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Extremal gaps in BP_3-*designs* *

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Abstract

In this paper we examine Voloshin's colorings of mixed hypergraphs derived from P_3-*designs* and construct families of P_3-*designs* having the chromatic spectrum with the *leftmost hole* and *rightmost hole*.

1 Introduction

A *mixed hypergraph* is a triple $\mathcal{H} = (X, \mathcal{C}, \mathcal{D})$, where X is the vertex set ($X \neq \emptyset$) and each of $\mathcal{C}$, $\mathcal{D}$ is a list of subsets of X, called the $\mathcal{C}$-*edges* and the $\mathcal{D}$-*edges* of $\mathcal{H}$, respectively. A *proper k-coloring* of a mixed hypergraph $\mathcal{H}$ is a function $f : X \mapsto \{1, 2, ..., k\}$ so that each $\mathcal{C}$-edge contains at least two vertices x, y, $x \neq y$, such that $f(x) = f(y)$, and each $\mathcal{D}$-edge contains at least two vertices x, y, $x \neq y$, such that $f(x) \neq f(y)$. A *strict k-coloring* of $\mathcal{H}$ is a proper *k-coloring* using all k colors. When a hypergraph admits a strict *k-coloring* it is said to be *k-colorable*.

The *minimum* (*maximum*) number of colors in a strict coloring of a mixed hypergraph $\mathcal{H}$ is called the *lower (upper) chromatic number* of $\mathcal{H}$ and is denoted by $\chi(\mathcal{H})$ ($\bar{\chi}(\mathcal{H})$).

For each k, $1 \leq k \leq n$, let r_k be the number of partitions of the vertex set into k nonempty parts (color classes) such that the coloring constraint is satisfied on each $\mathcal{C}$-edge and on each $\mathcal{D}$-edge. We call these partitions *feasible*. Thus r_k is the number of different strict *k-colorings* of $\mathcal{H}$ if we disregard permutations of colors. The vector

$$R(\mathcal{H}) = (r_1, \ldots, r_n) = (0, \ldots, 0, r_\chi, \ldots, r_{\bar{\chi}}, 0, \ldots, 0)$$

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is the *chromatic spectrum* of $\mathcal{H}$. The set of values k, such that $\mathcal{H}$ has a strict k-*coloring*, is the *feasible set* of $\mathcal{H}$ denoted by $S(\mathcal{H})$; this is the set of indices i such that $r_i > 0$. Observe that it $r_i = 0$ for every $i < \chi$, $i > \bar{\chi}$, and when $\chi = \bar{\chi}$ the mixed hypergraph can be colored only with k colors.
It may happen however, that $r_i = 0$ for some $\chi < i < \bar{\chi}$; in this case the chromatic spectrum and feasible set have gaps (are broken).
The *length of the gap* is the number of consecutive zeros in it. The gap of length one is called *hole*. Chromatic spectrum may have many holes. In this case there are *leftmost* and *rightmost* holes.
The concepts of *mixed hypergraphs*, *lower (upper) chromatic number*, *chromatic spectrum* and *gaps* were introduced in [6, 9, 10]. Interesting problems arise when the considered hypergraphs verify the condition to be *Steiner systems or Designs* [2, 7].
In [4] we constructed families of P_3-*designs* having *broken* chromatic spectrum containing many *gaps*. In this paper we construct families of P_3-*designs* having the chromatic spectrum with the *leftmost hole* and *rightmost hole*.

2 P_3-designs and BP_3-designs

Let λK_n be the complete multigraph on n vertices, where every edge is repeated λ times. If G is a graph, the multigraph λK_n is said to be *G-decomposable*, briefly $\lambda K \mapsto G$, if it is a union of edge-disjoint subgraphs of K_n, each of them being isomorphic to G. The multigraph λK_n is also said to admit a *G-decomposition*. This is denoted by the pair $\Sigma = (X, B)$, where X is the vertex set of λK_n and B is the edge-disjoint decomposition of λK_n into copies of G. A G-decomposition of λK_n, $\Sigma = (X, B)$, is also called a *G-design* of order n, block size $|X|$ and index λ [3]. A $P(n,k,\lambda)$-*design* is a G-design having order n, block-size k, index λ, where G is a path on k vertices having for vertices $x_1, x_2, ..., x_k$ and for edges all the pairs $\{x_i, x_{i+1}\}$, for every $i = 1, 2, ..., k-1$. A $P(n,k,1)$-*design* is also called a $P_k(n)$-*design*.

It is well known that:

i) *A necessary and sufficient condition for the existence of a* $P(n,k,\lambda)$-*design is:* $\lambda n(n-1) \equiv 0$, *mod* $2(k-1)$, $n \geq 2k$.

ii) *There exists a* $P_3(n)$-*design if and only if* $n(n-1) \equiv 0$ *mod* 4, $n \geq 3$.

It is possible to construct P_3-*designs* as follows.

A) If (X, B) is a $P_3(n)$-*design* of order n, for $n \equiv 0$ mod 4, and F is an 1-factor defined on $X = \{x_1, x_2, ..., x_n\}$, then consider:

   i) $X' = X \cup \{\infty\}$, where $\infty \notin X$;
   ii) $B' = B \cup C$,

   where C is the collection of all the P_3-*paths* having the vertex ∞ as *centre* and all the pairs of F as *terminal* points. We can prove immediately that (X', B') is a $P_3(n+1)$-*design*.

B) Let (X, B) be a $P_3(n)$-*design* of order n and let (Y, C) be a $P_3(4)$-*design*, being $Y = \{y_1, y_2, y_3, y_4\}$. If $F_1, F_2, ..., F_{n-1}$ are 1-factors (non necessarily distinct or disjoint) defined on $X = \{x_1, x_2, ..., x_n\}$, consider:

   i) $X' = X \cup Y$;
   ii) $B' = B \cup C \cup D_1 \cup D_2 \cup D_3 \cup D_4$,

   where, for $i = 1, 2, 3, 4$, D_i is the collection of all the P_3-*paths* having the vertex $y_i \in Y$ as *centre* and all the pairs of the 1-factor F_i as *terminal* points. It is easy to prove that (X', B') is a $P_3(n+4)$-*design*.

