About researches of V. A. Andrunakievich (on the structural theory and theory of radicals. II)

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Аннотация

This is the second part of overview published in Russian in the book devoted Academician V.A. Andrunakievich (*Academicianul Vladimir Andrunachievichi: Bibliografie*, Institutul de Matematica si Informatica, Chisinau, 2009).

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3. Modules, one-sided ideals and radicals

It has already been noted that in describing the structure of algebras often are used **representations**, i.e., homomorphisms of algebras into the algebras $pounds(M)$ of linear transformations of spaces, that is, $\rho : A \to \pounds(M)$.

If the space is finite-dimensional, then matrix representations are obtained, since $\rho(A)$ turns out to be a subalgebra of the algebra $[\ Phi]_n$ of matrices isomorphic to the algebra $\pounds(M)$ (see the theorems in the first section).

Therefore, one can speak simply about "matrix representations"and in the general case, but in this case we obtain matrices that have not necessarily a finite size.

Some possibilities and refinements of such representations were considered in the papers of V. A. Andrunakievich [6, 9, 11] and the works of many other authors.

If the representation $\rho : A \to \pounds(M)$ is given, then the space M turns into the **module** M_a (right), according to the rule $xa = x\rho(a) \in M$.

This means that the element $xa \in M$ is defined for all $x \in M$, $a \in A$, and the resulting multiplication is connected in the module by the laws $x(ab) = (xa)b$, $x(a + b) = xa + xb$, $(x + y)a = xa + ya$ as for spaces, but with changing of multiplication by elements of field Φ by multiplication by elements of algebra A.

On the other hand, according to the indicated laws, if the module M_A is given, then according to the rule $x\rho(a) = xa \in M$ the corresponding representation $\rho : A \to \pounds(M)$ is obtained.

In other words, the definition of a module is equivalent to specifying (definition) a representation, and we can assume that the algebra A has the unit $e \neq 0$ (since it is always possible to "attach") - in this case it turns out that $\rho(e) = \varepsilon_M$, i.e., $xe = x$ for all $x \in M$.

By symmetry, with the help of multiplication $x \to ax$ the left modules ${}_AM$ are determined and the corresponding representations, too.

From the above it turns out that for the algebra A **kernels** $Ker\rho = \{a \in A \mid xa = 0 \; for \; x \in M_A\} = (0 : M)_A$ runs through all ideals of the algebra A.

The corresponding factor-algebra $A/(0 : M_A) = \overline{A}$ already is **isomorphically** embedded in the algebra $\pounds(M)$ - the rule $x\overline{a} = xa$ transforms the module M_A to the module $M_{\overline{A}}$ with annihilator $(0 : M)_{\overline{A}} = 0$, i. e., **exact** representation of the algebra $\overline{A}$ is constructed.

Considering submodules in the module M_A one can see that for the cyclic submodule $xA = \{xa \mid a \in A\}$ **annihilator** $(0 : x)_A = \{A \in A \mid xa = 0\}$ is a **right ideal** of the algebra A, and $xA \approx A/(0 : x)_A$ (when algebra A has a unity).

By “symmetry”we obtain a relationship between the left ideals of the algebra A and the annihilators of the left A-modules.

In addition, one can take into account that the algebra A naturally converts to the modules A_A and ${}_AA$, and the submodules of the module A_A are right ideals of the algebra A, and the submodules of the module ${}_AA$ are left ideals in A (and so on).

It is thanks to these relationships that algebras are often described with the help of appropriate modules (or representations).

In particular, one can take into account **simple or irreducible modules** M_A for which M is the only submodule except the zero one, with $xa \neq 0$ for some $x \in M$, $a \in A$, and therefore $M_A = xA$ for $x \neq 0$.

At the same time, it turns out that for an irreducible module its algebra

$E = End(M_A)$ of endomorphisms is a skew- field (that is, it coincides with the group of automorphisms) according to the classical Schur's lemma.

