fost introdus conceptul de (n, m)-unitate. În cercetările realizate sunt descrise quasigrupurile topologice cu (n, m)-identităţi, care se obţin utilizând izotopiile grupurilor topologice. Astfel de quasigrupuri au fost numite quasigrupuri (n, m)-omogene. În lucrare s-a reuşit extinderea unor afirmaţii fundamentale din clasa grupurilor topologice în clasa quasigrupurilor topologice (n, m)-omogene. Astfel, utilizând noţiunile noi introduse: (n, m)-identitate, (n, m)-izotop omogen, quasigrup (n, m)-omogen, s-a reuşit, de exemplu, să se construiască şi să se demonstreze unicitatea măsurii Haar pe quasigrupuri mediale. Metodologia propusă pentru cercetare poate fi utilizată la investigarea n-quasigrupurilor cu unităţi multiple.

6. A fost introdus conceptul de algebră universală fuzzy şi omomorfisme fuzzy. Cercetările efectuate au condus la determinarea soluţiei generale a problemei omomorfismelor pentru algebrele universale fuzzy şi au fost găste condiţiile pentru care există o bază omogenă pentru L-fuzzy E-grupoizi cu diviziune de stânga ori de dreapta. Menţionăm că rezultatele obţinute generalizează unele din teoremele demonstrate de S. Nanda, Wang-Jin Liu pentru groupoizi, grupuri, module şi inele. Pentru realizarea obiectivului propus s-a introdus noţiunile: L-fuzzy E-algebră, fuzzy omomorfism al L-fuzzy E-algebrelor, fuzzy s-factor omomorfism. Metodele elaborate şi rezultatele obţinute permit studierea ulterioară a diverselor aspecte ale L-fuzzy E-algebrelor.

Rezultatele obţinute, construcţiile şi metodele elaborate pot fi cu succes aplicate:
- la cercetarea obiectelor libere generate de diverse spaţii topologice;
- la examinarea proprietăţilor topologice ale diverselor clase de grupoizi cu unităţi multiple;
- la investigarea structurilor algebrice şi topologice ale diferitor clase de algebre topologice;
- la studierea anumitor clase de automate şi semi-automate;
- la construirea algebrelor topologice libere, cu anumite proprietăţi topologice, în varietăţi complete;
- la studierea proprietăţilor algebrice ale algebrelor universale fuzzy;
- la elaborarea cursurilor opţionale pentru masteranzi şi doctoranzi.

Având în vedere rolul algebrelor topologice universale în algebra abstractă, topologie, algebra topologică, topologie algebrică, analiza armonică, teoria automatelor şi semi-automatelor, algebre fuzzy putem considera că teoria şi conceptele elaborate pot fi aplicate eficient în cercetările din domeniile menţionate mai sus cât şi în alte direcţii de cercetare.

Obiective de perspectivă. În perspectivă:
- se vor studia proprietăţile topologice ale spaţiilor care admit anumite structuri algebrice;
- se vor studia tipurile de topologii care pot fi introduse pe algebre universale şi o transformă în algebră topologică;
- va fi studiat mai profund conceptul de unitate multiplă pentru diverse clase de algebre;
- va fi studiat rolul algebrelor topologice universale în teoria automatelor şi semi-automatelor şi diverse sisteme informaţionale;
- se va elabora un curs opţional pentru masteranzi, doctoranzi în domeniul algebrelor topologice universale.

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DECLARAŢIA PRIVIND ASUMAREA RĂSPUNDERII

Subsemnatul, declar pe propria mea răspundere că materialele prezentate în teza de doctorat se referă la propriile activităţi şi realizări, în caz contrar urmând să suport consecinţele, în conformitate cu legislaţia în vigoare.

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