About researches of V. A. Andrunakievich (on the structural theory and theory of radicals. I)

Yu. M. Ryabukhin

February 10, 2018

Abstract

This is the first part of overview published in Russian in the book devoted Academician V.A. Andrunakievich (*Academicianul Vladimir Andrunachievichi: Bibliografie*, Institutul de Matematica si Informatica, Chisinau, 2009).

2000 Mathematics Subject Classification: 16Nxx

Key words and phrases: ring, radical, algebra.

Results of the works of V. A. Andrunakievich are very diverse, but they are mainly related to the structural theory of associative algebras (or rings) and the theory of radicals as one of the instruments of structural theory.

Most of the main results are published in such journals as "Izvestiya AN SSSR" and "Reports of the Academy of Sciences of the USSR", "Uspekhi Matematicheskikh Nauk" and "Matematicheskii Sbornik", "Izvestiya AN MSSR" and are well known to specialists working in the indicated fields of modern algebra and close to them.

Moreover, already in the first papers of V. A. Andrunakievich, close interconnection between theory of radicals and the structural theory is revealed, and the developed by him theory of special radicals allowed us to generalize almost all known (by the appropriate time) structural theorems and to prove a number of new theorems, which reflect specific features of rings or algebras being examined.

In fact, even a fluent review or analysis of the works of V. A. Andrunakievich shows that the results obtained by him reveal different possibilities of the development of the structural theory of algebras, starting with the foundations of this theory and continuing with the study of the varieties of close algebraic systems.

Moreover, the ideas even of the first works of V. A. Andrunakievich continue to work even now, and many of the results obtained by him are reflected in monographs, including the most recent ones.

All this allows us to assert that V. A. Andrunakievich was one of the world's leading experts in the developed by him structural theory and theory of radicals.

At the same time, many working statements and theorems proved by V. A. Andrunakievich, have already become classical and understandable to all those interested in construction or description of algebras, which we try to "prove" or explain.

We will consider associative algebras over some field F.

Such algebras as algebra $F[t]$ of polynomials and algebras of series, algebras of linear transformations of spaces (over the field F), algebra $[F]_n$ of matrices we consider as the well-known ones.

In fact, algebras are very diverse and we can, for example, take into account more specialized algebras of triangular or Block-triangular matrices, and in addition to fields, allow skew-fields to be included (for example, the skew-field of quaternions) in which nonzero elements form a group of invertible elements, but multiplication is not commutative.

1. Structural theory and theory of radicals of algebras

In description a finite-dimensional algebra A, one can apply its embeddings or homomorphisms into the algebras $[\Phi]_n$ of matrices.

In addition to ideals (as kernels of homomorphisms), one-sided ideals also arise: right ideals as sums of principal right ideals of the form $aA = \{ax \mid x \in A\}$ and left ideals.

Due to the finite dimensionality, in the algebra A, the **classical radical** $rad(A)$ is constructed as the largest of ideals N that are nilpotent algebras, that is, such that $x_1x_2\ldots x_n = 0$ for all $X_i \in N$ and some natural number $n \geq 2$. Passing to the quotient algebra $\overline{A} = A/rad(A)$ we get that $rad\overline{A} = 0$, and, on the other hand, if $R = rad(A)$, then, of course, $rad(R) = R$.

In classical structural theory of finite-dimensional algebras (Wedderburn, Molin, A. I. Mal'tsev and many others), the semisimple algebras Q for which

$rad(Q) = 0$ are completely described, and **simple** algebras with unity element (in which the only non-zero ideal is the whole algebra) turned out to be the leading special case.

Namely, simple algebras have the form of algebras of $[T]_n$ matrices over skew-fields T (and if the ground field Φ is algebraically closed as a field of complex numbers, then $T = \Phi$).

After this it turns out that for $rad(Q) = 0 \neq Q$, $Q = \bigoplus_1^m Q_i$ is a finite direct sum of simple algebras $Q_i = [T_i]_{n_i}$. This finishes **a description of all classical semisimple algebras** (up to the description of algebras that are skew-fields).

The finite-dimensional radical algebras $R = rad(R)$ receive only "some description" - in such algebras all elements of $r \in R$ **are nilpotent**, i. e., $r^n = 0$, and therefore nilalgebras are obtained.

But from the finite dimensionality it follows that all these nilalgebras are nilpotent, and therefore coincide with their radical.