C) If (X, B) is a $P_3(n)$-*design* of order n, (Y, C) is a $P_3(4)$-*design* defined on $Y = \{y_1, y_2, y_3, y_4\}$, and F_1, F_2, F_3 are 1-factors (non necessarily distinct or disjoint) defined on Y, consider:

   i) $X' = X \cup Y$;

ii) $B' = B \cup C \cup D_1 \cup D_2 \cup D_3 \cup D_4$,

where, for $i = 1, 2, 3, 4$, D_i is the collection of all the P_3-*paths* having the vertex $x_i \in X$ as *centre* and all the pairs of the 1-factor F_i as *terminal* points. It is immediate to prove that (X', B') is a $P_3(n+4)$-*design* .

A path of a P_3-*design* having *centre* y and *terminal* vertices x, z, is indicated by $\langle x, y, z \rangle$. Further, we consider hypergraphs $\mathcal{H} = (X, E)$ associated with P_3-*designs* $\Sigma = (X, B)$: they have vertex set X and an edge $\{x, y, z\} \in E$ if and only if the path $\langle x, y, z \rangle \in B$. In what follows we will consider P_3-*designs* with particular Voloshin-colorings. In fact, in the case $\mathcal{C} = \mathcal{D}$, *3-uniform* mixed-hypergraphs have a singular property: in every edge there are two vertices with the same color and two vertices with distinct colors.
We will consider always P_3-*designs* having the property that $\mathcal{C} = \mathcal{D}$ and we will call them BP_3-*designs* (designs with bicolored blocks), in analogy to BSTS considered in [2, 7].

3 BP_3-designs with minimum or maximum possible gap

The following Theorem was proved in [4].

Theorem 1 *If $\Sigma = (X, B)$ is a BP_3-design of order n, then:*

$$2 \leq \chi \leq \bar{\chi} \leq \lceil n/2 \rceil$$

and the bounds are the best possible.

From Theorem 1 it follows:

Corollary 1 *In the chromatic spectrum of a $BP_3(n)$-design, the leftmost hole is possible at 3 and $n \geq 8$, the rightmost hole is possible at $i = \lceil n/2 \rceil - 1$ and $n \geq 8$.*

Proof.
In a BP_3-*design* always $r_1 = 0$. Therefore, if $r_2 \neq 0$, the first possibility for a gap in the chromatic spectrum is for $r_3 = 0$, $r_4 \neq 0$, with a gap at $i = 3$. Further, $r_4 \neq 0$ implies $n \geq 8$. Since $r_i = 0$, for every $i = \lceil n/2 \rceil + 1, \ldots, n$, the last possibility for a gap is for $r_{\lceil n/2 \rceil} \neq 0$, $r_{\lceil n/2 \rceil - 1} = 0$, $r_{\lceil n/2 \rceil - 2} \neq 0$, with a gap at $i = \lceil n/2 \rceil - 1$. From $r_{\lceil n/2 \rceil - 2} \neq 0$, $\lceil n/2 \rceil - 2 \geq 2$, it follows, also, $n \geq 8$.

□

Theorem 2 *For every admissible* n*,* $n \geq 8$*, there exists* $BP_3(n)$-design *having the only hole and it is at 3.*

Proof.
To prove the statement, it suffices to construct a family of $BP_3(n)$-*designs*, for every $n \equiv 0$, 1 mod 4, $n \geq 8$, having only hole and it is at 3.
If $n = 4$, $n = 5$, no gap is possible.
If $n = 8$, the following BP_3-*design* $\Sigma_8 = (X, B)$ is *2-colorable*, *4-colorable*, but it is not *3-colorable*. The chromatic spectrum is $R = (0, r_2, 0, r_4, 0, 0, 0, 0)$, for $r_2 \neq 0$, $r_4 \neq 0$:

$\Sigma_8 = (X, B)$,

$X = \{x_0, x_1, ..., x_7\}$,

B:

$$
\begin{array}{lll}
\langle x_0, x_1, x_2 \rangle & \langle x_0, x_2, x_3 \rangle & \langle x_0, x_3, x_1 \rangle \\
\langle x_4, x_5, x_6 \rangle & \langle x_4, x_6, x_7 \rangle & \langle x_4, x_7, x_5 \rangle \\
\langle x_0, x_4, x_1 \rangle & \langle x_2, x_4, x_3 \rangle & \langle x_0, x_5, x_1 \rangle \\
\langle x_2, x_5, x_3 \rangle & \langle x_0, x_6, x_1 \rangle & \langle x_2, x_6, x_3 \rangle \\
\langle x_0, x_7, x_1 \rangle & \langle x_2, x_7, x_3 \rangle &
\end{array}
$$

