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T-invariants for jumping Petri nets *

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Abstract

This paper introduces the notions of T-invariants for the class of finite jumping Petri nets, and extends the results concerning T-invariants from classical Petri nets.

Keywords: parallel/distributed systems, Petri nets, jumping Petri nets, invariants, verification of properties.

1 Introduction

A Petri net is a mathematical model used for the specification and the analysis of parallel/distributed systems. An introduction about Petri nets can be found in [4]. The place and transition invariants are a formal analysis method for Petri nets, which was introduced in [3].

The basic idea behind place invariants is to construct equations which are satisfied for all reachable markings, an idea which is very similar to that of invariants in program verification. First we formulate some equations, which we postulate to be satisfied independently of the steps that occur. Then we prove that the equations are indeed satisfied, and finally we use them to prove dynamic properties of the modelled system.

Place and transition invariants are useful to prove dynamic properties, like reachability, boundedness, home, liveness and fairness properties. Another advantage of invariants is that they can be constructed during the design of a system, and this will usually lead to an improved design.

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It is well-known that the behaviour of some distributed systems cannot be adequately modelled by classical Petri nets. Many extensions which increase the computational and expressive power of Petri nets have been thus introduced. One direction has led to various modifications of the firing rule of nets. One of these extensions is that of jumping Petri nets, introduced in [5].

A *jumping Petri net* is a classical Petri net Σ equipped with a binary relation R on the markings of Σ. The meaning of a pair $(M, M') \in R$ is that the net Σ may spontaneously "jump" from the marking M to the marking M' (this is similar to λ-moves in automata theory). A jumping Petri net is *finite* if the set of jumps R is finite.

This paper defines the notions of transition invariants for finite jumping Petri nets, and shows that all the results about T-invariants from classical Petri nets (i.e. P/T-nets) hold in the case of finite jumping Petri nets, too.

The paper is organized as follows. Section 2 presents the basic terminology, notation and results concerning Petri nets and jumping Petri nets. Section 3 introduces the notion of transition invariants for finite jumping Petri nets and extends the results concerning invariants from P/T-nets. Finally, section 4 concludes this paper and formulates some open problems.

2 Preliminaries

We will assume to be known the basic terminology and notation about sets and relations, vectors and matrixes, and formal languages. Let us just say that the notation $\#(a, w)$ will denote the number of occurences of the symbol a in the word w.

This section will establish the basic terminology, notation, and results concerning Petri nets in order to give the reader the necessary prerequisites for the understanding of this paper (for details the reader is referred to [1], [4], [2]). Mainly, it will follow [2], [5].

2.1 Petri nets

A *Place/Transition net*, shortly *P/T-net* or *net*, (finite, with infinite capacities), abbreviated *PTN*, is a 4-tuple $\Sigma = (S, T, F, W)$, where S and T are two finite non-empty sets (of *places* and *transitions*, resp.), with $S \cap T = \emptyset$, $F \subseteq (S \times T) \cup (T \times S)$ is the *flow relation* and $W : (S \times T) \cup (T \times S) \to \mathbb{N}$ is the *weight function* of Σ verifying $W(x, y) = 0$ iff $(x, y) \notin F$.

For any $x \in S \cup T$, the sets ${}^\bullet x = \{y \in S \cup T | (y, x) \in F\}$ and, resp., $x^\bullet = \{y \in S \cup T | (x, y) \in F\}$ are called the *preset*, resp. *postset*, of x. Moreover, for any $X \subseteq S \cup T$, ${}^\bullet X = \cup\{{}^\bullet x | x \in X\}$ and $X^\bullet = \cup\{x^\bullet | x \in X\}$. An element $x \in S \cup T$ is called *isolated* iff ${}^\bullet x \cup x^\bullet = \emptyset$.

A *marking* of a *PTN* Σ is a function $M : S \to \mathbb{N}$; it will be sometimes identified with a $|S|$-dimensional vector. The operations and relations on vectors are componentwise defined. $\mathbb{N}^S$ denotes the set of all markings of Σ.

A *marked PTN*, abbreviated *mPTN*, is a pair $\gamma = (\Sigma, M_0)$, where Σ is a *PTN* and M_0, called the *initial marking* of γ, is a marking of Σ.

In the sequel the term "Petri net"(*PN*) or "net" will be often used to denote a *PTN* or a *mPTN* whenever it is not necessary to specify its type (i.e. marked or unmarked).