Applying matrix representations, we obtain corresponding descriptions in the form of algebras of block-triangular matrices "with zeros on the main diagonal ". All this is visually portrayed

$$\begin{pmatrix} [T_1]_{n_1} & 0 & 0 \\ 0 & [T_2]_{n_1} & 0 \\ 0 & 0 & \ddots \end{pmatrix} \qquad \begin{pmatrix} 0 & * & * \\ 0 & 0 & * \\ \cdots & 0 & 0 \end{pmatrix}$$

$$Q = \bigoplus_1^m Q_i,\ rad(Q) = 0 \qquad R = rad(R),\ R^m = 0$$

in the form of written matrix representations, where the diagonal blocks of the semisimple algebra Q are its minimal ideals, which are always simple algebras.

In this case, with some refinement of the specifics of the ground field, the splitting of any algebra A is obtained - it turns out that $A = S + R$, where $R = rad(A)$, the algebra S is a semisimple subalgebra, $S \approx A/R$ (and this is refined in the corresponding classical structural theorems of Wedderburn-Molin-Mal'tsev).

In the course of development of the structural theory, algebraists began to apply weaker restrictions instead of the finite-dimensionality condition.

In particular, after Artin and Emma Noether algebraists began to consider the **minimality condition** for left ideals, when strictly decreasing chains of left ideals break off at a finite step (and similarly for right ideals).

It turned out that classical nilpotent radical always exists, and the above matrix representations are obtained, that is, semisimple algebras are again described as **finite** direct sums of matrix algebras over matrices (i.e., simple algebras), and for radical algebras, as the nilpotent ones, we again obtain representations in the form of algebras of triangular matrices (and so on), but the skew-fields are not necessarily finite-dimensional.

At the same time, the corresponding minimal left (or right) ideals began to be actively applied.

We have already noted that the most important case of left ideals is **the main left ideals** - they have the form $Aa = \{xa \mid x \in A\}$.

On the other hand, the algebras of matrices arise as algebras of linear transformations of the corresponding spaces P, but if the space P is infinite-dimensional, then it is necessary to specify the specificity of the emerging matrices, since their size can turn out to be infinite.

All this was taken into account in the series of papers [5-12, 16, 18] of V.A. Andrunakievich, and it turned out that to prove the structural theorems "almost the same as the classical ones", the conditions of minimality for the principal left ideals are sufficient.

Theorem. Let the algebra A satisfy the chain termination condition $Aa_1 \supset Aa_2 \supset \dots$ of principal left ideals (that is, the minimality condition only for such "very special" left ideals), then the following statements hold:

a. If there are nonzero nilpotent ideals in the algebra A, then its non-zero **classical radical** $rad(A)$ (or nilradical) also arises , and it is nilpotent, and therefore the algebra$R = rad(A)$ has a corresponding "triangular representation" with a finite number of blocks and nonzero diagonal blocks. Moreover, if $A \neq rad(A)$, then $rad(A/rad(A)) = 0$ and the corresponding non-zero algebra $\overline{A} = A/rad(A)$ has no nilpotent ideals except the zero ideal 0.

b. If A is a simple algebra (i.e., $xy \neq 0$ for some elements $x, y \in A$ and the only ideal in A is the algebra A), then A is the algebra of matrices of linear transformations of **finite** rank of the corresponding space over some skew-field T, naturally related to algebra A (and all this is refined), but the size of the matrices is not necessarily finite (as for the algebra of linear transformations of finite rank, but for an infinite-dimensional space $P =_\Phi P$). If the algebra A has a unity element, then $A \approx [T]_n$ for the corresponding skew-field T and some natural number $n \geq 1$.

c. If the algebra A **is semisimple** (that is, it does not have nilpotent ideals), but not simple, then $A = \bigoplus_i Q_i$ is the direct sum of simple algebras (described in **b**) and these algebras Q_i run through **all** minimal ideals of the algebra A, but the number of these minimal ideals Q_i can be infinitely large (i.e., in the corresponding matrix representation it can turn out to be "infinitely many" diagonal blocks - compare with the scheme outlined above). However, if the algebra A has a unity element then $A = \bigoplus_1^m Q_i$ for some natural number $m \geq 2$ and in this situation all $Q_i = [T_i]_{n_i}$ for some $n_i \geq 1$ and T_i.

When proving these and even more general structural theorems (see [6-12, 15, 16, 18]) V.A. Andrunakievich used, of course, many of the results of other authors, which is proved by the analysis that he had carried out beginning in [5] and continued in the following papers. This is what allowed him to generalize the well-known structural theorems. In particular, the role of **idempotents** - elements $e = e^2 \neq 0$ (see [18]) has been revealed, since the minimal left ideal L of a semisimple algebra Q always has the form $L = Ae$ for some idempotent $e \in L$, and eAe turns out to be skew-field, that the "classics" already noticed as well.

