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On strong quasistability of a vector problem on substitutions

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Abstract

A type of the stability of the Pareto, Smale, and Slater sets for a problem of minimizing linear forms over an arbitrary set of substitutions of the symmetric group is investigated. This type of stability assumes that at least one substitution preserves corresponding efficiency for "small" independent perturbations of coefficients of the linear forms. Quantitative bounds of such a type of stability are found.

In the paper [1], two types of stability for a vector integer linear programming (ILP) problem are investigated. This problem consists in finding the Pareto set. Note that these types of stability are first introduced for a scalar trajectory problem in [2]. In [1], a formula for the strong quasistability radius is deduced and a necessary and sufficient condition of such a stability for a vector ILP problem is obtained. The aim of this paper is to extend these results to vector combinatorial problems of finding the Pareto, Smale, and Slater sets among substitutions of the symmetric group.

1 Preliminaries

Let $m, n \in \mathbb{N}$, $m \geq 2$, $A = [a_{ij}]_{n\times m}$ and $B = [b_{ij}]_{n\times m}$ be the pair of real matrices (throughout the paper, $\mathbb{N}$ denotes the set of natural numbers). Let S_m be the symmetric group of substitutions acting on the set $N_m = \{1, 2, ..., m\}$. On a nonempty set of substitutions $T \subseteq S_m$, we specify the vector criterion

$$f(t, A, B) = (f_1(t, A_1, B_1), f_2(t, A_2, B_2), ..., f_n(t, A_n, B_n)) \to \min_{t \in T}$$

with partial criteria of the form

$$f_i(t, A_i, B_i) = \sum_{j=1}^{m} a_{ij} b_{it(j)}, \ i \in N_n,$$

where $t = \left(\begin{smallmatrix} 1 & 2 & \cdots & m \\ t(1) & t(2) & \cdots & t(m) \end{smallmatrix}\right)$. Here and subsequently, a lower index at a matrix (vector) points to the corresponding row (component) of the matrix (vector). For example, $A_i = (a_{i1}, a_{i2}, ..., a_{im})$.

In this context, traditional definitions (see for instance [3]) of the set of strongly efficient substitutions (Smale set), set of truly efficient substitutions (Pareto set), and set of weakly efficient substitutions (Slater set) have, respectively, the form:

$$T_k^n(A, B) = \{t \in T : \ \tau_k(t, A, B) = \emptyset\}, \ k \in N_3,$$

where

$$\tau_1(t, A, B) = \{t' \in T \setminus \{t\} : \ q(t, t', A, B) \geq 0_{(n)}\},$$

$$\tau_2(t, A, B) = \{t' \in T : \ q(t, t', A, B) \geq 0_{(n)}, \ q(t, t', A, B) \neq 0_{(n)}\},$$

$$\tau_3(t, A, B) = \{t' \in T : \ \forall i \in N_n \ (q_i(t, t', A_i, B_i) > 0)\},$$

$$q(t, t', A, B) = (q_1(t, t', A_1, B_1), q_2(t, t', A_2, B_2), ..., q_n(t, t', A_n, B_n)),$$

$$q_i(t, t', A_i, B_i) = f_i(t, A_i, B_i) - f_i(t', A_i, B_i), \ i \in N_n,$$

$$0_{(n)} = (0, 0, ..., 0) \in \mathbb{R}^n.$$

It follows directly from these definitions that

$$T_1^n(A, B) \subseteq T_2^n(A, B) \subseteq T_3^n(A, B). \tag{1}$$

For any number $k \in N_3$, let us denote by $Z_k^n(A, B)$ the problem of finding the set of efficient substitutions $T_k^n(A, B)$. As we assumed the nonemptiness of the set T, it is evident that $T_2^n(A,B) \neq \emptyset$ and $T_3^n(A,B) \neq \emptyset$ for any $A,B \in \mathbb{R}^{nm}$. Notice that the set of strong efficient substitutions $T_1^n(A,B)$ can be empty. In the sequel, speaking about the problem $Z_1^n(A,B)$ we suppose that $T_1^n(A,B) \neq \emptyset$.