Observe that Σ_8 contains two *sub* $-$ BP_3-*designs* of order 4 defined on $Y' = \{x_0, x_1, x_2, x_3\}$ and $Y'' = \{x_4, x_5, x_6, x_7\}$, both *2-colorable*. A coloring f of Σ_8, assigning the same color to x_0, x_1, uses 4 colors necessarily: $f(x_0) = f(x_1) = \alpha_0$ implies $f(x_2) = f(x_3) = \alpha_1$, from which $\{f(x_4), f(x_5), f(x_6), f(x_7)\} = \{\alpha_2, \alpha_3\}$, with $\{\alpha_0, \alpha_1\} \cap \{\alpha_2, \alpha_3\} = \emptyset$. A coloring f of Σ_8, assigning different colors to x_0,x_1, uses 2 colors necessarily: $f(x_0) = \alpha_0$, $f(x_1) = \alpha_1$

implies $\{f(x_2), f(x_3)\} = \{\alpha_0, \alpha_1\}$, with $f(x_2) \neq f(x_3)$, from which $\{f(x_4), f(x_5), f(x_6), f(x_7)\} = \{\alpha_0, \alpha_1\}$. The chromatic spectrum is $R = (0, r_2, 0, r_4, 0, 0, 0, 0)$, for $r_2 \neq 0$, $r_4 \neq 0$.
If $n = 9$, consider the $BP_3(9)$-*design* $\Sigma_9 = (X', B')$, obtained from Σ_8 by Construction A), adding a vertex ∞ and all the blocks $\langle x_{2i}, \infty, x_{2i+1}\rangle$, for every $i = 0, 1, 2, 3$. If f is a coloring of Σ_8 such that $f(x_0) = f(x_1) = \alpha_0$, then $f(x_2) = f(x_3) = \alpha_1$, Σ_8 is *4-colorable* and Σ_9 can be *4-colorable* or *5-colorable*, respectively if $f(x_4) = \alpha_2$, $f(x_5) = \alpha_3$, or if $f(x_4) = f(x_5) = \alpha_2$ and $f(x_6) = f(x_7) = \alpha_3$. If $f(x_0) = \alpha_0$, $f(x_1) = \alpha_1$, Σ_9 is *2-colorable*, but not *3-colorable*. Its chromatic spectrum is: $R = (0, r_2, 0, r_4, r_5, 0, 0, 0, 0)$, for $r_2 \neq 0$, $r_4 \neq 0$, $r_5 \neq 0$.
If $n = 12$, consider the $BP_3(12)$– *design* $\Sigma_{12} = (X, B)$ defined on $X = \{x_0, x_1, ..., x_{11}\}$, containing Σ_8 and having the further blocks:

$$\begin{array}{llll}\langle x_8, x_9, x_{10}\rangle & \langle x_8, x_{10}, x_{11}\rangle & \langle x_8, x_{11}, x_9\rangle & \langle x_5, x_{11}, x_7\rangle \\ \langle x_4, x_{10}, x_6\rangle & \langle x_4, x_{11}, x_6\rangle & \langle x_5, x_{10}, x_7\rangle & \end{array}$$

and $\langle x_{2i}, x_j, x_{2i+1}\rangle$, for $i = 0, 1, 2, 3$, $j = 8, 9$ and for $i = 0, 1$, $j = 10, 11$. Σ_{12} is *2-colorable* , if f is a coloring such that $f(x_0) = \alpha_0$, $f(x_1) = \alpha_1$. In the case $f(x_0) = f(x_1) = \alpha_0$, $f(x_2) = f(x_3) = \alpha_1$, Σ_{12} is *4-colorable* or *5-colorable.* The chromatic spectrum is $R = (0, r_2, 0, r_4, r_5, 0, 0, ..., 0, 0)$, for $r_2 \neq 0$, $r_4 \neq 0$, $r_5 \neq 0$.
If $n = 13$, we consider a $BP_3(13)$-*design* $\Sigma_{13} = (X', B')$ obtained from Σ_{12} by Construction A) with the further blocks $\langle x_{2i}, \infty, x_{2i+1}\rangle$, for every $i = 0, 1, 2, 3$, and $\langle x_8, \infty, x_{10}\rangle$, $\langle x_9, \infty, x_{11}\rangle$. Σ_{13} is *2 - colorable*, *4-colorable*, *5-colorable*, but not *3-colorable.* Its chromatic spectrum is $R = (0, r_2, 0, r_4, r_5, 0, 0, ..., 0, 0)$, for $r_2 \neq 0$, $r_4 \neq 0$, $r_5 \neq 0$.
Now, let $n = 4h$, $h \geq 4$. Let $\Sigma_n = (X, B)$ be the $BP_3(n) - design$ defined on $X = \{x_0, x_1, ..., x_{n-1}\}$, containing h $BP_3(4) - designs$, having vertex set $X_i = \{x_{4i}, x_{4i+1}, x_{4i+2}, x_{4i+3}\}$, for $i = 0, 1, 2, ..., h-1$, and blocks $\langle x_{4i}, x_{4i+1}, x_{4i+2}\rangle$, $\langle x_{4i}, x_{4i+2}, x_{4i+3}\rangle$, $\langle x_{4i}, x_{4i+3}, x_{4i+1}\rangle$, with the following further blocks:
for $i = 1, 2, ..., h-2$, and $j = 4i$, $u = 0, 1, \ldots, 2i-1$,

$$\begin{gathered}\langle x_{2u}, x_j, x_{2u+1}\rangle\\ \langle x_{2u}, x_{j+1}, x_{2u+1}\rangle\\ \langle x_{2u}, x_{j+2}, x_{2u+1}\rangle\\ \langle x_{2u}, x_{j+3}, x_{2u+1}\rangle\end{gathered}$$

for $i = 0, 1, 2, \ldots, 2h-3$,

$$\langle x_{2i}, x_{4h-4}, x_{2i+1}\rangle \quad \langle x_{2i}, x_{4h-2}, x_{2i+1}\rangle$$

for $i = 4, 5, \ldots, 2h-3$,

$$\begin{gathered}\langle x_{2i}, x_{4h-3}, x_{2i+1}\rangle\\ \langle x_{2i}, x_{4h-1}, x_{2i+1}\rangle\end{gathered}$$

and

$$\begin{array}{ll}\langle x_0, x_{4h-1}, x_1\rangle & \langle x_2, x_{4h-1}, x_3\rangle\\ \langle x_4, x_{4h-1}, x_6\rangle & \langle x_5, x_{4h-1}, x_7\rangle\\ \langle x_0, x_{4h-3}, x_1\rangle & \langle x_2, x_{4h-3}, x_3\rangle\\ \langle x_4, x_{4h-3}, x_6\rangle & \langle x_5, x_{4h-3}, x_7\rangle.\end{array}$$