One of the obvious corollaries of this theorem is obtained for the case when in the algebra A **there are no nilpotent elements**, i.e., when $a^2 \neq 0$ also follows from $a \neq 0$. In this situation, of course, the radical is equal to 0, and under the condition of minimality for the principal left ideals, it turns out that A is a direct sum of not necessarily finite number of skew-fields.

The best-known particular case is Dedekind's classical theorem on finite-dimensional commutative algebras, which turn out to be "finite" direct sums of fields (extensions of the fundamental one).

In fact, V.A. Andrunakievich has proved many other structural theorems, since algebraists by this time (50th years of the last century) have already begun to study also the algebras, in which "the classical radical no longer exists", since the nilalgebras (where all the elements are nilpotent) are not necessarily nilpotent.

In connection with this situation, various generalizations of the classical radical arose, and sometimes "radicals opposite to the classical".

Therefore, there was a need for a "general theory of radicals", which was created in the works of A.G. Kurosh, the scientific supervisor of the first investigations of V. A. Andrunakievich.

Thanks to the research of V. A. Andrunakievich, the **theory of hereditary radicals** was developed, among which special radicals were allocated by him - these radicals are most often used when proving structural theorems.

More about this (and the history of the development of the structural theory of rings and algebras) is said in monographs [44, 62, 65] and in [4-17, 22] by V.A. Andrunakievich, but we note some details and basic ideas that led to special radicals of associative algebras.

Instead of simple algebras, V.A. Andrunakievich proposed to consider the **primary** algebras A in which the inequality $0 \neq xAy$ is always true for nonzero $x, y \in A$.

In such algebras **for nonzero ideals** $J(J \triangleleft A)$ **always** $J^2 \neq 0$, since for nonzero ideals B, C we always have$BC = \{\sum b_i c_j \mid b_i \in B, c_j \in C\} \neq 0$.

Namely from primary algebras **special classes** of M algebras are constructed such that from $A \in M$ and $J \triangleleft A$ it follows always $J \in M$, and for the primary algebra C that contains the algebra $A \in M$, $A \neq 0$ as an ideal, we always obtain $C \in M$.

After this (according to the construction indicated by A.G. Kurosh and Amitsur, who also constructed a general theory of radicals), the **upper radical** S_M defined by class M is constructed.

For the special class M this means that in each algebra A its ideal $S_M(A) = \bigcap\{J \triangleleft A \mid A/J \in M\}$ is constructed as the intersection of all the indicated M-ideals of algebra A.

In this case, of course, the indicated algebras A/J are primary, since all algebras from M are primary.

As a result, there arises **special** (according to V.A. Andrunakievich) **radical** S_M, defined by the given special class M.

In this case, always $S_M(A/S_M(A)) = 0$ and always $S_M(S_M(A)) = S_M(A)$ for all radicals in the sense of A.G. Kurosh, but in addition it turns out that when $r = S_M$ for the ideal $J \triangleleft A$ it is always $r(J) = J \bigcap r(A)$.

At the same time (according to A.G. Kurosh) for the radical $r = S_M$ we construct the class $\Re(r)$ of all r-radical algebras $R = r(R)$ and the class $\Im(r)$ of all r-semi-simple algebras Q for which $r(Q) = 0$.

These classes always determine each other, since for any algebra A the equality $\bigcap\{J \triangleleft A \mid S_r(A/J) = 0\} = r(A) = \sum\{R \triangleleft A \mid S_r = r(R)\}$ is true.

In particular, for a special radical in the S_M-semi-simple algebra Q, the only nilpotent ideal is 0, since in the primary algebra for ideals we always have $J^m = 0 \Rightarrow J = 0$ for $m \geq 2$.

Ideals of the algebra Q also turn out to be semisimple algebras, and the ideals of radical algebras are radical, by specifics of S_M.

Remark 1. A non-zero prime algebra Q can have many different non-zero ideals - the most famous example is the algebra $\Phi[t] = A$ of polynomials and all of its nonzero ideals, always having the form $gA = \{gf \mid f \in A\}$ (they all are integral domains, i.e., algebras without divisors of zero - if $x \neq 0 \neq y$, then $xy \neq 0$).

However, the primary finite-dimensional algebra $Q \neq 0$ is a simple algebra with unity and has the form $Q = [T]_n$ for some skew-field T and some natural number $n \geq 2$ if Q is not a skew-field. And in the algebra T, which is a skew-field, there are no nonzero one-sided ideals (right or left), since $Tq = T = qT$ for $0 \neq q \in T$.

Remark 2. Among the special radicals there is the smallest one - this is the **lower nilradical** $b = S_\Pi$, constructed from the class Π of all prime algebras.