Let Σ be a net, $t \in T$ and $w \in T^*$. The functions $t^-, t^+ : S \to \mathbb{N}$ and $\Delta t, \Delta w : S \to \mathbb{Z}$ are defined by: $t^-(s) = W(s, t)$, $t^+(s) = W(t, s)$, $\Delta t(s) = t^+(s) - t^-(s)$, and

$$\Delta w(s) = \begin{cases} 0, & \text{if } w = \lambda \\ \sum_{i=1}^{n} \Delta t_i(s), & \text{if } w = t_1 t_2 \dots t_n \ (n \geq 1) \end{cases} \text{, for all } s \in S.$$

The sequential behaviour of a net Σ is given by the so-called *firing rule*, which consists of

- the *enabling rule*: a transition t is *enabled* at a marking M in Σ (or t is *fireable* from M), abbreviated $M[t\rangle_\Sigma$, iff $t^- \leq M$;
- the *computing rule*: if $M[t\rangle_\Sigma$, then t may *occur* yielding a new marking M', abbreviated $M[t\rangle_\Sigma M'$, defined by $M' = M + \Delta t$.

The notation "$[.\rangle_\Sigma$" will be simplified to "$[.\rangle$" whenever Σ is understood from the context.

In fact, any transition t of Σ establishes a binary relation on $\mathbb{N}^S$, denoted by $[t\rangle_\Sigma$ and given by: $M[t\rangle_\Sigma M'$ iff $t^- \leq M$ and $M' = M + \Delta t$. If $t_1, t_2, \ldots, t_n$ ($n \geq 1$) are transitions of Σ, $[t_1 t_2 \ldots t_n\rangle_\Sigma$ will denote the classical product of the relations $[t_1\rangle_\Sigma, \ldots, [t_n\rangle_\Sigma$, i.e. $[t_1 t_2 \ldots t_n\rangle_\Sigma = [t_1\rangle_\Sigma \circ \ldots \circ [t_n\rangle_\Sigma$. Moreover, the relation $[\lambda\rangle_\Sigma$ is considered, by defining $[\lambda\rangle_\Sigma = \{(M, M) | M \in \mathbb{N}^S\}$.

Let $\gamma = (\Sigma, M_0)$ be a marked Petri net, and $M \in \mathbb{N}^S$. The word $w \in T^*$ is called a *transition sequence* from M in Σ if there exists a marking M' of Σ such that $M[w\rangle_\Sigma M'$. Moreover, the marking M' is called *reachable* from M in Σ. $TS(\Sigma, M) = \{w \in T^* | M[w\rangle_\Sigma\}$ denotes the set of all transition sequences from M in Σ, and $RS(\Sigma, M) = [M\rangle_\Sigma = \{M' \in \mathbb{N}^S | \exists w \in TS(\Sigma, M) : M[w\rangle_\Sigma M'\}$ denotes the set of all reachable markings from M in Σ. In the case $M = M_0$, the set $TS(\Sigma, M_0)$ is abbreviated by $TS(\gamma)$ and it is called *the set of all transition sequences* of γ, and the set $RS(\Sigma, M_0)$ is abbreviated by $RS(\gamma)$ (or $[M_0\rangle_\gamma$) and it is called *the set of all reachable markings* of γ.

A place $s \in S$ is *bounded* if there exists an integer $k \in \mathbb{N}$ such that $M(s) \leq k$, for all $M \in [M_0\rangle_\gamma$. The net γ is *bounded* if all its places are bounded.

A transition $t \in T$ is *quasi-live* if there exists a reachable marking $M \in [M_0\rangle_\gamma$ such that t is fireable from M, i.e. $M[t\rangle_\gamma$. The net γ is *quasi-live* if all its transitions are quasi-live.

A transition $t \in T$ is *live* if for any reachable marking $M \in [M_0\rangle_\gamma$, there exists a marking M' reachable from M, i.e. $M' \in [M\rangle_\gamma$, such that t is fireable from M', i.e. $M'[t\rangle_\gamma$. The net γ is *live* if all its transitions are live.

2.2 Jumping Petri nets

Jumping Petri nets ([5]) are an extension of P/T-nets, which allows them to do "spontaneous jumps" from one marking to another one (this is similar to λ-moves in automata theory).

A *jumping P/T-net*, abbreviated *JPTN*, is a pair $\gamma = (\Sigma, R)$, where Σ is a *PTN* and R is a binary relation on the set of markings of Σ (i.e. $R \subseteq \mathbb{N}^S \times \mathbb{N}^S$), called the *set of (spontaneous) jumps* of γ. In what follows the set R of jumps of any *JPTN* will be assumed *recursive*, that is for any couple of markings (M, M') it can be effectively decided whether or not (M, M') is a member of R.

Let $\gamma = (\Sigma, R)$ be a *JPTN*. The pairs $(M, M') \in R$ are referred to as *jumps* of γ. Σ is called the *underlying P/T-net* of γ. A *marking* of γ is any marking of its underlying P/T-net. If γ has finitely many jumps (i.e. R is finite), then γ is called a *finite jumping Petri net*, abbreviated *FJPTN*.

A *marked jumping P/T-net* is defined similarly as a marked P/T-net, by changing "Σ" into "Σ, R". The abbreviations used will be mY, with $Y \in \{JPTN, FJPTN\}$.