After that we get a description of the primitive algebras, i.e., the algebras Q which have exact irreducible representation or exact simple module M_Q (this is the Jacobson "density theorem" that gives a description of the algebra Q as a "very special" subalgebra in the algebra of linear transformations of the space $M_Q =_E M$ over the corresponding skew-field E).

As a result, there arises one of the most special radicals – the Jacobson **radical** j, naturally associated with irreducible modules.

In this, V.A. Andrunakievich noticed that the class π of primitive algebras is special, and therefore the radical $j = S_\pi$ is **special** too.

At the same time, we obtain a well-known description of radical $j(A)$ as the intersection of the kernels of all irreducible representations of the algebra A or of the annihilators $(0 : M)_A$ of irreducible modules.

In fact, it turned out that the class of irreducible modules is one of the special classes of modules (analogues of special classes of algebras), which was shown by V.A. Andrunakievich by introducing **primary** modules – such modules M_A that $xa \neq 0$ for some $x \in M$, $a \in A$ and $(0 : M)_A = (0 : N)_A$ for any nonzero submodule N_A of the module M_A.

In this, special classes of modules consist of primary modules (compare with special classes of algebras).

Theorem. Primary algebras are precisely algebras that have an exact primary module.

Therefore, for the lower nilradical of b and any algebra A, the radical $b(A)$ coincides with the intersection of annihilators of the primary modules M_A.

A similar construction is obtained **for all** special radicals by means of the corresponding special class of modules (among which there is the class of irreducible modules, as it was already noted).

Moreover, the analogous representations are obtained for all hereditary radicals (and then for all radicals), using appropriate classes of modules.

The arising representations of radicals were obtained in a series of works by V.A. Andrunakievich (together with his pupil and co-author) [24, 28, 29, 30].

At the same time, it turned out that radicals of algebras are represented as the intersection of very special one-sided ideals – it suffices to note that $(0 : M)_A = \bigcap\{(0 : x)_A \mid 0 \neq x \in M\}$ for right modules, which is indicated by the corresponding right ideals $(0 : x)_A$.

In particular, for the radical $j = S_\pi$, the representation $j(A)$ is obtained as the intersection of the right ideals $(0 : x)_A$ for elements of irreducible A modules.

If algebra A has a unity, then the right ideals $(0 : x)_A$ (from the indicated representation) are all maximal right ideals of the algebra A.

Due to this, we obtain structural theorems on primitive algebras, including simple algebras with unity.

Representations of radicals with the help of respective classes of modules are considered in great detail in the monograph [65], where many other results of V.A. Andrunakievich (and his pupils) are mentioned.

On the other hand, in the last series of papers of V.A. Andrunakievich [56-60], radicality and primitivity "modulo" right ideal P have been considered – for $P = 0$, from the proved general theorems of V.A. Andrunakievich the well-known structural theorems on primitive algebras and simple algebras with unity are obtained.

4. Radical algebras and adjoint multiplication

In the proof of structural theorems, as a rule, only semisimple algebras having zero radicals are sufficiently well described (under appropriate restrictions). It is well seen for special radicals and described in great detail in a monograph [44] based on the papers of V.A. Andrunakievich.

However, even in the first papers, V.A. Andrunakievich also proved a number of theorems about radical algebras in the sense of radical j, that is, algebras $R = j(R)$.

Moreover, thanks to the research carried out by V.A. Andrunakievich in his Ph.D. thesis, a certain "parallelism" arose between such radical algebras and skew-fields – algebras Q for which $Q \setminus 0$ is a multiplicative group of invertible elements.

At the same time, there arose various radical algebras, and in the course of subsequent studies, the "variety of radical algebras" also arose.

By the beginning of these studies (1946-1947), the embeddings of algebras without zero divisors into skew-fields were already actively used (similarly to the embeddings of the algebras $\Phi[t]$ of polynomials into the fields $\Phi(t)$ of fractions of the form fg^{-1}, where f, $g \in \Phi[t]$, and $g \neq 0$) as it was for embeddings of semigroups into the groups.