Obviously, if we go over the single-criterion case ($n = 1$, $A,B \in \mathbb{R}^m$), the sets $T_2^1(A,B)$ and $T_3^1(A,B)$ coincide and turn into the set of optimal substitutions whereas our problem turns into a well-known scalar problem of minimizing linear forms over an arbitrary set of substitutions (see, e.g., the monographs [4,5] and the review [6] with its bibliography). In the case when an optimal substitution t^* of the problem $Z_2^1(A,B)$ is unique, we obviously see that $T_1^1(A,B) = T_2^1(A,B) = T_3^1(A,B) = \{t^*\}$. Otherwise the set of strongly efficient substitutions of the scalar problem is empty.

As it was known before, to carry out the solution sensitivity analysis to variation of problem's parameters is one of the important elements of solving practical optimization problems. In this paper, we study the behavior of the set of efficient substitutions for perturbation of elements of the matrix A. Now the question is: how much strongly can one vary these parameters independently from each other such that at least one substitution preserves corresponding efficiency in any perturbed problem? Such a type of stability of a vector problem is accepted to call strong quasistability. Note that the sense of this term is explained in [1]. A quantitative characteristic of similar stability naturally leads to the concept of the strong quasistability radius of the problem. Before giving the strong definition of such a radius, following [2] we introduce the following notation.

For a substitution $t \in T_k^n(A,B)$, $k = 1,2$, let

$$W_k^n(t,B) = \{t' \in T : \ I^n(t,t',B) \neq \emptyset\};$$

for a substitution $t \in T_3^n(A,B)$, let

$$W_3^n(t,B) = \{t' \in T : \ I^n(t,t',B) = N_n\},$$

where

$$I^n(t,t',B) = \{i \in N_n : \ \delta_i(t,t',B_i) > 0\},$$

$$\delta_i(t,t',B_i) = \sum_{j=1}^{m} |b_{it(j)} - b_{it'(j)}|.$$

Clearly, for any substitution $t \in T_1^n(A,B)$, we have

$$T \setminus \{t\} = W_1^n(t,B) = W_2^n(t,B) \supseteq W_3^n(t,B); \qquad (2)$$

for $t \in T_2^n(A,B)$, we have

$$T \setminus \{t\} \supseteq W_2^n(t,B) \supseteq W_3^n(t,B).$$

For any $k \in N_3$, we call a problem $Z_k^n(A,B)$ *nontrivial* if the set $W_k^n(t,B)$ is not empty for any efficient substitution $t \in T_k^n(A,B)$. In the case when there exists a substitution $t \in T_k^n(A,B)$ such that $W_k^n(t,B) = \emptyset$, a problem $Z_k^n(A,B)$ is called *trivial.*

In the above notation, we give the following evident properties.

Property 1 . *The problem $Z_1^n(A,B)$ of finding the set $T_1^n(A,B)$ is nontrivial if and only if $|T| > 1$.*

Property 2 . *If $|T| = 1$, then any problems $Z_2^n(A,B)$ and $Z_3^n(A,B)$ are trivial.*

Property 3 . *If*

$$\exists t' \in T \ \forall i \in N_n \ (q_i(t,t',A_i,B_i) > 0),$$

then $t \notin T_3^n(A,B)$.