In the sequel the term "*jumping net*" (JN) will be often used to denote a *JPTN* or a *mJPTN* whenever it is not necessary to specify its type (i.e. marked or unmarked).

Pictorially, a jumping net will be represented as a classical net and, moreover, the relation R will be separately listed.

Let γ be a jumping net, and $r = (M', M'') \in R$. The function $\Delta r : S \to \mathbb{Z}$ is defined by: $\Delta r(s) = M''(s) - M'(s)$, for all $s \in S$.

The behaviour of a jumping net γ is given by the *j-firing rule*, which consists of

- the *j-enabling rule*: a transition t is *j-enabled* at a marking M (in γ), abbreviated $M[t\rangle_{\gamma,j}$, iff there exists a marking M_1 such that $MR^*M_1[t\rangle_\Sigma$ (Σ being the underlying P/T-net of γ and R^* the reflexive and transitive closure of R);

- the *j-computing rule*: if $M[t\rangle_{\gamma,j}$, then the marking M' is *j-produced* by occurring t at M, abbreviated $M[t\rangle_{\gamma,j}M'$, iff there exists two markings M_1, M_2 such that $MR^*M_1[t\rangle_\Sigma M_2R^*M'$.

The notation "$[.\rangle_{\gamma,j}$" will be simplified to "$[.\rangle_j$" whenever γ is understood from the context.

The notions of *transition j-sequence* and *j-reachable marking* are defined similarly as for Petri nets (the relation $[\lambda\rangle_{\gamma,j}$ is defined by $[\lambda\rangle_{\gamma,j} = R^* = \{(M, M') | M, M' \in \mathbb{N}^S, MR^*M'\}$). The *set of all* *j-reachable markings* of a marked jumping Petri net γ is denoted by $RS(\gamma)$ or by $[M_0\rangle_{\gamma,j}$ (M_0 being the initial marking of γ).

All other notions from P/T-nets (i.e. bounded place, bounded net, quasi-live transition, quasi-live net, live transition, live net, etc.) are defined for jumping Petri nets similarly as for P/T-nets, by considering the notion of *j-reachability* instead of *reachability* from P/T-nets.

Some jumps of a marked jumping Petri net may be never used. Thus a *mJPTN* $\gamma = (\Sigma, R, M_0)$ is called *R-reduced* iff, for any jump $(M, M') \in R$, $M \neq M'$ and $M \in [M_0\rangle_{\gamma,j}$.

3 T-invariants for finite jumping Petri nets

This section will present some linear algebraic techniques for analysing the properties of jumping Petri nets. More exactly, it will show how T-invariants can be defined for finite jumping Petri nets.

In the sequel, the matrixes and the vectors will be considered to have integer numbers as components. The linear combinations will be also considered to have integer numbers as coefficients. The (row or column) vector with all its components 0, no matter its size, will be abbreviated by **0**. The inequality on vectors will be understood as the inequalities on the components, and the strict inequality as inequality with strict inequality on at least one component.

First, we will briefly give a presentation of the notions of incidence matrix and S-invariants, which we introduced in [6]. And then we will introduce the notion of T-invariants for finite jumping Petri nets and extend the results concerning T-invariants from P/T-nets.

3.1 Incidence matrix

As in the case of P/T-nets, in order to be able to define the notion of the incidence matrix for a finite jumping Petri net $\gamma = (\Sigma, R)$, where $\Sigma = (S, T, F, W)$ is the underlying P/T-net of γ, it is necessary to have a total ordering of the sets S, T and R. Without loss of generality, it will be assumed that, if these sets are of the form

$$S = \{s_1, \ldots, s_m\},\ T = \{t_1, \ldots, t_n\},\ \text{and}\ \ R = \{r_1, \ldots, r_p\},$$

then they are totally ordered by the natural order on the indexes of the elements:

$$S: \ s_1 < \ldots < s_m, \ T: \ t_1 < \ldots < t_n, \ \text{and} \ R: \ r_1 < \ldots < r_p.$$

Now, let us recall the definition of the incidence matrix from [6]:

Definition 3.1.1 *Let $\gamma = (\Sigma, R)$ be a finite jumping Petri net. The $m \times (n+p)$-dimensional matrix I_γ defined by*

$$I_\gamma(i,j) = \begin{cases} I_\Sigma(i,j) & ,\forall \ 1 \le j \le n \\ I_R(i,j-n) & ,\forall \ n+1 \le j \le n+p \end{cases} ,\forall \ 1 \le i \le m,$$

is called the incidence matrix *of the net γ, abbreviated by $I_\gamma = (I_\Sigma, I_R)$, where:*

1) I_Σ is the $m \times n$-dimensional matrix given by

$$I_\Sigma(i,j) = \Delta t_j(s_i), \ \forall \ 1 \le i \le m, \ \forall \ 1 \le j \le n,$$

i.e. it is the incidence matrix *of the underlying P/T-net of γ;*

2) I_R is the $p \times n$-dimensional matrix given by

$$I_R(i,j) = \Delta r_j(s_i), \ \forall \ 1 \le i \le m, \ \forall \ 1 \le j \le p \ ;$$