We see that, in a coloring f of Σ_n, if $f(x_0) = \alpha_0$ and $f(x_1) = \alpha_1$, the only possibility for Σ_n is a *2-coloring*. This exists for $\{f(x_0),\ f(x_1)\} = \{\alpha_0,\ \alpha_1\}$, $\{f(x_2),\ f(x_3)\} = \{\alpha_0,\ \alpha_1\}$, $f(x_4) = f(x_7) = \alpha_0$, $f(x_5) = f(x_6) = \alpha_1$, and $f(x_{2i}) = \alpha_0$ and $f(x_{2i+1}) = \alpha_1$ for $i = 4, 5, ..., 2h-1$. If $f(x_0) = f(x_1) = \alpha_0$, then $f(x_2) = f(x_3) = \alpha_1$ and $\{f(x_4), f(x_5), f(x_6), f(x_7)\} \cap \{\alpha_0, \alpha_1\} = \emptyset$. In the case $f(x_4) = f(x_5) = \alpha_2$, it is $f(x_6) = f(x_7) = \alpha_3$ and, since $\langle x_4, x_{4h-3}, x_6\rangle$, $\langle x_5, x_{4h-3}, x_7\rangle$, $\langle x_4, x_{4h-1}, x_6\rangle$, $\langle x_5, x_{4h-1}, x_7\rangle$ are blocks of Σ_n, $f(x_{4h-3}) = f(x_{4h-1}) = \alpha_2$ or α_3. The only possibility is $f(x_{2i}) = f(x_{2i+1}) = \alpha_i$, for every $i = 0, 1, 2, ..., 2h-3$, and $f(x_{4h-4}) = f(x_{4h-2}) = \alpha_{2h-2}$. So, f is a *(2h-1)-coloring*. In the case $f(x_4) = \alpha_2$, $f(x_5) = \alpha_3$, Σ_n is *4-colorable* or *5-colorable*: it is $f(x) = \alpha_2$ or α_3 for every $x \neq x_{4h-3}$, $x \neq x_{4h-1}$ and Σ_n is *4-colorable* if $f(x_6) = \alpha_3$, $f(x_7) = \alpha_2$, Σ_n is *5-colorable* if $f(x_6) = \alpha_2$, $f(x_7) = \alpha_3$. The chromatic spectrum of Σ_n is $R = (0, r_2, 0, r_4, r_5, 0, ..., 0, r_{2h-1}, 0, 0, ..., 0, 0)$, for $r_2 \neq 0$, $r_4 \neq 0$, $r_5 \neq 0$, $r_{2h-1} \neq 0$ and $r_i = 0$, for $5 < i < 2h-1$.

Observe that Σ_n contains a $sub-BP_3(n-3)-design$ $\Sigma'_{n-3} = (X', B')$, defined on $X' = X - \{x_{n-3}, x_{n-2}, x_{n-1}\}$: Σ'_{n-3} is obtained from Σ_n by deleting the vertices x_{n-3}, x_{n-2}, x_{n-1} and all the blocks containing them. All the colorings of Σ'_{n-3} are obtained from colorings of Σ_n, by a restriction. Σ'_{n-3} is 2-colorable, *4-colorable*, *(2h-1)-colorable*. Its chromatic spectrum $R = (0, r_2, 0, r_4, 0, 0, ..., 0, r_{2h-1}, 0, 0, ..., 0, 0)$, for $r_2 \neq 0$, $r_4 \neq 0$, $r_{2h-1} \neq 0$. This completes the proof. □

Theorem 3 *For every admissible* n*,* $n \geq 8$*, there exist* $BP_3(n)$*-designs having the only rightmost hole at* $\lceil n/2 \rceil - 1$ *in the chromatic spectrum.*

Proof. To prove the statement, it suffices to construct a family of $BP_3(n)$-*designs*, for every $n \equiv 0$, 1 mod 4, $n \geq 8$, having chromatic spectrum $R = (0, \ldots, r_{\lceil n/2 \rceil - 2}, 0, r_{\lceil n/2 \rceil}, 0, \ldots, 0)$, for $r_{\lceil n/2 \rceil - 2} \neq 0$, $r_{\lceil n/2 \rceil} \neq 0$, with the only rightmost hole at $\lceil n/2 \rceil - 1$. No gap is possible for $n = 4$, $n = 5$.

Case 1: Let $n \equiv 0$ mod 4, $n \geq 8$. For $n = 8$, the $BP_3(8)$-*design* defined in Theorem 2 has the requested spectrum.
If $n = 12$, the following BP_3-*design* $\Sigma_{12} = (X, B)$ is *4 -colorable*, *6-colorable*, but it is not *5-colorable*. Its chromatic spectrum $R = (0, 0, 0, r_4, 0, r_6, 0, 0, 0, 0, 0, 0)$, for $r_4 \neq 0$, $r_6 \neq 0$:

$\Sigma_{12} = (X, B)$,
$X = \{x_0, x_1, ..., x_{11}\}$,
B:

$$
\begin{array}{lll}
\langle x_0, x_1, x_2 \rangle & \langle x_0, x_2, x_3 \rangle & \langle x_0, x_3, x_1 \rangle \\
\langle x_4, x_5, x_6 \rangle & \langle x_4, x_6, x_7 \rangle & \langle x_4, x_7, x_5 \rangle \\
\langle x_1, x_4, x_3 \rangle & \langle x_1, x_5, x_3 \rangle & \langle x_2, x_6, x_3 \rangle \\
\langle x_2, x_7, x_3 \rangle & \langle x_4, x_0, x_5 \rangle & \langle x_4, x_2, x_5 \rangle \\
\langle x_6, x_0, x_7 \rangle & \langle x_6, x_1, x_7 \rangle &
\end{array}
$$

and

$$
\begin{array}{llll}
\langle x_1, x_8, x_2\rangle & \langle x_4, x_8, x_6\rangle & \langle x_5, x_8, x_7\rangle & \langle x_1, x_9, x_2\rangle \\
\langle x_4, x_9, x_6\rangle & \langle x_5, x_9, x_7\rangle & \langle x_1, x_{10}, x_2\rangle & \langle x_4, x_{10}, x_6\rangle \\
\langle x_5, x_{10}, x_7\rangle & \langle x_1, x_{11}, x_2\rangle & \langle x_4, x_{11}, x_6\rangle & \langle x_5, x_{11}, x_7\rangle \\
\langle x_8, x_0, x_9\rangle & \langle x_{10}, x_0, x_{11}\rangle & \langle x_8, x_3, x_9\rangle & \langle x_{10}, x_3, x_{11}\rangle .
\end{array}
$$

Observe that Σ_{12} contains three *sub-*$BP_3(4)$*-designs* defined on $Y_1 = \{x_0, x_1, x_2, x_3\}$, $Y_2 = (x_4, x_5, x_6, x_7\}$, $Y_3 = \{x_8, x_9, x_{10}, x_{11}\}$. It is not possible that a coloring f of Σ_{12}, assigns the same color to x_0, x_1. In fact, if $f(x_0) = f(x_1) = \alpha_0$, then $f(x_2) = f(x_3) = \alpha_1$, from which $\{f(x_4), f(x_5), f(x_6), f(x_7)\} = \{\alpha_0, \alpha_1\}$, with $\{f(x_4), f(x_5)\} = \{\alpha_0, \alpha_1\}$, $f(x_6) \neq \alpha_1$, $f(x_7) \neq \alpha_1$ and this gives a monochromatic block. So, $f(x_0) = \alpha_0$, $f(x_1) = \alpha_1$ and $\{f(x_2), f(x_3)\} \subseteq \{\alpha_0, \alpha_1\}$. If $f(x_2) = f(x_3)$, then $f(x_2) = f(x_3) = \alpha_1$ and $f(x_i) \notin \{\alpha_0, \alpha_1\}$ for every $i = 4, 5, ..., 11$. Further $f(x_4) = f(x_5)$, $f(x_6) = f(x_7)$. It follows $f(x_4) = f(x_5) = \alpha_2$, $f(x_6) = f(x_7) = \alpha_3$ and then $\{f(x_8), f(x_9), f(x_{10}), f(x_{11})\} = \{\alpha_4, \alpha_5\}$. Therefore, Σ_{12} is *6-colorable.* If $\{f(x_2), f(x_3)\} = \{\alpha_0, \alpha_1\}$, then $\{(f(x_4), f(x_5)\} = \{f(x_6), f(x_7)\} = \{\alpha_0, \alpha_1\}$. We conclude $f(x_3) = \alpha_0$. In fact, for the existence of the blocks $\langle x_1, x_4, x_3\rangle$, $\langle x_1, x_5, x_3\rangle$, $f(x_3) = \alpha_1$ implies $f(x_4) = f(x_5) = \alpha_0$, with a monochromatic block in Σ_{12}. Hence $f(x_2) = \alpha_1$. Further, $\langle x_1, x_i, x_2\rangle \in B$ for every $i = 8, 9, 10, 11$, implies $f(x_i) \neq \alpha_1$, from which $f(x_i) \in \{\alpha_0, \alpha_2, \alpha_3\}$. At last, from $\langle x_8, x_0, x_9\rangle$, $\langle x_8, x_3, x_9\rangle \in B$ it follows $f(x_8) = f(x_9) = \alpha_2$ and from $\langle x_{10}, x_0, x_{11}\rangle$, $\langle x_{10}, x_3, x_{11}\rangle \in B$ it follows $f(x_{10}) = f(x_{11}) = \alpha_3$.

Therefore, Σ_{12} is *4-colorable* and the chromatic spectrum of Σ_{12} is $R = (0, 0, 0, r_4, 0, r_6, 0, 0, 0, 0, 0, 0)$, for $r_4 \neq 0$, $r_6 \neq 0$.

Let $n = 4h$, $h \geq 4$, and let $\Sigma_n = (X, B)$ be the BP_3-*design* defined as follows. $X = \{x_0, x_1, ..., x_{n-1}\}$. The family B contains the following blocks:

for every $r = 0, ..., h - 1$:

$$
\begin{array}{c}
\langle x_{4r}, x_{4r+1}, x_{4r+2}\rangle \\
\langle x_{4r}, x_{4r+2}, x_{4r+3}\rangle \\
\langle x_{4r}, x_{4r+3}, x_{4r+1}\rangle
\end{array}
$$

for every $i = 2, ..., h-2$:

$$\begin{array}{ccc} \langle x_{4i}, x_0, x_{4i+1}\rangle & \langle x_{4i+2}, x_0, x_{4i+3}\rangle & \langle x_{4i}, x_3, x_{4i+1}\rangle \\ \langle x_{4i+2}, x_3, x_{4i+3}\rangle & \langle x_1, x_{4i}, x_2\rangle & \langle x_1, x_{4i+1}, x_2\rangle \\ \langle x_1, x_{4i+2}, x_2\rangle & \langle x_1, x_{4i+3}, x_2\rangle, & \end{array}$$

for every $u = 2, ..., 2i-1$:

$$\begin{array}{c} \langle x_{2u}, x_{4i}, x_{2u+1}\rangle \\ \langle x_{2u}, x_{4i+1}, x_{2u+1}\rangle \\ \langle x_{2u}, x_{4i+2}, x_{2u+1}\rangle \\ \langle x_{2u}, x_{4i+3}, x_{2u+1}\rangle, \end{array}$$