Property 4 . *Let a problem $Z_k^n(A,B)$, $k \in N_3$, be nontrivial. Then we have*

$$\forall t \in T_k^n(A,B) \ \forall t' \in W_k^n(t,B) \ (I^n(t,t',B) \neq \emptyset).$$

Property 5 . *It follows that*

$$\delta_i(t,t',B_i) = 0 \ \Rightarrow q_i(t,t',A_i,B_i) = 0,$$

$$I^n(t,t',B_i) = I^n(t,t'',B_i) = \emptyset \ \Rightarrow I^n(t',t'',B_i) = \emptyset.$$

Property 6 . *If* $I^n(t,t',B) \neq N_n$*, then*

$$\forall i \in N_n \setminus I^n(t,t',B)\ \forall A'_i \in \mathbb{R}^m\ (q_i(t,t',A_i + A'_i, B_i) = 0).$$

Property 7 . *Let* $t \in T^n_k(A,B)$*,* $k = 2,3$*,* $|T| > 1$*, and all elements of each row in the matrix* B *pairwise different. Then the following three assertions are true:*

- $\forall t' \neq t\ (I^n(t,t',B) = N_n)$;
- $W^n_k(t,B) = T \setminus \{t\}$;
- *the problem* $Z^n_k(A,B)$ *is nontrivial.*

For any natural number d, by the norm of a vector $x = (x_1, x_2, \ldots, x_d) \in \mathbb{R}^d$ we mean the norm l_∞ as follows:

$$\|x\| = \max\{|x_i| :\ i \in N_d\}.$$

By the norm of a matrix we mean the norm of the vector constrained from the elements of the matrix.

Property 8 . *If different substitutions* $t, t' \in T$*, an index* $i \in N_n$*, and a vector* $A'_i \in \mathbb{R}^m$ *are such that*

$$q_i(t,t',A_i,B_i) + \|A'_i\|\delta_i(t,t',B_i) < 0,$$

then it follows that

$$q_i(t,t',A_i + A'_i, B_i) < 0.$$

Actually, on account of the obvious inequality

$$q_i(t,t',A'_i,B_i) \leq \|A'_i\|\delta_i(t,t',B_i),$$

deduce $q_i(t,t',A_i + A'_i,B_i) = q_i(t,t',A_i,B_i) + q_i(t,t',A'_i,B_i) \leq q_i(t,t', A_i,B_i) + \|A'_i\|\delta_i(t,t',B_i) < 0$.

As stipulated above, perturbation of elements of the matrix A is realized by addition with corresponding elements of a matrix A' of the same dimension. For any number $\varepsilon > 0$, consider the set of perturbation matrices

$$\mathcal{A}(\varepsilon) = \{A' \in \mathbb{R}^{nm} :\ \|A'\| < \varepsilon\}.$$

A problem $Z_k^n(A+A', B)$, where $A' \in \mathcal{A}(\varepsilon)$, obtained from an initial problem $Z_k^n(A, B)$ by addition of matrices A and A' is called *perturbed*.

For arbitrary fixed number $k \in N_3$, the *radius of strong quasistability* of the problem $Z_k^n(A, B)$ will be denoted by $\rho_k^n(A, B)$ and defined by

$$\rho_k^n(A, B) = \begin{cases} \sup\ \Omega, & \text{if } \Omega \neq \emptyset, \\ 0 & \text{otherwise}, \end{cases}$$

where $\Omega = \{\varepsilon > 0 : \ \exists t \in T_k^n(A, B) \ \forall A' \in \mathcal{A}(\varepsilon) \ (t \in T_k^n(A + A', B))\}$.

Thus the radius of strong quasistability of a problem assigns the limit of independent perturbations of elements of the matrix A, with at least one substitution preserving corresponding efficiency.

Due to inclusions (1), we see that

$$\rho_2^n(A, B) \leq \rho_3^n(A, B)$$

for all problems $Z_2^n(A, B)$ and $Z_3^n(A, B)$; if matrices A and B are such that the Smale set $T_1^n(A, B)$ is not empty, then

$$\rho_1^n(A, B) \leq \rho_2^n(A, B) \leq \rho_3^n(A, B). \tag{3}$$

It is natural to consider that the radius of strong quasistability of a problem $Z_k^n(A, B)$ is infinite whenever there exists an efficient substitution $t \in T_k^n(A, B)$ such that $t \in T_k^n(A + A', B)$ for all matrices $A' \in \mathbb{R}^{nm}$.