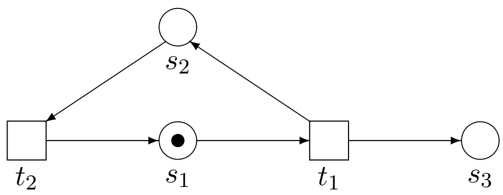
it is called the incidence matrix *of the jumps of γ.*

The notion of incidence matrix is extended also to marked finite jumping Petri nets (Σ, R, M_0) through the unmarked underlying net (Σ, R).

The main result from [6] about the incidence matrix is the following:

Theorem 3.1.1 *Let $\gamma = (\Sigma, R)$ be a FJPTN, and M_1, M_2 two markings of γ. If M_2 is j-reachable from M_1, then there exists a positive column vector f such that $M_2 = M_1 + I_\gamma \cdot f$.*

Proof. See [6]. The vector f is equal with the sum of the "effects" (i.e. $f = \sum \Delta t + \sum \Delta r$) of all the transitions and jumps through which the marking M_2 is j-reachable from M_1. $\square$



$$R = \{((1, 0, 5), (1, 0, 0))\}$$

Figure 1:The jumping net from example 3.1.1

Example 3.1.1 *Let γ be the mFJPTN from figure 1. The incidence matrix of γ is*

$$I_\gamma = \begin{pmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 1 & 0 & -5 \end{pmatrix}.$$

The transition sequence t_1t_2 is j-enabled at the marking $M = (1, 0, 5)$ and the marking j-produced by the occuring of t_1t_2 at M is

$$M' = M + I_\gamma \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix},$$

if the jump occurs before the occuring of transition t_1, resp.

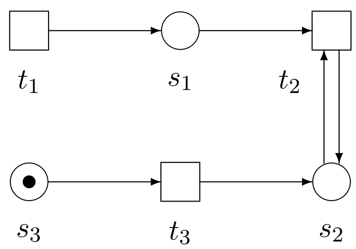
$$M'' = M + I_\gamma \cdot \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 6 \end{pmatrix},$$

otherwise (i.e. the jump does not occur at all).

3.2 S-invariants

In the modelling of real systems by Petri nets it is very important to know if the number of tokens lying in the places of the net is preserved or not during the evolution of the system; uncontrolled losses of tokens are unwanted. The dynamic behaviour of marked finite jumping Petri nets depends on the structure of the net and on the initial marking. Both factors are known *a priori* and, thus, they can be investigated independently of the dynamic behaviour of the net.

Example 3.2.1 *Let γ be the mFJPTN from figure 2. In the net γ the total number of tokens is not preserved, because the number of tokens which can appear in the place s_1 is unlimited. But, on the other hand, the initial marking of γ can be reproduced, using the jump of the net.*



$$R = \{((0,1,0),(0,0,1))\}$$

Figure 2:The jumping net from example 3.2.1

Now, let us recall the definition of S-invariants from [6]:

Definition 3.2.1 *Let $\gamma = (\Sigma, R)$ be a finite jumping Petri net, with $\Sigma = (S,T,F,W)$ being the underlying P/T-net of γ. An* S-invariant *of γ is any $|S|$-dimensional vector J of integer numbers which satisfies the equation $J^t \cdot I_\gamma = \mathbf{0}$, where I_γ is the incidence matrix of γ.*
The S-invariant $J > \mathbf{0}$ is called minimal *if there exists no S-invariant J' such that $\mathbf{0} < J' < J$.*

Example 3.2.2 *The net from figure 1 has the minimal S-invariant:*

$$J = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix},$$

any other S-invariant of the net being a linear combination of the form $z \cdot J$, with $z \in \mathbb{Z}$. Similarly, for the jumping net from figure 2 there is one minimal S-invariant:

$$J' = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}.$$

The main results from [6] about the S-invariants are the following:

Theorem 3.2.1 *If J is an S-invariant of a mFJPTN $\gamma = (\Sigma, R, M_0)$, then*

$$J^t \cdot M = J^t \cdot M_0, \text{ for any } M \in [M_0\rangle_{\gamma,j}.$$

This theorem says that any S-invariant of a *mFJPTN* γ gives the weights for the places of a subnet of γ in which the tokens are preserved (through these weights).

The converse of theorem 3.2.1 holds under a supplementary hypothesis:

Theorem 3.2.2 *Let $\gamma = (\Sigma, R, M_0)$ be a mFJPTN quasi-live and R-reduced. If J is a vector of integers satisfying the property*

$$J^t \cdot M = J^t \cdot M_0, \text{ for any } M \in [M_0\rangle_{\gamma,j},$$

then J is an S-invariant of γ.