for every $j = 2, ..., 2h-3$,

$$\begin{array}{c} \langle x_{2j}, x_{4h-4}, x_{2j+1}\rangle \\ \langle x_{2j}, x_{4h-3}, x_{2j+1}\rangle \\ \langle x_{2j}, x_{4h-2}, x_{2j+1}\rangle \\ \langle x_{2j}, x_{4h-1}, x_{2j+1}\rangle \end{array}$$

and

$$\begin{array}{cccc} \langle x_1, x_4, x_3\rangle & \langle x_1, x_5, x_3\rangle & \langle x_4, x_2, x_5\rangle & \langle x_2, x_6, x_3\rangle \\ \langle x_2, x_7, x_3\rangle & \langle x_6, x_1, x_7\rangle & \langle x_4, x_0, x_5\rangle & \langle x_6, x_0, x_7\rangle, \end{array}$$

further

$$\begin{array}{cc} \langle x_{4h-4}, x_0, x_{4h-3}\rangle & \langle x_{4h-2}, x_0, x_{4h-1}\rangle \\ \langle x_{4h-4}, x_1, x_{4h-3}\rangle & \langle x_{4h-2}, x_1, x_{4h-1}\rangle \\ \langle x_{4h-4}, x_2, x_{4h-3}\rangle & \langle x_{4h-2}, x_2, x_{4h-1}\rangle \\ \langle x_{4h-4}, x_3, x_{4h-3}\rangle & \langle x_{4h-2}, x_3, x_{4h-1}\rangle. \end{array}$$

Let f be a coloring of Σ_n. Consider the *sub-*$BP_3(12)$*-design* defined on $X_0 \cup X_1 \cup X_2$. It is $f(x_0) \neq f(x_1)$, necessarily.

In fact, if $f(x_0) = f(x_1) = \alpha$, then $f(x_2) = f(x_3) = \beta$, from which $\{f(x_4), f(x_5)\} \subseteq \{\alpha, \beta\}$. But, since $\langle x_4, x_0, x_5\rangle$, $\langle x_4, x_2, x_5\rangle \in B$, then $\{f(x_4), f(x_5)\} = \{\alpha, \beta\}$ and, from this, also $\{f(x_6), f(x_7)\} = \{\alpha, \beta\}$. But the existence of the blocks $\langle x_2, x_6, x_3\rangle$, $\langle x_2, x_7, x_3\rangle$ implies $f(x_6) = f(x_7) = \alpha$, with the monochromaticity of the block $\langle x_6, x_0, x_7\rangle$.
So, $f(x_0) = \alpha_0$, $f(x_1) = \alpha_1$, and this implies $\{f(x_2), f(x_3)\} \subseteq \{\alpha_0, \alpha_1\}$. We see that $f(x_2) = f(x_3) = \alpha_1$, necessarily.
In fact, if $f(x_2) \neq f(x_3)$, since $\langle x_6, x_0, x_7\rangle$, $\langle x_6, x_1, x_7\rangle$ are blocks of Σ_n and $f(x_0) = \alpha_0$, $f(x_1) = \alpha_1$, it follows $\{f(x_6), f(x_7)\} = \{\alpha_0, \alpha_1\}$. This implies $\{f(x_4), f(x_5)\} = \{\alpha_0, \alpha_1\}$, from which $f(x_3) = \alpha_0$ and $f(x_2) = \alpha_1$, because of the existence of the blocks $\langle x_1, x_4, x_3\rangle$, $\langle x_1, x_5, x_3\rangle$. It follows $\{f(x_8), f(x_9), f(x_{10}), f(x_{11})\} = \{\alpha_0, \alpha_1\}$, with $f(x_8) = f(x_9) = \alpha_0$, from which $f(x_{10}) = f(x_{11}) = \alpha_1$. But this implies the monochromaticity of the blocks $\langle x_1, x_{10}, x_2\rangle$, $\langle x_1, x_{11}, x_2\rangle$. So, $f(x_2) = f(x_3)$. Since $\langle x_0, x_2, x_3\rangle \in B$, then $f(x_2) = f(x_3) = \alpha_1$.
Therefore: $f(x_0) = \alpha_0$, $f(x_1) = f(x_2) = f(x_3) = \alpha_1$. It is immediate to see that no vertex of $X_1 \cup X_2$ can be colored by α_1. It is not possible that $f(x_5) = \alpha_0$, otherwise also $f(x_4) = \alpha_0$, for the existence of the block $\langle x_4, x_2, x_5\rangle$, with the monochromaticity of the block $\langle x_4, x_0, x_5\rangle$. It is not possible that $f(x_6) = \alpha_0$ (resp. $f(x_7) = \alpha_0$), otherwise also $f(x_7) = \alpha_0$ (resp. $f(x_6) = \alpha_0$), for the existence of the block $\langle x_6, x_1, x_7\rangle$, with the monochromaticity of the block $\langle x_6, x_0, x_7\rangle$.
Also $f(x_4) \neq \alpha_0$, otherwise $f(x_5) \in \{\alpha_0, \alpha_1\}$, for the existence of the block $\langle x_4, x_2, x_5\rangle$, so $\{f(x_4), f(x_5), f(x_6), f(x_7) \cap \{\alpha_0, \alpha_1\} = \emptyset$.
There is the same conclusion for the vertices of X_2, and we have $\{f(x_8), f(x_9), f(x_{10}), f(x_{11})\} \cap \{\alpha_0, \alpha_1\} = \emptyset$.
We can see that the existence of the blocks $\langle x_4, x_2, x_5\rangle$ and $\langle x_6, x_1, x_7\rangle$ implies $f(x_4) = f(x_5) = \alpha_2$ and $f(x_6) = f(x_7) = \alpha_3$. It follows $\{f(x_8), f(x_9), f(x_{10}), f(x_{11})\} = \{\alpha_4, \alpha_5\}$. If we consider that $\langle x_4, x_2, x_5\rangle$ and $\langle x_6, x_1, x_7\rangle$ are blocks of Σ_n, we can see also that, $f(x_8) = f(x_9)$, $f(x_{10}) = f(x_{11})$, necessarily.
In general we have, for every $i = 1, 2, ..., h-2$:
$\{f(x_{4i}), f(x_{4i+1}), f(x_{4i+2}), f(x_{4i+3})\} = \{\alpha_{2i}, \alpha_{2i+1}\}$ with $f(x_{4i}) = f(x_{4i+1}) = \alpha_{2i}$, $f(x_{4i+2}) = f(x_{4i+3}) = \alpha_{2i+1}$. So, the *sub-*$BP_3$*-design* defined on $X - X_{h-1}$ is uniquely *(2h-2)-colorable.*