Lemma 1 . *The radius of strong quasistability of any trivial problem $Z_k^n(A, B)$, $n \in \mathbb{N}$, $k \in N_3$, is equal to infinity.*

Proof. It follows from the triviality of the problem $Z_k^n(A, B)$ that there exists a substitution $t \in T_k^n(A, B), \ k \in N_3$, such that $W_k^n(t, B) = \emptyset$. If $k = 1$, then $|T| = 1$ follows by Property 1 and the lemma is evident in this case.

Further, let $k = 2$. Then, in accordance with the definition of the set $W_2^n(t, B)$, the set $I^n(t, t', B) = \emptyset$ for any substitution $t' \in T$. Therefore Property 6 yields

$$\forall t' \in T \ \forall i \in N_n \ \forall A'_i \in \mathbb{R}^m \ (q_i(t, t', A_i + A'_i, B_i) = 0).$$

This means that t is a truly efficient substitution of the problem $Z_2^n(A+A', B)$ for any perturbation matrix $A' \in \mathbb{R}^{nm}$, i.e. $\rho_2^n(A,B) = \infty$.

Let $k = 3$. Then, according to the definition of the set $W_3^n(t,B)$, for any substitution $t' \in T$, we get $I^n(t,t',B) \neq N_n$. Using Property 6, we therefore obtain

$$\forall t' \in T \ \forall i \in N_n \setminus I^n(t,t',B) \ \forall A'_i \in \mathbb{R}^m \ (q_i(t,t',A_i + A'_i, B_i) = 0).$$

The above means that $t \in T_3^n(A+A', B)$ for any matrix $A' \in \mathbb{R}^{nm}$, i.e. $\rho_3^n(A,B) = \infty$. $\square$

In view of Lemma 1, let us deduce a formula for the radius of quasistability for a nontrivial problem.

The two next properties follow immediately from the definition of $\rho_k^n(A,B)$.

Property 9 . *Let a problem $Z_k^n(A,B)$, $k \in N_3$, be nontrivial and $t \in T_k^n(A,B)$. If there exists a number $\varphi > 0$ such that the inclusion $t \in T_k^n(A+A',B)$ holds for any perturbation matrix $A' \in \mathcal{A}(\varphi)$, then $\rho_k^n(A,B) \geq \varphi$.*

Property 10 . *Suppose a problem $Z_k^n(A,B)$, $k \in N_3$, is nontrivial, $\varphi \geq 0$, and for any number $\varepsilon > \varphi$ there exists a perturbation matrix $A' \in \mathcal{A}(\varepsilon)$ such that $t \in T_k^n(A,B) \Rightarrow t \notin T_k^n(A+A',B)$. Then $\rho_k^n(A,B) \leq \varphi$.*

The following lemma is needed for the sequel

Lemma 2[9]. *Suppose a problem $Z_k^n(A,B)$, $k \in N_3$, is nontrivial, $t \in T_k^n(A,B)$, $\varphi > 0$, and the formula*

$$\forall A' \in \mathcal{A}(\varphi) \ \forall t' \in W_k^n(t,B) \ \exists i \in I^n(t,t',B) \ (q_i(t,t',A_i + A'_i, B_i) < 0)$$

is true. Then $t \in T_k^n(A+A',B)$ for any matrix $A' \in \mathcal{A}(\varphi)$.