In this case semisimple algebras are exactly algebras without nonzero nilpotent ideals. **Upper nilradical** k is a special one too, for which all nilalgebras are radical, i.e., algebras consisting only of nilpotent elements.

At the same time various **nilradicals** arise, i.e., such radicals s in the sense of A.G. Kurosh, that $b(A) \subseteq s(A) \subseteq k(A)$ for all algebras A.

In this case many nilradicals are special, i.e., the corresponding class $\Pi \bigcap \Im(s)$ of primary s -semi-simple algebras turns out to be a special class of algebras.

In particular, **locally nilpotent radical** l is a special one too, for which all locally nilpotent algebras are radical (that is, algebras in which all finitely generated algebras are nilpotent).

It follows from the above that special radicals are very diverse, and according to the natural order it turns out that $b \leq l \leq k$.

At the same time, according to Remarks 1 and 2, for finite-dimensional algebras A we obtain the classical radical $rad(A) = b(A) = k(A)$; i.e., in this case all nilradicals coincide.

Moreover, there are other special radicals that coincide in the finite-dimensional case with the classical (nilpotent) radical, many of which are indicated or determined by V.A. Andrunakievich.

For example, if a special class M consists of only algebras with unity element, then $M \subseteq \Pi_1$ for the class of all simple algebras with unity element and all classes $M \subseteq\ Pi_1$ are always special.

Of course, the radical S_M for $M \subseteq \Pi_1$ coincides in finite-dimensional algebras with classical radical (according to Remark 1), and many of the special radicals have the same property - this was noticed by V.A. Andrunakievich in [14, 17, 22] and continued in the works of many other authors, including the first monograph [62] on the theory of radicals.

On the other hand, if in the algebra Q there is the smallest nonzero ideal C, and $C^2 = C$, then subdirectly indecomposable **algebras with idempotent core** C form a special radical $S_{\Pi_0} \leq S_{\Pi_1}$, since the core C is a simple algebra (but not necessarily having the unity element, see the theorem).

It is easy to see that all locally nilpotent algebras S_{Π_0} are radical and therefore it turns out that $L \leq S_{\Pi_0} \leq S_{\Pi_1}$. More complicate to see that the following is true:

Proposition. If S_{Π_0} is a radical algebra R satisfying the maximality condition for ideals (that is, strictly increasing chains of ideals break off at a finite step), then the algebra R is **nilpotent**, i.e., $R^m = 0$ for some natural number $m \geq 2$.

In particular, under this condition for break, locally nilpotent algebras are nilpotent, and therefore the special radicals S_{Pi_0}, l, S_{Π_1} coincide in finite-dimensional algebras with classical (nilpotent) radical.

This is one of the well-known "working statements" of V.A. Andrunakievich, and if we apply (following V.A. Andrunakievich) **annihilators**, similar results are obtained under weaker restrictions.

That is why the radical S_{Π_0} is called the radical of Andrunakievich (see the monographs [63, 65]), and various special radicals lead to various structural theorems under "comparatively weak restrictions".

More details can be found in the works of V. A. Andrunakievich [8-19] and in monographs [44, 65], where there are many results, theorems and "working statements" of V.A. Andrunakievich.

It is very surprising, but many of these "working statements" can be very simply proved and are very often used (even in the works of many other authors, and sometimes, after some refinement, in arbitrary not necessarily associative algebras).

The most famous is

Lemma of V.A. Andrunakievich (see the monograph [65], published in 2004). Let$J \triangleleft B \triangleleft A$, i.e., J is an ideal of the algebra B, and B is an ideal of the algebra A. Then:

a. If J_A is an ideal of algebra A generated by J, then $J_A^3 \subseteq J$.

b. If the quotient algebra B/J is a semi-prime one (or without nonzero nilpotent ideals), then J is an ideal of the algebra A.

In fact, $J_A = J + AJ + JA + AJA$ and therefore $J_A^3 \subseteq BJB \subseteq J$, that is, **a.** is true.

But then **b** is also true because of the specifics of algebras without nilpotent ideals.

This is used to prove the equality $b(B) = B \bigcap b(A)$ for the lower nilradical B (see Remark 2) and for all special or supernilpotent radicals.

In the case under consideration (associative algebras over a field) it turns out that the hereditary radical r is either **hypernilpotent** i.e., all nilpotent algebras are r-radical, i.e. $r \geq b$, or **sub-idempotent**, i.e., all r-radical algebras $R = R^2$ and this is equivalent to the fact that all nilpotent algebras turn out to be r-semisimple.

Moreover, all special radicals are hypernilpotent and the sub-idempotent radicals are opposite to hypernilpotent ones.