All other results concerning S-invariants from P/T-nets hold also for finite jumping Petri nets (see [6]).

3.3 T-invariants

As in the case of P/T-nets, another important aspect in the analyse of jumping Petri nets, besides the preserving of the number of tokens during the evolution of the system, is the reproducibility of the markings.

The notations used will be those from subsection 3.1.

Definition 3.3.1 *A marking M of a jumping Petri net $\gamma = (\Sigma, R)$ is called* reproducible *if there exists a transition j-sequence $w \in T^*$ such that $M[w\rangle_{\gamma,j}M$ and, moreover, $w \neq \lambda$ or $(M, M) \in R^+$ (where R^+ is the transitive closure of the relation R).*

Let us notice that not every reproducible marking M of a *JPTN* γ satisfies the property that any marking $M' \geq M$ of γ is also reproducible. Therefore, the result about the monotony of reproducible markings from P/T-nets does not hold for jumping Petri nets. The justification of this remark follows from the fact that the property about the monotony of transitions' firings from P/T-nets does not hold in the case of jumping Petri nets, i.e.

$$M_1[t\rangle_{\gamma,j}M_2 \;\wedge\; M_1' \geq M_1 \;\not\Rightarrow\; M_1'[t\rangle_{\gamma,j}M_2', \text{ with } M_2' = M_1' + M_2 - M_1.$$

Another remark is that, if M is reproducible, then $M[w\rangle_{\gamma,j}M$, and, proceeding from theorem 3.1.1, there exists a positive vector f such that $M + I_\gamma \cdot f = M$, i.e. $I_\gamma \cdot f = \mathbf{0}$.

Definition 3.3.2 *Let $\gamma = (\Sigma, R)$ be a finite jumping Petri net, with $\Sigma = (S, T, F, W)$ being the underlying P/T-net of γ.*

(1) A T-invariant *of γ is any $(n + p)$-dimensional vector J of integer numbers which satisfies the equation $I_\gamma \cdot J = \mathbf{0}$, where $I_\gamma = (I_\Sigma, I_R)$ is the incidence matrix of γ.*
Remark: $I_\gamma \cdot J = \mathbf{0}$ is equivalent to $I_\Sigma \cdot J_\Sigma + I_R \cdot J_R = \mathbf{0}$, where J_Σ is the n-dimensional vector defined by $J_\Sigma(i) = J(i), 1 \leq i \leq n$, and J_R is the p-dimensional vector defined by $J_R(i) = J(n + i)$, $1 \leq i \leq p$, and we will abbreviate this by $J = (J_\Sigma, J_R)$.

(2) If J is a T-invariant of γ, then the set

$$P_J = \{t_i \in T \mid J(i) \neq 0\} \cup \{r_i \in R \mid J(n + i) \neq 0\}$$

is called the support *of J.*

(3) The T-invariant J is called positive *if $J \geq \mathbf{0}$.*

(4) The T-invariant $J > \mathbf{0}$ *is called* minimal *if there exists no T-invariant* J' *such that* $\mathbf{0} < J' < J$.

(5) The finite jumping Petri net induced *by the T-invariant* J *is defined by*

$$\gamma' = (\Sigma', R'), \text{ with } \Sigma' = (S', T', F', W'),$$

where:

a) $T' = P_J \cap T$;
b) $S' = {}^\bullet T' \cup T'^\bullet$;
c) $F' = F \cap ((S' \times T') \cup (T' \times S'))$;
d) $W' = W|_{F'}$;
e) $R' = P_J \cap R$.

It is easy to notice that any linear combination of T-invariants is a T-invariant:

Lemma 3.3.1 *If* J_1 *and* J_2 *are T-invariants of a finite jumping Petri net* γ, *and* $z \in \mathbb{Z}$, *then* $J_1 + J_2$ *and* $z \cdot J_1$ *are T-invariants of* γ, *too.*

Obviously, any finite jumping Petri net has at least one T-invariant, $J = \mathbf{0}$, but this one is trivial. Thus, a jumping net is said *to have T-invariants* if it has at least one non-null T-invariant.

Example 3.3.1 *The net from figure 1 has the minimal T-invariant*

$$J = \begin{pmatrix} 5 \\ 5 \\ 1 \end{pmatrix},$$

any other T-invariant of the net being a linear combination of the form $z \cdot J$, *with* $z \in \mathbb{Z}$.