2 Main result

For arbitrary number $k \in N_3$, by definition, put

$$\varphi_k^n(A,B) = \max_{t \in T_k^n(A,B)} \ \min_{t' \in W_k^n(t,B)} \ \max_{i \in I^n(t,t',B)} \Gamma_i(t,t',A_i,B_i),$$

$$\Gamma_i(t,t',A_i,B_i) = -\frac{q_i(t,t',A_i,B_i)}{\delta_i(t,t',B_i)}.$$

Theorem. *For any numbers $k \in N_3$ and $n \in \mathbb{N}$, the formula for the radius of strong quasistability of a nontrivial vector problem $Z_k^n(A,B)$ is as follows:*

$$\rho_k^n(A,B) = \varphi_k^n(A,B).$$

Proof. Let $t \in T_k^n(A,B)$, $k \in N_3$. On account of the nontriviality of the problem $Z_k^n(A,B)$, we then conclude that $W_k^n(t,B) \neq \emptyset$. It follows from Property 4 that the set $I^n(t,t',B)$ is empty for each substitution $t' \in W_k^n(t,B)$. It is not hard to see that $\varphi_k := \varphi_k^n(A,B) \geq 0$.

First we prove $\rho_k^n(A,B) \geq \varphi_k$. It is natural to assume that $\varphi_k > 0$ (in the case $\varphi_k = 0$ there is nothing to prove). Then the set of perturbation matrices $\mathcal{A}(\varphi_k) \neq \emptyset$ and according to the definition of the number φ_k, for any matrix $A' \in \mathcal{A}(\varphi_k)$, we get

$$\exists t \in T_k^n(A,B)\ \forall t' \in W_k^n(t,B)\ \exists i \in I^n(t,t',B)$$

$$(\|A_i'\| < \varphi_k \leq \Gamma_i(t,t',A_i,B_i)).$$

Hence Property 8 gives

$$q_i(t,t',A_i + A_i',B_i) < 0.$$

Therefore, applying Lemma 2, one can be sure that $t \in T_k^n(A+A',B)$ for any matrix $A' \in \mathcal{A}(\varphi_k)$. Thus, by Property 9, we finally obtain $\rho_k^n(A,B) \geq \varphi_k$.

Now we prove the inequality $\rho_k^n(A,B) \leq \varphi_k$. According to the definition of the number φ_k, we have

$$\forall t \in T_k^n(A,B)\ \exists t' \in W_k^n(t,B) \forall i \in I^n(t,t',B)$$
$$(\varphi_k \geq \Gamma_i(t,t',A_i,B_i)). \tag{4}$$

Note that $I^n(t,t',B) \neq \emptyset$. Let $\varepsilon > \varphi_k$. Consider the perturbation matrix $A^* = [a_{ij}^*]_{n\times m}$ such that

$$a_{ij}^* = \begin{cases} \alpha & \text{for } b_{it(j)} \geq b_{it'(j)}, \\ -\alpha & \text{for } b_{it(j)} < b_{it'(j)}, \end{cases}$$

where $\varphi_k < \alpha < \varepsilon$. Obviously, the matrix $A^* \in \mathcal{A}(\varepsilon)$.

Taking into account (4) and the structure of the matrix A^*, we easily get

$$\forall i \in I^n(t,t',B)\ (q_i(t,t',A_i+A_i^*,B_i) = q_i(t,t',A_i,B_i)+ \\ +\alpha\delta_i(t,t',B_i) \geq (\alpha-\varphi_k)\delta_i(t,t',B_i) > 0). \tag{5}$$

Let us consider 2 cases.

Case 1: $I^n(t,t',B) = N_n$. By (5) and Property 3, we then have

$$\forall \varepsilon > \varphi_k\ \exists A^* \in \mathcal{A}(\varepsilon)\ (t \notin T_3^n(A+A^*,B)).$$

From (1), we therefore get $t \notin T_k^n(A+A^*,B)$. Thus, according to Property 10, we conclude that $\rho_k^n(A,B) \leq \varphi_k$.

Case 2: $I^n(t,t',B) \neq N_n$. Since $W_k^n(t,B) \neq \emptyset$, it follows from the definition of the set $W_3^n(t,B)$ that $k \neq 3$ in our case. Therefore we have 2 subcases.