On the other hand, the "circular operation", or **adjoint multiplication**, given by the rule $x \circ y = x + y - xy$ has been just started to be used.

It is almost obvious that for any algebra A we obtain an "adjoint" monoid $A(\circ)$, where zero plays the role of the unity according to the equalities $x \circ 0 = x = 0 \circ x$.

Moreover, we have already "noticed" that $R = j(R)$ if and only if $R(\circ)$ is a group.

It is this operation that V.A. Andrunakievich used to prove theorems on embeddings of algebras into radical algebras, which led to the construction of various and very interesting radical algebras.

According to V.A. Andrunakievich, the element $c \in A$ is radical if $c \circ c^* = 0 = c^* \circ c$ for the element $c^* \in A$ (which is quasi-inverse to the element c).

The element $c \in A$ is **semi-radical** if for all $a, b \in A$ the equalities $a \circ c = b \circ c$, $a = b$, $c \circ a = c \circ b$ are equivalent.

Algebra A is semi-radical if all its elements are semi-radical, and according to the already noted, algebra R is radical, i.e., $R = j(R)$, if and only if all elements of R are radical.

In particular, it turns out that all radical algebras are semi-radical, and there arises the problem of embedding of semi-radical algebras into the radical ones (analogous to the problem of embedding into the skew-fields).

Theorem. There exist semiradical algebras that are not subalgebras of radical algebras (relevant example is analogous to the classical example of algebra without zero divisors that is not embeddable into skew-fields of A.I. Malcev). However, if a semiradical algebra satisfies the condition $(^*)d, g \in R \Rightarrow \exists x,\ y \in R \mid d \circ x = g \circ y$, then algebra R is a subalgebra of algebra $\hat{R} = R \circ R^*$ consisting of added fractions of the form $a \circ b^*$ with the equality rule $a \circ d^* = c \circ g^* \Leftrightarrow d \circ x = c \circ y$ (see $(^*)$). In this, operations with added fractions are performed according to the written out rules:

$b_1 \circ z = a_2 \circ t \Rightarrow (a_1 \circ b_1^*) \circ (a_2 \circ b_2^*) = (a_1 \circ z) \circ (b_2 \circ t)^*$,
$c = d \circ x = g \circ y \Rightarrow a \circ d^* + b \circ g^* = (a \circ x - c + b \circ y) \circ c^*$,
$\forall \alpha \in \Phi \mid \alpha(a \circ b^*) = (\alpha(a - b) + b) \circ b^* \in R \circ R^*$,

then it should be always taken into account that $xy = x + y - x \circ y$. In this situation, **algebra** $\hat{R} = R \circ R^*$ is **radical**, since the equalities $(a \circ b^*)^* = b \circ a^*$ are always true. The embedding is performed according to the rule $r = r \circ 0^* = (r \circ t) \circ t^*$. Moreover, for any radical algebra Q each homomorphism $\varphi : R \to Q$ of algebras **always and uniquely** extends to the homomorphism $\hat{\varphi} : hatR \to \bigcup$ of radical algebras now. Therefore, the radical algebra $\hat{R}$ is uniquely defined to within identical on R isomorphism.

In the course of the proof of this theorem, V.A. Andrunakievich noted that algebras can be considered as "new" algebraic systems in which, instead of multiplication, the adjoint multiplication is considered, due to which the adjoint monoid $A(\circ)$ is obtained. In this, the distributivity laws are rewritten in the "more complicated" form $x \circ (y - t + z) = x \circ y - x \circ t + x \circ z$ and similarly $(y - t + z) \circ x = y \circ x - t \circ x + z \circ x$. This is what led to the above construction $R \subseteq \hat{R} = R \circ R^*$ of embedding algebras into radical algebras under the indicated restrictions.