The jumping net from figure 2 has two minimal T-invariants

$$J_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \; J_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix},$$

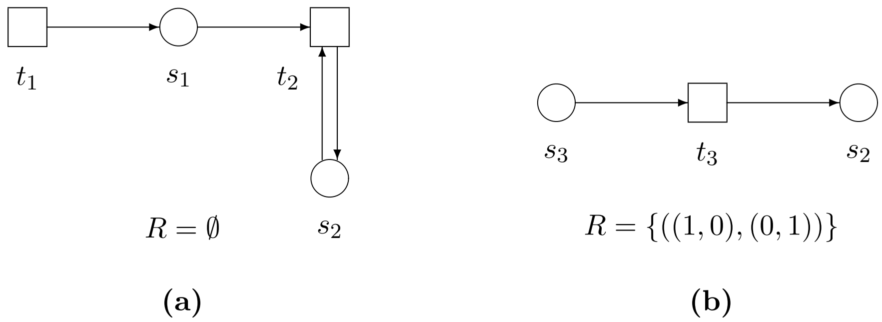


Figure 3:The induced nets from example 3.3.1

any other T-invariant of the net being a linear combination of the form $z_1 \cdot J_1 + z_2 \cdot J_2$*, with* $z_1, z_2 \in \mathbb{Z}$*. The jumping nets induced by the T-invariants* J_1 *and* J_2 *are represented in figure 3(a) and, resp., 3(b).*

Unfortunately, there exists no connection between the T-invariants of a finite jumping Petri net and those of its underlying P/T-net, despite the fact that there exists such a connection for S-invariants ([6]).

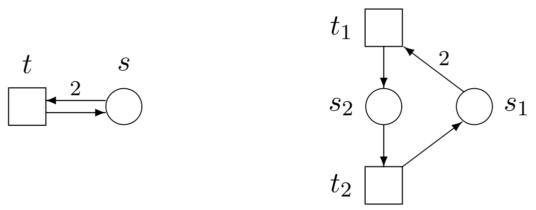
Clearly, if a net has reproducible markings, then it has T-invariants.

Theorem 3.3.1 *Let* γ *be a finite jumping Petri net. If* γ *has reproducible markings, then* γ *has positive T-invariants.*

Proof. Let M be a reproducible marking of γ. Then, there exists a transition j-sequence $w \in T^*$ with $M[w\rangle_{\gamma,j}M$, and, proceeding from theorem 3.1.1, there exists a positive vector f such that $M + I_\gamma \cdot f = M$. It follows that $I_\gamma \cdot f = \mathbf{0}$. Thus, f is a positive T-invariant of γ. □

The converse of this theorem does not hold, unfortunately, for jumping Petri nets, although it holds in case of P/T-nets. The justification of this remark follows from the fact that the property about the monotony of transitions' firings from P/T-nets does not hold in the case of jumping Petri nets, i.e.

$$M_1[t\rangle_{\gamma,j}M_2 \ \wedge \ M_1' \geq M_1 \ \not\Rightarrow \ M_1'[t\rangle_{\gamma,j}M_2', \text{ with } M_2' = M_1' + M_2 - M_1.$$



$R_1 = \{((1),(0))\}$ $R_2 = \{((1,0),(0,0))\}$

(a) **(b)**

Figure 4:The jumping nets from example 3.3.2

The following example ilustrates this remark.

Example 3.3.2 *Let us consider the marked R-reduced jumping Petri nets* $\gamma_1 = (\Sigma_1, R_1)$ *and* $\gamma_2 = (\Sigma_2, R_2)$ *represented in figure 4(a) and, resp., 4(b). The vectors*

$$J_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \text{ resp. } J_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix},$$

are the minimal T-invariants of the nets γ_1 *and, resp.,* γ_2*. However, on the other hand, it is easy to see that neither one from the nets* γ_1 *and* γ_2 *has reproducible markings.*

Let us remark that, in the particularly case of a finite jumping Petri net $\gamma = (\Sigma, R)$ which has a T-invariant $J = (J_\Sigma, J_R) > \mathbf{0}$ (this means that $I_\Sigma \cdot J_\Sigma + I_R \cdot J_R = \mathbf{0}$), such that J satisfies supplementary the restrictive condition $I_\Sigma \cdot J_\Sigma = I_R \cdot J_R = \mathbf{0}$, then γ has reproducible markings. Indeed, from $I_\Sigma \cdot J_\Sigma = \mathbf{0}$, we conclude that Σ, the underlying P/T-net of γ, has reproducible markings, i.e. there exists a marking M and a transition sequence $w \in T^*$ such that $M[w\rangle_\Sigma M$. Therefore, we have also $M[w\rangle_{\gamma,j} M$ (without any jump), so γ has reproducible markings.