2.1. Let $k=2$. Then $t \in T_2^n(A,B)$. By Property 6, we now deduce that

$$q_i(t,t',A_i+A_i^*,B_i) = 0$$

for any index $i \in N_n \setminus I^n(t,t',B)$. Using (5), we hence obtain

$$\forall \varepsilon > \varphi_2\ \exists A^* \in \mathcal{A}(\varepsilon)\ (t \notin T_2^n(A+A^*,B)).$$

Now, taking into account Property 10, we have

$$\rho_2^n(A,B) \leq \varphi_2.$$

2.2. Let $k=1$. Then $t \in T_1^n(A,B)$ and it follows from (1) that

$$\forall \varepsilon > \varphi_1\ \exists A^* \in \mathcal{A}(\varepsilon)\ (t \notin T_1^n(A+A^*,B)).$$

Application of Property 10 concludes $\rho_1^n(A,B) \leq \varphi_1$. $\square$

3. Corollaries

The following assertions are extracted immediately from the theorem.

Corollary 1 . *The radius of strong quasistability $\rho_k^n(A,B)$, $n \in \mathbb{N}$, $k \in N_3$, of any nontrivial problem $Z_k^n(A,B)$ is a finite number.*

Corollary 2 . *For any nontrivial problem $Z_2^n(A,B)$, $n \in \mathbb{N}$, the following conditions are equivalent:*
$\bullet \rho_2^n(A,B) = 0$,
$\bullet \forall t \in T_2^n(A,B) \ \exists t' \in W_2^n(t,B) \ (q(t,t',A,B) = 0_{(n)})$.

Corollary 3 . *For any nontrivial problem $Z_3^n(A,B)$, $n \in \mathbb{N}$, the following conditions are equivalent:*
$\bullet \rho_3^n(A,B) = 0$
$\bullet \forall t \in T_3^n(A,B) \ \exists t' \in W_3^n(t,B) \ (q(t,t',A,B) \geq 0_{(n)})$.

For $n = 1$, the theorem passes on to the following claim.

Corollary 4 . *For the radius of strong quasistability of a nontrivial scalar problem $Z_2^1(A,B)$,*

$$A = (a_1, a_2, ..., a_m), \ B = (b_1, b_2, ..., b_m), \ m \geq 2,$$

the formula is as follows:

$$\rho_2^1(A,B) =$$

$$= \max_{t \in T_2^1(A,B)} \min_{t' \in W_2^1(t,B)} (\sum_{j=1}^{m} a_j(b_{t'(j)} - b_{t(j)}))(\sum_{j=1}^{m} |b_{t(j)} - b_{t'(j)}|)^{-1}.$$

A problem $Z_k^n(A,B)$ is called *strongly quasistable* if its strong quasistability radius is positive.

Corollary 5 . *Any nontrivial problem $Z_1^n(A,B)$, $n \in \mathbf{N}$, is strongly quasistable.*

Proof. Since the problem $Z_1^n(A,B)$ is nontrivial, it follows from Property 1 that $|T| > 1$. Let $t \in T_1^n(A,B)$. Then we have

$$\forall t' \in T \setminus \{t\} \ \exists i \in N_n \ (q_i(t,t',A_i,B_i) < 0). \qquad (6)$$

According to Property 4, $I^n(t,t',B) \neq \emptyset$. Now we show that the index i pointed in (6) belongs to the set $I^n(t,t',B)$. Suppose the contrary, i.e. $i \in N_n \setminus I^n(t,t',B)$. Then $\delta_i(t,t',B_i) = 0$. By Property 5, we now obtain $q_i(t,t',A_i,B_i) = 0$. This contradicts (6).