As a simple corollary, it turns out that the commutative algebra K without zero divisors and without unity is **always** isomorphically embedded into the radical algebra $\hat{K} = K \circ K^*$, since for K the conditions of $(^*)$ and semiradicality are performed. The constructed algebra $\hat{K}$ is the subalgebra of arising field $Q_{cl}(K)$ of fractions of the algebra K. This leads to a variety of radical algebras that are algebras without zero divisors. On the other hand, if N is a **nilalgebra**, i.e., for $a \in N$ it is always $a^{n+1} = 0$ for some natural number $n = n(a)$, then the equalities $a \circ a^* = 0 = a^* \circ a$ are also obtained when $-a^* = a + a^2 + \ldots + a^n$. Therefore, all the nilpotent elements are radical and all nilalgebras are radical algebras. After this, it can be noticed that in the finite-dimensional algebras the radical $j = S_\pi$ coincides with the classical nilpotent radical, since all finite-dimensional radical algebras are nilalgebras. Using the construction from the theorem, more facts can be proved.

Proposition. According to algebra $K = tA = \langle t \rangle$ of polynomials with zero free term, radical algebra $J = \langle t \rangle^* = \hat{K} = K \circ K^*$ is constructed (according to the theorem).

In this:

a. Algebra R is radical if and only if for each $r \in R$ there exists a homomorphism $\varphi_r : J \to R$ for which $\varphi_r(t) = r$, which takes into account the specificity of algebras $K = \langle t \rangle$ and $J = \langle t \rangle^*$ (already radical).

b. For algebra J in the field $Q_c l(K)$ of fractions the following equality is true:

$J = \{a(e - b)^{-1} \mid a, b \in K\}$, because $c \circ b^* = (c - b)(e - b)^{-1}$ for all $c, b \in K$ and unity $e \notin K = tA$.

Therefore, only the powers $J^{n+1} = t^n J$ of algebra J that form a strictly decreasing chain are nonzero ideals of J.

c. Radical algebra R is a nilalgebra if and only if algebra J **is not** isomorphically embedded into R algebra. In particular, if the radical algebra R is a subalgebra of a finitely generated algebra, and the main field Φ is uncountable (as the field of real numbers is), then R is a nilalgebra, since the dimension is $dimJ \geq \mid \Phi \mid$.

The statements **a, b** are in fact proved in the first papers of V.A.

Andrunakievich [1-4] and are a simple consequence of the above theorem. It is not less obvious that the algebra J is infinitely dimensional, and applying **a, b**, we see that if the algebra J **is not** embedded isomorphically into the radical algebra R (for example, when the algebra R is a finite dimensional one), then for $r \in R$ and the related homomorphism φ_r the following inclusion is always true: $J^n \subseteq Ker\varphi_r$, and therefore, $r^n = 0$, i.e. R is a nilalgebra. Moreover, applying the well-known basis of the field $Q_cl(K) = \Phi(t)$, we obtain the linearly independent $\{t(e - \alpha t)^{-l} \mid \alpha \in \Phi\} \subseteq J$, and therefore it is always $dimJ \geq \mid \Phi \mid$. It remains to note that all finitely-generated (and countably generated) algebras have at most countable dimension.

This result was "rediscovered" by other authors 10 years after the work of V.A. Andrunakievich.

After 25 years, the attention was paid to the fact that radical algebras form a "manifold of algebras" – with the additional operation $x \to x^*$ of taking a quasi-inverse. On the other hand, in the joint papers of V.A. Andrunakievich, the construction of arising theory of variety of radical algebras was continued, which led to the construction of very interesting radical algebras that are "free in some manifold". In fact, we did not notice much else (for example, the theory of algebras without nilpotent elements), but it is already clear that the ideas of even the first works of V.A. Andrunakievich "continue to work".