At the same time, "there is a duality for hereditary radicals", introduced into consideration by V. A. Andrunakievich (but more on this later), and the corresponding sub-idempotent radicals are also "very often" used together with special radicals to prove structural theorems.

Theorem. Consider only the hereditary radicals r, i.e., such that $r(B) = B \bigcap r(A)$, when B is an ideal of the algebra A. Then:

a. Among the radicals s such that for a given radical r the equalities $r(A) \bigcap s(A) = 0$ hold for all algebras A, there always exists a **largest** radical r'. Moreover, the radical $s = r'$ is dual, that is, $s = s'' = (s')'$. The class $\Re(r')$ of all r'-radical algebras coincides with the class of all strongly r-semisimple algebras, such that $r(\overline{Q}) = 0$ for all homomorphic images of $\overline{Q}$ of the corresponding algebra $Q = r'(Q)$. The equality written above can be rewritten in the form $r(s(A)) = s(r(A)) = 0$ by symmetry.

b. The largest sub-idempotent radical is the hereditarily idempotent radical f, i.e., algebras $R = f(R)$ are algebras such that $F = F^2$ for all ideals F of the algebra R. Therefore, for a hypernilpotent radical s, the dual radical s' is always sub-idempotent, i. e., equality $s' \leq b' - f = f''$ holds (according to **a**), since $s \geq b$ for the lower nilradical $b = S_{\Pi}$. For sub-idempotent radicals r the dual radical r'' is always hypernilpotent and **is special** - the equality $r' = S_{\Pi(r)}$ holds for the special class $\Pi(r)$ of all subdirectly indecomposable algebras with an idempotent core $C = r(C)$.

In other words, $r' = S_{\Pi(r)} \geq S_{\Pi_0} = a = a'' = b'' \geq b$ (but $a \neq b$), i.e., the radical a is the smallest dual hypernilpotent (and special) radical, since $a = f'$.

All of the above is proved in the papers of V. A. Andrunakievich [17, 19, 20, 21, 22] and his doctoral dissertation, and then applied to prove a number of structural theorems related to the corresponding sub-idempotent radicals, which is reflected in the monograph [44].

The results obtained were applied or generalized by many authors, as it is shown in the monographs [62-65], where the lattices of radicals were studied and many of the results of V. A. Andrunakievich (and sometimes of his pupils too) are given in many details.

Moreover, it turned out that the ideas of V. A. Andrunakievich and his "working statements" also work in "not necessarily associative algebras".

2. Additive theory of ideals

One of the most famous is the theorem of arithmetic - the natural number is always represented in a unique way in the form $n = p_1^{k_1} \dots p_r^{k_r}$ of products of powers of simple (pairwise distinct) numbers.

Translating this theorem into the language of ideals $mZ = \{mzSz \in Z\}$ of the ring Z of integers, we find that there exist unique representations $nZ = p_1^{k_1}Z \bigcap \dots \bigcap p_r^{k_r}Z$ of the corresponding ideals in the form of intersection of **primary** ideals - the ideals of the form p^kZ (for prime numbers p).

Moreover, for the ideal nZ its radical or root $\sqrt{nZ} = \{z \in Z \mid z^m \in nZ$ $for\ m \geq 1\}$ is constructed, and for the primary ideal p^kZ its radical is the **unique** maximal ideal containing p^kZ, and this is the ideal pZ for the corresponding prime number p.

It can be seen that it always follows from $xy \in pZ$ that $x \in pZ$ or $y \in pZ$, and if $xy \in p^kZ$ and $y \notin p^kZ$, $x \neq 0$, then $x^s \in p^kZ$ for some $s \geq 1$ - this characterizes the primary ideals and their radicals.

It is not less clear that $\sqrt{nZ} = \bigcap_1^r p_iZ$.

It turned out that similar results are obtained for commutative rings (or algebras) with the maximality condition for ideals (the best known example, apart from the ring Z, is the algebra of polynomials of a finite number of variables, according to Hilbert's theorem).

In this ring A for ideal B the radical $\sqrt{B}$ consisting of $a \in A$ such that $a^m \in B$ for some $m \geq 1$ is always constructed.

If we take into account products of ideals, then, thanks to the maximality condition, it turns out that always $(\sqrt{B})^m \subseteq B$ for some sufficiently large number $m \geq 1$.

After this, there arise **simple** ideals P, i.e., such that $B \subseteq P$ or $C \subseteq P$ follows from $BC \subseteq P$ (these are analogues of prime ideals), and then **primary** ideals of Q, for which $\sqrt{Q}$ is a prime ideal (and this is an analogue of the primary ideals p^kZ).

According to Emma Noether the following theorems are true:

Existence theorem. For an ideal B, there always exists a representation in the form of an intersection $\bigcap_1^r Q_i$ of a finite number of primary ideals Q_i.