Definition 3.3.3 *A T-invariant J of a mFJPTN $\gamma = (\Sigma, R, M_0)$ is called* realizable *iff there exists a marking $M \in [M_0\rangle_{\gamma,j}$ and a transition j-sequence $M[w\rangle_{\gamma,j}M'$ such that $J(i) = \#(t_i, w)$, for all $1 \leq i \leq n$, and, moreover, for any $n+1 \leq i \leq n+p$, $J(i)$ is equal with the number of ("hidden") appearances of the jump r_{i-n} in the transition j-sequence $M[w\rangle_{\gamma,j}M'$.*

In other words, the T-invariant J is *realizable* iff there exists a marking $M \in [M_0\rangle_{\gamma,j}$ and a transition j-sequence

$$Mu'_1M'_1[t_{i_1}\rangle_\Sigma M''_1u''_1M_1u'_2M'_2[t_{i_2}\rangle_\Sigma M''_2u''_2M_2 \dots u'_hM'_h[t_{i_h}\rangle_\Sigma M''_hu''_hM_h,$$

with $h \geq 0$, $t_{i_1}, \dots, t_{i_h} \in T$, and $u'_1, u''_1, \dots, u'_h, u''_h \in R^*$, such that

$$J(i) = \begin{cases} \#(t_i, t_{i_1} \cdots t_{i_h}) & \text{, for any } 1 \leq i \leq n \\ \#(r_{i-n}, u'_1u''_1 \cdots u'_hu''_h) & \text{, for any } n+1 \leq i \leq n+p \end{cases}$$

Remark: $h = 0$ iff $P_J \cap T = \emptyset$; in this case, the transition j-sequence is of the form MuM', with $u \in R^+$, and the requirement which must be fulfill is

$$J(i) = \begin{cases} 0 & \text{, for any } 1 \leq i \leq n \\ \#(r_{i-n}, u) & \text{, for any } n+1 \leq i \leq n+p \end{cases}$$

It is easy to remark that not every positive T-invariant of a marked finite jumping Petri net is realizable.

Obviously, the following converse of theorem 3.3.1 holds, with a stronger hypothesis:

Theorem 3.3.2 *Let γ be a mFJPTN. If γ has realizable T-invariants, then γ has reproducible markings.*

Proof. Let J be a realizable T-invariant of a *mFJPTN* γ. Thus, there exists a marking $M \in [M_0\rangle_{\gamma,j}$ and a transition j-sequence $M[w\rangle_{\gamma,j}M'$ such that $J(i) = \#(t_i, w)$, for all $1 \leq i \leq n$, and, moreover, for any $n+1 \leq i \leq n+p$, $J(i)$ is equal with the number of ("hidden") appearances of the jump r_{i-n} in the transition j-sequence $M[w\rangle_{\gamma,j}M'$. From this fact, and accordingly to the meaning of the incidence matrix of a net, it is easy to notice that the marking M' can be computed from the marking M as (see theorem 3.1.1) :

$$M' = M + I_\gamma \cdot J.$$

But, we have $I_\gamma \cdot J = 0$, because J is a T-invariant of γ. It follows that $M' = M$, and, therefore, M is a reproducible marking of γ. $\square$

In the sequel, we will show that the result from P/T-nets, which says that the bounded and live nets are covered by T-invariants, holds also for finite jumping nets.

Definition 3.3.4 *A FJPTN γ is said to be* covered by T-invariants *if, for each transition $t \in T$, there exists a positive T-invariant J_t of γ with $t \in P_{J_t}$, and, for each jump $r \in R$, there exists a positive T-invariant J_r of γ with $r \in P_{J_r}$.*

Example 3.3.3 *The nets from figure 1 and figure 2, are covered by T-invariants.*

Lemma 3.3.2 *If γ is a FJPTN covered by T-invariants, then there exists a T-invariant J with $P_J = T \cup R$.*

Proof. By the hypothesis, for each $t \in T$, there exists a positive T-invariant J_t with $t \in P_{J_t}$, and, for each jump $r \in R$, there exists a positive T-invariant J_r with $r \in P_{J_r}$. Using lema 3.3.1, the vector

$$J = \sum_{t \in T} J_t + \sum_{r \in R} J_r$$

is a T-invariant fulfilling the requirements. $\square$

We will introduce now a notion of liveness of a jump, for jumping Petri nets, similarly with the notion of liveness of a transition.

Definition 3.3.5 *Let $\gamma = (\Sigma, R, M_0)$ be a marked jumping Petri net. A jump $r=(M_1, M_2) \in R$ is called* R-live *if for any j-reachable marking $M \in [M_0\rangle_{\gamma,j}$, the marking M_1 is j-reachable from M, i.e. $M_1 \in [M\rangle_{\gamma,j}$. The net γ is called* R-live *if all its jumps are R-live.*

Theorem 3.3.3 *Any marked finite jumping Petri net, bounded, live and R-live, is covered by T-invariants.*

Proof. Let $\gamma = (\Sigma, R, M_0)$ be a *mFJPTN*, which is bounded, live and R-live.
1) Since γ is bounded, there exists an integer number k such that for all $M \in [M_0\rangle_{\gamma,j}$ and for all $s \in S$, $M(s) \leq k$. We conclude that the reachability set $[M_0\rangle_{\gamma,j}$ is finite, because it can have at most $(k+1)^{|S|}$ elements. Let $q = |[M_0\rangle_{\gamma,j}| \in \mathbb{N}$.