Now, consider the vertices of X_{h-1}. It is immediate to see that $\{f(x_{4h-4}), f(x_{4h-3}), f(x_{4h-2}), f(x_{4h-1}\} \cap \{\alpha_2, ..., \alpha_{2h-3}\} = \emptyset$ for the existence of the blocks

$$\begin{array}{ccc} \langle x_{4i}, x_{4h-4}, x_{4i+1}\rangle & \langle x_{4i}, x_{4h-3}, x_{4i+1}\rangle & \langle x_{4i}, x_{4h-2}, x_{4i+1}\rangle \\ \langle x_{4i}, x_{4h-1}, x_{4i+1}\rangle & \langle x_{4i+2}, x_{4h-4}, x_{4i+3}\rangle & \langle x_{4i+2}, x_{4h-3}, x_{4i+3}\rangle \\ & \langle x_{4i+2}, x_{4h-2}, x_{4i+3}\rangle & \langle x_{4i+2}, x_{4h-1}, x_{4i+3}\rangle \end{array}$$

for every $i = 1, 2, ..., h-2$. We see that it is possible to color the vertices of X_{h-1} as follows: $f(x_{4h-4}) = f(x_{4h-2}) = \alpha_{2h-2}$, $f(x_{4h-3}) = f(x_{4h-1}) = \alpha_{2h-1}$. So, Σ_n is *2h-colorable.*
It is also possible to color the vertices of X_{h-1} as follows: $\{f(x_{4h-4}), f(x_{4h-3})\} = \{f(x_{4h-2}), f(x_{4h-1})\} = \{\alpha_0, \alpha_1\}$. So, Σ_n is *(2h-2)-colorable.*
No other case can be verified. In fact, if $f(x_{4h-4}) = \alpha_0$ (resp. α_1), necessarily $f(x_{4h-3}) = \alpha_1$ (resp. α_0) for the existence of the blocks $\langle x_{4h-4}, x_1, x_{4h-3}\rangle$, $\langle x_{4h-4}, x_0, x_{4h-3}\rangle$. Since $\{f(x_{4h-4}), f(x_{4h-3})\} = \{\alpha_0, \alpha_1\}$, then also $\{f(x_{4h-2}), f(x_{4h-1})\} = \{\alpha_0, \alpha_1\}$. If $f(x_{4h-4}) \notin \{\alpha_0, \alpha_1\}$, then $f(x_{4h-4}) = \alpha_{2h-2}$, necessarily $f(x_{4h-3}) = f(x_{4h-4})$, $f(x_{4h-2}) = f(x_{4h-1})$, from which $f(x_{4h-2}) = f(x_{4h-1}) = \alpha_{2h-1}$, because it is not possible $f(x_{4h-2}) = f(x_{4h-1}) = \alpha_0$ or $f(x_{4h-2}) = f(x_{4h-1}) = \alpha_1$.

Case 2: Let $n \equiv 1 \bmod 4$, $n \geq 9$.
For $n = 9$, consider the $BP_3(9)$-*design* Σ_9, obtained by Construction A), starting from the following $BP_3(8)$-*design* $\Sigma'_8 = (Y, C)$:
$\Sigma'_8 = (Y, C)$,
$Y = \{y_0, y_1, ..., y_7\}$,
C:

$$\begin{array}{ccc} \langle y_0, y_1, y_2\rangle & \langle y_0, y_2, y_3\rangle & \langle y_0, y_3, y_1\rangle \\ \langle y_4, y_5, y_6\rangle & \langle y_4, y_6, y_7\rangle & \langle y_4, y_7, y_5\rangle \\ \langle y_0, y_4, y_1\rangle & \langle y_0, y_5, y_1\rangle & \langle y_0, y_6, y_1\rangle \\ \langle y_0, y_7, y_1\rangle & \langle y_4, y_2, y_5\rangle & \langle y_4, y_3, y_5\rangle \\ \langle y_6, y_2, y_7\rangle & \langle y_6, y_3, y_7\rangle & \end{array}$$