Intersection theorem. Intersection of primary ideals with the same radical P is a primary ideal Q with the same radical $P = \sqrt{Q}$.

Uniqueness theorem. For an ideal B, there exists an irreducible representation $B = \bigcap_1^r Q_i$ in the form of intersection of primary ideals, i.e., such that all $\bigcap_{i \neq j} Q_i \neq B$.

The irreducible representation is unique; therefore, the set of prime ideals $P_i = \sqrt{Q_i}$ is also unique, for which the unique and irreducible representation $\sqrt{B} = \bigcap_i P_i$ for radicals is also obtained.

These theorems are fundamental for Noetherian primarity, and in fact, many other beautiful "work" statements about primary and simple ideals are also obtained.

After that, a situation appeared that resembled something that happened in the structural theory: the search for "generalizations" of classical Noetherian primarity to the noncommutative case began, but under the condition of maximality for the ideals (or unilateral ideals) of the rings under consideration.

However, the necessary generalizations were not obtained for relatively long time (about fifty years), then "tertiarity" arose (in the works of the French algebraists Leonce Lesieur and Robert Croisot) as one of the possible generalizations, and numerous "almost generalizations" arose either not coincident in commutative rings with Noetherian primarity or such that one of the "defining" theorems mentioned above was violated.

V. A. Andrunakievich has joined the search for possible generalizations, and then his disciples (I.M. Goian was one of the first) too.

After clarifying the statement of problems, V. A. Andrunakievich explained the emerging "difficulties" (with the active help of pupils) - a few unexpectedly it turned out that it was true the following

Theorem. When considering the generalizations of classical primarity to a non-commutative case, there is only one generalisation – primarity, for which the existence theorem, the intersection theorem and the uniqueness theorem hold.

In this case, the ideals Q arise as primary ideals (in corresponding already non-commutative rings), that are irreducible relatively intersection (i.e., such that $B \supseteq Q$ or $C \supseteq Q$ follows from $B \cap C \supseteq Q$).

This was proved in a series of works by V. A. Andrunakievich [31-35] and "everything explained".

Moreover, in the definition of irreducible ideals, only the specificity of the lattice of ideals is just taken into account (and if the lattice satisfies the maximality or minimality condition, then this already allows us to prove the "Existence theorem").

Therefore, in the "final" paper [35] (published in "Izvestia of the Academy of Sciences of the USSR"), an analogous theorem was proved exactly for lattices. In this case the right or left quotients (of ideals or elements of the emerging multiplicative lattices) were the main tool, according to the formulation by V. A. Andrunakievich the way of solving the problem.

We note that the corresponding "quotient" or conditions for the termination of chains of quotients have already been applied by V.A. Andrunakievich in the proof of structural theorems (see, for example, [15, 18, 26, 27], and in more detail - a monograph [45]).

At the same time, it turned out that an appropriate "primary theory" can be constructed for many algebraic systems (for subgroups of groups, subsemigroups of semigroups, submodule of modules, etc.).

On the other hand, restrictions can be weakened, for example, only the ideals of an algebra with the maximality condition for ideals can be considered, and to weaken the requirements of "defining theorems".

As a result, various generalizations of the "diprimarity" type are obtained only for two-sided ideals (and in the commutative case the classical Noetherian primarity is obtained), and sometimes (for stronger restrictions), generalizations of classical Artin-Rees theorems are obtained.

In this area, I. M. Goyan - the pupil of V. A. Andrunakievich (there are also others) worked most actively and works. He considered generalizations that do not necessarily coincide in the commutative case with classical primarity.

References

1. Andrunakievich V. A. Semi-radical rings. Thesis of the Candidate of Phys.-Math. sciences. Moscow, Moscow State University, 1946.

2. Andrunakievich V. A. Semi-radical and radical rings. UMN, 1947, Vol. 2, no. 1.

3. Andrunakievich V. A. Semi-radical and radical rings. Reports of the USSR Academy of Sciences, 1947, vol. 55, no. 1.

4. Andrunakievich V. A. Semi-radical rings. Izvestiya AN SSSR, Ser. Mat., 1948, Vol. 12, no. 2.

5. Andrunakievich V. A. To the definition of a radical of a ring. Izvestiya AN SSSR, Ser. Mat., 1952, Vol. 16, no. 2.

6. Andrunakievich V. A. Linear equations in infinite-dimensional spaces. Uch. Notes of Chisinau Univesity, 1952, Vol. 5.

7. Andrunakievich V. A. The radical in generalized Q-sets. Izvestiya AN SSSR, Ser. Mat., 1954, Vol. 18, no. 5.

8. Andrunakievich V. A. Rings with the minimality condition for ideals. DAN SSSR, 1954, Vol. 98, no. 3.

9. Andrunakievich V. A. Rings with minimal two-sided ideals. DAN USSR, 1955, vol. 100, no. 3.

10. Andrunakievich V. A. Rings with the minimality condition for two-sided ideals. UMN, 1955, Vol. 10, ed. 2.

11. Andrunakievich V. A. Rings with annular condition. Izvestia of the Academy of Sciences of the USSR, ser. Mat., 1956, vol. 20, no. 4.