Let $t \in T$ be an arbitrary transition of γ. The transition t is live, because γ is live, and therefore we have that $\forall M \in [M_0\rangle_{\gamma,j}$, $\exists M' \in [M\rangle_{\gamma,j}$ such that $M'[t\rangle_{\gamma,j}$, or equivalent:

$$(*)\ \ \forall M \in [M_0\rangle_{\gamma,j}, \exists M'' \in [M_0\rangle_{\gamma,j}, \exists w \in T^* \text{ such that } M[wt\rangle_{\gamma,j}M'',$$

where $M'' = M' + \Delta t$.

Let $M_1 \in [M_0\rangle_{\gamma,j}$ be an arbitrary marking. Applying $(*)$ for $M = M_1$, there exists $M_2 \in [M_0\rangle_{\gamma,j}$ and $w_1 \in T^*$ such that $M_1[w_1t\rangle_{\gamma,j}M_2$. Using $(*)$ for $M = M_2$, we conclude that there exists $M_3 \in [M_0\rangle_{\gamma,j}$ and $w_2 \in T^*$ such that $M_2[w_2t\rangle_{\gamma,j}M_3$.

By iterating q times this reasoning, we obtain that there exists the markings $M_2, M_3, \ldots, M_{q+1} \in [M_0\rangle_{\gamma,j}$ and $w_1, w_2, \ldots, w_q \in T^*$ such that

$$M_1\ [w_1t\rangle_{\gamma,j}\ M_2\ [w_2t\rangle_{\gamma,j}\ M_3\ \ldots\ M_q\ [w_qt\rangle_{\gamma,j}\ M_{q+1}.$$

Thus, we have $q+1$ markings $M_1, M_2, \ldots, M_{q+1}$. Since $|[M_0\rangle_{\gamma,j}| = q$, it has to exist two indexes l, k with $1 \leq l < k \leq q$ such that $M_l = M_k$.

Now let us consider the subsequence

$$M_l\ [w_lt\rangle_{\gamma,j}\ M_{l+1}\ [w_{l+1}t\rangle_{\gamma,j}\ \ldots\ M_{k-1}\ [w_{k-1}t\rangle_{\gamma,j}\ M_k,$$

in which t appears at least once because $l < k$; thus we have

$$M_l[w\rangle_{\gamma,j}M_k, \text{ with } w = w_ltw_{l+1}t\ldots w_{k-1}t.$$

Proceeding from theorem 3.1.1, there exists a positive column vector $J_t : \{1, \ldots, n+p\} \to \mathbb{Z}$ such that $M_k = M_l + I_\gamma \cdot J_t$; moreover, we have that $J_t(j) = \#(t_j, w)$, for all $1 \le j \le n$.

Since $M_l = M_k$, we obtain $I_\gamma \cdot J_t = \mathbf{0}$. Thus, J_t is a positive T-invariant of γ, and, moreover, $t \in P_{J_t}$ because t appears at least once in the sequence w.

Since $t \in T$ was chosen arbitrary, we conclude that for any $t \in T$ there exists a positive T-invariant J_t such that $t \in P_{J_t}$.

2) Now we will make a similar reasoning about the jumps of the net.

Let $r = (M_1, M_2) \in R$ be an arbitrary jump of γ. The jump r is R-live, because γ is R-live, and therefore we have that $\forall M \in [M_0\rangle_{\gamma,j}$, $M_1 \in [M\rangle_{\gamma,j}$, or equivalent:

$$(**) \ \ \forall M \in [M_0\rangle_{\gamma,j}, \exists w \in T^* \text{ such that } M[w\rangle_{\gamma,j} M_1 r M_2.$$

Using $(**)$ for $M = M_2$, we obtain that there exists $w_1 \in T^*$ such that

$$M_2[w_1\rangle_{\gamma,j} M_1 r M_2.$$

For this transition j-sequence, proceeding from theorem 3.1.1, there exists a positive column vector $J_r : \{1, \ldots, n+p\} \to \mathbb{Z}$ such that $M_2 = M_2 + I_\gamma \cdot J_r$. Thus, we obtain $I_\gamma \cdot J_r = \mathbf{0}$, which means that J_r is a positive T-invariant of γ, and, moreover, $r \in P_{J_r}$ because r appears at least once in the above transition j-sequence.

Since the jump $r \in R$ was chosen arbitrary, we conclude that for any $r \in R$ there exists a positive T-invariant J_r such that $r \in P_{J_r}$.

From 1) and 2) we conclude that γ is covered by T-invariants. $\square$

4 Conclusion

In this paper the notion of T-invariants was introduced for the class of finite jumping Petri nets. Also, the paper extended the results concerning T-invariants from the class of P/T-nets to the class of finite jumping Petri nets.

Some problems remain to be studied, for example: i) extending the notion of T-invariants to the entire class of jumping Petri nets; ii) making some case studies on models of real-world systems.

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