and adding a vertex ∞, with all the possible paths $\langle y_{2i}, \infty, y_{2i+1}\rangle$, for every $i = 0, 1, 2, 3$. It is possible to see that Σ_9 is *2-colorable*, *3-colorable*, *5-colorable*, but not *4 -colorable*. Its chromatic spectrum $R = (0, r_2, r_3, 0, r_5, 0, 0, 0, 0)$, for $r_2 \neq 0$, $r_3 \neq 0$, $r_5 \neq 0$.
In fact, if f is a coloring of Σ_9, assigning different colors to y_0, y_1, $f(y_0) = \alpha_0$, $f(y_1) = \alpha_1$, then $f(y_i) \in \{\alpha_0, \alpha_1\}$, for every vertex $y_i \in Y - \{y_0, y_1\}$, because of the existence of the block $\langle y_0, y_i, y_1\rangle \in C$. A *2-coloring* of Σ_9 exists for $f(y_{2i}) = \alpha_0$, $f(y_{2i+1}) = f(\infty) = \alpha_1$.
If f assigns the same color to y_0, y_1, $f(y_0) = f(y_1) = \alpha_0$, then $f(y_2) = f(y_3) = \alpha_1$, $f(\infty) = \alpha_2$ and no one of the others can be colored by α_0. If $f(y_i) = \alpha_1$, for some $i = 4, 5, 6, 7$, then Σ_9 is *3-colorable*: in this case necessarily $\{f(y_4), f(y_5)\} = \{\alpha_1, \alpha_2\}$, $\{f(y_6), f(y_7)\} = \{\alpha_1, \alpha_2\}$, $f(\infty) = \alpha_1$. If $f(y_i) \neq \alpha_1$, for every $i = 4, 5, 6, 7$, then Σ_9 is *5-colorable*: necessarily $f(y_4) = f(y_5) = \alpha_2$, $f(y_6) = f(y_7) = \alpha_3$, $f(\infty) = \alpha_4$.
Let $n = 4h + 1$, $n \geq 13$. Consider the $BP_3(n)$-*design* $\Sigma_n = (Y, C)$, defined on $Y = \{y_0, y_1, ..., y_{4h-1}\} \cup \{\infty\}$, consisting of h $BP_3(4)$-*designs* defined on $Y_i = \{y_{4i}, y_{4i+1}, y_{4i+2}, y_{4i+3}\}$ for every $i = 0, 1, 2, ..., h-1$, containing the previous $BP_3(9)$-*design* Σ_9 with all its blocks, and having the further blocks:

$$\begin{array}{lll}
\langle y_8, y_9, y_{10}\rangle, & \langle y_8, y_{10}, y_{11}\rangle, & \langle y_8, y_{11}, y_9\rangle \\
\dots & \dots & \dots \\
\langle y_{4h-4}, y_{4h-3}, y_{4h-2}\rangle, & \langle y_{4h-4}, y_{4h-2}, y_{4h-1}\rangle, & \langle y_{4h-4}, y_{4h-1}, y_{4h-3}\rangle
\end{array}$$

for every $i = 2, 3, ..., h-1$:

$$\begin{array}{llll}
\langle y_0, y_{4i}, y_1\rangle & \langle y_0, y_{4i+1}, y_1\rangle & \langle y_0, y_{4i+2}, y_1\rangle & \langle y_0, y_{4i+3}, y_1\rangle \\
\langle y_2, y_{4i}, y_3\rangle & \langle y_2, y_{4i+1}, y_3\rangle & \langle y_2, y_{4i+2}, y_3\rangle & \langle y_2, y_{4i+3}, y_3\rangle,
\end{array}$$

and for every $j = 1, 2, \dots, i-1$:

$$\begin{array}{cc} \langle y_{4i}, y_{4j}, y_{4i+1}\rangle & \langle y_{4i}, y_{4j+2}, y_{4i+1}\rangle \\ \langle y_{4i+2}, y_{4j}, y_{4i+3}\rangle & \langle y_{4i+2}, y_{4j+2}, y_{4i+3}\rangle \\ \langle y_{4i}, y_{4j+1}, y_{4i+1}\rangle & \langle y_{4i}, y_{4j+3}, y_{4i+1}\rangle \\ \langle y_{4i+2}, y_{4j+1}, y_{4i+3}\rangle & \langle y_{4i+2}, y_{4j+3}, y_{4i+3}\rangle \end{array}$$

with $\langle y_{2i}, \infty, y_{2i+1}\rangle$, for every $j = 4, 5, ..., 2h - 1$.
We have examined the possible colorings of Σ_9. We try to extend them to all Σ_n. When the coloring f assigns different colors to y_0, y_1, Σ_n is uncolorable or *2-colorable.* A *2-coloring* of Σ_n exists for $f(y_{2i}) = \alpha_0$, $f(y_{2i+1}) = f(\infty) = \alpha_1$. If f assigns the same colour to y_0, y_1, $f(y_0) = f(y_1) = \alpha_0$, then $f(y_2) = f(y_3) = \alpha_1$, $f(\infty) \notin \{\alpha_0, \alpha_1\}$. In the case that $f(y_i) = \alpha_1$, for some $i = 4, 5, 6, 7$, then Σ_n is *(2h-1)-colorable.* In fact, necessarily $\{f(y_4), f(y_5)\} = \{\alpha_1, \alpha_2\}$, $\{f(y_6), f(y_7)\} = \{\alpha_1, \alpha_2\}$, $f(\infty) = \alpha_2$, and $\{f(y_{2j}), f(y_{2j+1})\} = \{\alpha_{j-1}\}$, for every $j = 4, 5, ..., 2h-$ 1 and $f(\infty) = \alpha_2$. If $f(y_i) \neq \alpha_1$, for every $i = 4, 5, 6, 7$, then Σ_n is *(2h+1)-colorable.* In fact, necessarily $f(y_4) = f(y_5) = \alpha_2$, $f(y_6) = f(y_7) = \alpha_3$, and from this $f(y_8) = f(y_9) = \alpha_4$, $f(y_{10}) = f(y_{11}) = \alpha_5, \ldots,$ $f(y_{4h-2}) = f(y_{4h-1}) = \alpha_{2h-1}$, $f(\infty) = \alpha_{2h}$.
This completes the proof. □

4 Concluding remark

In this paper we constructed families of BP_3-*designs* having only one hole: the leftmost or the rightmost. In [4] we constructed families of BP_3-*designs* having gaps of different lengths. We point out that, at the moment, it seems difficult to construct families of BP_3-*designs* having simultaneously the only leftmost and rightmost hole.

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