12. Andrunakievich V. A. Biregular rings. Mat. Sbornik, 1956, t. 39, no. 4.

13. Andrunakievich V. A. On some classes of associative rings. UMN, 1956, Vol. 11, ed., 4.

14. Andrunakievich V. A. To the theory of radicals of associative rings. DAN USSR, 1957, vol. 111, no. 3.

15. Andrunakievich V. A. Anti-simple and strongly idempotent rings. Proceedings of the USSR Academy of Sciences, ser. Mat.,1957, vol. 21, no. 1.

16. Andrunakievich V. A. Modular ideals, radicals, and semisimplicity of rings. UMN, 1957, Vol. 12, ed. 3.

17. Andrunakievich V. A. Radicals of associative rings, I. Mat. Sbornik, 1958, vol. 44, no. 2.

18. Andrunakievich V. A. On the separation of the radical of a ring as a direct summand. UMN, 1958, Vol. 13, ed., 5.

19. Andrunakievich V. A. Radicals of associative rings. The thesis of the doctor of f.-m. Sciences, Moscow, Moscow State University, 1958.

20. Andrunakievich V. A. Radicals of weakly associative rings. DAN USSR, 1960, vol. 134, no. 6.

21. Andrunakievich V. A. Radikals of weakly associative rings, I. Izvestiya of Mold. Branch of the Academy of Sciences of the USSR, 1960, no. 10.

22. Andrunakievich V. A. Radicals of associative rings, II. Examples of special radicals. Mat. Sbornik, 1961, vol. 55, no. 3.

23. Andrunakievich V. A. On one characteristic of the supernilpotent radical. UMN, 1961, Vol. 16, vol. 1.

24. Andrunakievich V. A. Primary modules and the Baer radical. Sib. Mat. Zh., 1961, vol. 2, no. 6.

25. Andrunakievich V. A. Radicals of weakly associative rings, II. Special radicals. Izvestiya of the Academy of Sciences of the MSSR,1962, no. 5.

26. Andrunakievich V. A. The radical and decomposition of the ring. DAN USSR, 1962, vol. 145, no.. 1.

27. Andrunakievich V. A. Radicals and the decomposition of a ring. Theses of Int. Congress. Math.,Stockholm, 1962.

28. Andrunakievich V. A., Ryabukhin Yu. M. Special modules and special radicals. DAN USSR, 1962, Vol. 147, no.. 6.

29. Andrunakievich V. A., Ryabukhin Yu. M. On the embedding of the modules. DAN USSR, 1963, vol. 153, no. 3.

30. Andrunakievich V. A., Ryabukhin Yu. M. Modules and radicals. DAN USSR, 1964, vol. 156, no.. 4.

31. Andrunakievich V. A., Ryabukhin Yu. M. Connected rings. DAN USSR, 1965, vol. 162, no.. 6.

32. Andrunakievich V. A., Ryabukhin Yu. M. Primary ideals are in non-commutative rings. DAN USSR, 1965, Vol. 165, no.. 1.

33. Andrunakievich V. A., Ryabukhin Yu. M. To the additive theory of ideals in rings and modules. DAN USSR, 1966, Vol. 168, no.. 3.

34. Andrunakievich V. A., Ryabukhin Yu. M. The general additive theory of ideals. Theses of Int. Congress. Math., Moscow, 1966.

35. Andrunakievich V. A., Ryabukhin Yu. M. Additive theory of ideals in systems with quotients. Izv. Akad. Nauk SSSR, Ser. Mat. 1967, Vol. 31, no. 5.

36. Andrunakievich V. A., Ryabukhin Yu. M. Rings without nilpotent elements, and completely prime ideals. DAN USSR, 1968, Vol. 180, no.. 1.

37. Andrunakievich V. A., Ryabukhin Yu. M. Direct sums of division algebras. DAN USSR, 1969, vol. 189, no.. 5.

38. Andrunakievich V. A., Marin V. G. Multi-operator linear algebras without nilpotent elements. DAN USSR, 1971, vol. 197, no.. 4.