5. Conclusions and comments

Many of the results of V.A. Andrunakievich have already been included in the monographs, beginning with the monograph [62] of Divinsky (published in Canada) and concluding with the last monograph [65] on the theory of radicals. The most well-known are the results of research that are included in the doctoral thesis of V.A. Andrunakievich (defended in 1958 at Moscow State University).

According to these researches [5] and the works [5-22], it turned out that it is just special and subidempotent radicals that lead to a variety of structural theorems. Since that time (early 60s of the last century) V.A. Andrunakievich has already become a leading world specialist in structural theory and the theory of radicals of algebras or rings (associative ones). Moreover, it turned out that the ideas and working statements of V.A. Andrunakievich are very useful in similar domains of algebra. This is reflected, for example, in monographs [63, 65] and in a number of papers of algebraists from Novosibirsk, where a number of structural theorems on alternative and Jordan algebras are proved with the help of "Andrunakievich lemma", "Andrunakievich radikal" and "Andrunakievich's variety, similar to the associative ones".

In fact, after the investigations of V.A. Andrunakievich and his pupils, the theory of special radicals and torsions has been developed by many authors for such algebraic systems as semigroups with zero, almost-rings, or even multi-operator algebras. For semigroups with zero, this was done by the V.A. Andrunakievich pupil R.S. Grigor (Florya), and continued by a number of Hungarian and German algebraists. For more general systems, the representations of radicals with representations in papers [28, 29, 30] were very useful, which is also reflected in the monographs [64, 65].

By this time (early 70s of the last century), due to the typical of V.A. Andrunakievich care of people a sufficiently large number of pupils was ensured by ideas and work, and many of the results obtained are reflected in the monograph [44], where the theory of hereditary radicals has been developed "in almost all good enough categories".

Thanks to the general theory, it has been shown that exactly special radicals are most naturally connected with M ideals and structural theory (see the first section). This interconnection is a bit weaker for supernilpotent or "weakly special"radicals. This was the continuation of the works of V.A. Andrunakievich [20-23] and allowed constructing theory of special radicals in semigroups with zero (where, of course, it is necessary to take into account the specificity too).

After the "duality theorems" of V.A. Andrunakievich (see the first section and the works [19-23, 25, 39-44]), the study of "lattices of radicals"continued (by many authors). In addition, in algebras with sufficiently weak "finiteness conditions", the supernilpotent radicals coincide with the special ones, according to V.A. Andrunakievich. However, in the general case "there are a lot" of supernilpotent but not special radicals, which is reflected in the monographs [44, 64, 65] and was the solution to the problem set in the monograph [62] in connection with the works of V.A. Andrunakievich.

The ideas of the first researches of V.A. Andrunakievich about radical algebras (from his Ph.D. dissertation defended in 1947), which are the beginning of the structural theory of radical algebras, continue to work. The interest to this direction increased significantly when it was found that "radical algebras form manifold" (and the free radical algebra was constructed by the English algebraist P.Cohn in view of the first works of V.A. Andrunakievich and his works on embeddings algebras into the bodies). In this way, very interesting "radical algebras that are free in some manifold" also appeared together with the concrete variety of radical algebras considered in the joint papers of V.A. Andrunakievich [51-55], which is reflected in [61] as well.

Naturally, the joint works on "radicality relative to right ideals", where generalizations of classical structural theorems are also obtained, are associated with mentioned above as well, as it is noted in the works [41-44, 47, 48, 56-59]. On the other hand, the study of some other special radicals has also been continued – in this way structural theory of algebras without nilpotent elements has been constructed, where variety of strictly regular algebras also appeared [36-39, 41, 46-52].

Studies of radical algebras and varieties are still far from complete and can be continued. A number of problems related to the structural theory of radicals of rings or algebras has been solved in the course of the investigations already carried out. However, problems still exist that are related to locally nilpotent radical that arose in the investigations of V.A. Andrunakievich (see [15, 25]) and reflected in monographs [63, 65]. Let's hope that these problems will be solved by his pupils or pupils of his pupils.

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