39. Andrunakievich V. A., Ryabukhin Yu. M. Torsions in algebras. DAN SSSR, 1973, vol. 208, no.. 2.

40. Andrunakievich V. A., Ryabukhin Yu. M. Radical closures of right ideals, and their complements.DAN USSR, 1974, vol. 219, no.. 2.

41. Andrunakievich V. A., Ryabukhin Yu. M. Certain classes of weakly Boolean rings. DAN SSSR, 1975, vol. 224, no. 1.

42. Andrunakievich V. A., Kracilov K. K., Ryabukhin Yu. M. Radical closures in complete lattices. General theory. Mat. Sbornik, 1976, vol. 100, no. 3.

43. Andrunakievich V. A., Kracilov K. K., Ryabukhin Yu. M. Lattice-complemented torsion in algebras over a Noetherian integral domain. DAN USSR, 1978, vol. 242, no. 5.

44. Andrunakievich V. A., Ryabukhin Yu. M. Radicals of algebras and structural theory. Moscow, "Nauka", 1979.

45. Andrunakievich V. A., Kitoroage I. D. Numere şi ideale. Chisinau, Lumina, 1979.

46. Andrunakievich V. A., Andrunakievich A. V. Strictly modular ring ideals. DAN SSSR, 1981, Vol. 257, no.. 1.

47. Andrunakievich V. A., Andrunakievich A. V. One-sided ideals and ring radicals.DAN SSSR, 1981, Vol. 259, no. 1.

48. Andrunakievich V. A., Andrunakievich A. V. One-sided ideals and radicals of rings. Algebra i Logika , 1981, vol. 20.

49. Andrunakievich V., A., Andrunakievich A. V. Abelian-regular ring ideals. DAN SSSR, 1982, Vol. 263, no. 6.

50. Andrunakievich V. A., Andrunakievich A. V. Subdirect products of division rings. DAN SSSR, 1983, Vol. 269, No 4.

51. Andrunakievich V. A. Completely primary ideals of a ring. Mat. Sb., 121(163) (1983), no. 3

52. Andrunakievich V. A., Andrunakievich A. V. A completely semiprime maximal right ideal of a ring is two-sided. DAN SSSR, 1984, Vol. 279, no. 2.

53. Andrunakievich V. A., Ryabukhin Yu. M. Free commutative quasiregular algebras and algebras without quasiregular subalgebras. UMN,1985, vol. 40, no. 4.

54. Andrunakievich V. A., Ryabukhin Yu. M. On varieties of quasiregular algebras. Mat. Sb. 1985, Vol. 127, no. 8.

55. Andrunakievich V. A., Ryabukhin Yu. M. Varieties of quasiregular algebras. DAN SSSR, 1986, vol. 287, No. 5.

56. Andrunakievich V. A., Ryabukhin Yu. M. Primitivity and quasiregularity with respect to the right ideals of a ring. Mat. Sb., 1987, Vol. 134, no. 4.

57. Andrunakievich V. A., Ryabukhin Yu. M. Ideals relative to a right ideal P, and C-identities for C?R/P. Izvestiya of the Academy of Sciences of the MSSR, ser. mat., 1990, no. 2.

58. Andrunakievich V. A., Ryabukhin Yu. M. Idempotents and quasiregularity with respect to right ideals of algebras. DAN Russia, 1992, vol. 325, no. 5.

59. Andrunakievich V., A., Ryabukhin Yu. M. Minimal right ideals of a ring with respect to a fixed right ideal. DAN Russia, 1993, vol. 331, No 6.

60. Andrunakievich V. A., Ryabukhin Yu. M. On reduced right ideals of a ring. Buletinul A Ş a RM, Matem., 1995, no. 1.

61. Ryabukhin Yu. M. Quasiregular algebras, modules, groups, and varieties. Buletinul AŞ a RM, Mat., 1997, no. 1.

62. Divinsky N. Rings and radicals. Allen & Unwin, London, 1965.

63. Zhevlakov K. A., Slinko A. M., Shestakov I. G., Shirshov A., I.,. Rings close to associative.Moscow, "Science", 1978.

64. Szasz V. A. Radicals in rings. Akademiai Kiado, Budapest and John Wiley & Sons, 1981.

65. Gardner B. J., Wiegandt R. Radical Theory of Rings. Marcel Dekker Inc., New York-Basel, 2004.

Yu. M. Reabukhin
Faculty of Physics and Mathematics
Shevchenko Transnistria State University
25 October str., 128, Tiraspol, MD-3300
Moldova
E-mail: *loretar@mail.ru*

Institute of Mathematics and Computer Science
Academy of Sciences of Moldova
5 Academiei str., Chişinău MD-2028
Moldova