Thus $i \in I^n(t,t',B)$. Therefore $\delta_i(t,t',B_i) > 0$, i.e., with account of (6), $\Gamma_i(t,t',A_i,B_i) > 0$. Hence we conclude that

$$\forall t' \in W_1^n(t,B)\ \exists i \in I^n(t,t',B)\ (\Gamma_i(t,t',A_i,B_i) > 0).$$

Combining this and the definition of the number $\varphi_1^n(A,B)$, on the basis of the theorem, we obtain

$$\rho_1^n(A,B) = \varphi_1^n(A,B) \geq \Gamma_i(t,t',A_i,B_i) > 0.$$

Finally, the problem $Z_1^n(A,B)$ is strongly quasistable. $\square$

The next claim follows from Corollary 5 and (3).

Corollary 6 . *Suppose the Smale set $T_1^n(A,B)$, $n \in \mathbb{N}$, is not empty. Then the nontrivial problems $Z_2^n(A,B)$ and $Z_3^n(A,B)$ are strongly quasistable.*

Now we cite an example illustrating that the proposition inverse to Property 6 is, in general, not true.

Example. Given $n = 2$, $m = 3$,

$$A = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix},\ B = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix},\ T = \{t_1, t_2, t_3\},$$

where

$$t_1 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix},\ t_2 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix},\ t_3 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}.$$

Then

$$f(t_1,A,B) = f(t_2,A,B) = (1,1),\ f(t_3,A,B) = (3,3),$$

$$T_1^2(A,B) = \emptyset,\ T_2^2(A,B) = T_3^2(A,B) = \{t_1,t_2\}.$$

Using the theorem, one can easily be sure that $\rho_2^2(A,B) = \rho_3^2(A,B) = 1 > 0$, i.e. the problems $Z_2^2(A,B)$ and $Z_3^2(A,B)$ are strongly quasistable whereas the Smale set $T_1^2(A,B)$ is empty.

Thus, unlike a vector integer linear programming problem [1], the nonemptiness of the Smale set is not a necessary condition for the strong quasistability of the vector problems $Z_2^n(A,B)$ and $Z_3^n(A,B)$.

For any nontrivial problem $Z_k^n(A,B)$, $k = 2,3$, by definition, put

$$\dot{T}_k^n(A,B) =$$

$$\{t \in T_k^n(A,B) : \ \forall t' \in W_k^n(t,B)\ \exists i \in I^n(t,t',B)\ (q_i(t,t',A_i,B_i) < 0)\}.$$

Corollary 7 . *Let $k = 2,3$. A necessary and sufficient condition that a nontrivial problem $Z_k^n(A,B)$, $n \in \mathbb{N}$ be strongly quasistable is that the set $\dot{T}_k^n(A,B)$ be nonempty.*

Proof. Necessity. Suppose a nontrivial problem $Z_k^n(A,B)$, $k = 2,3$, is strongly quasistable. Then $\rho_k^n(A,B) > 0$. By the theorem, we now get

$$\exists t \in T_k^n(A,B)\ \forall t' \in W_k^n(t,B)\ \exists i \in I^n(t,t',B)\ (-\frac{q_i(t,t',A_i,B_i)}{\delta_i(t,t',B_i)} > 0).$$

Since $i \in I^n(t,t',B)$, it follows that $\delta_i(t,t',B_i) > 0$. Therefore $q_i(t,t',A_i,B_i) < 0$. Hence $t \in \dot{T}_k^n(A,B)$, i.e. $\dot{T}_k^n(A,B) \neq \emptyset$.

Sufficiency. Let $t \in \dot{T}_k^n(A,B)$. According to the definition of the set $\dot{T}_k^n(A,B)$, we then have

$$\forall t' \in W_k^n(t,B)\ \exists i \in I^n(t,t',B)\ (q_i(t,t',A_i,B_i) < 0). \tag{7}$$

If we combine this and $\delta_i(t,t',B_i) > 0$ (since $i \in I^n(t,t',B)$), we obtain

$$\forall t' \in W_k^n(t,B)\ \exists i \in I^n(t,t',B)\ (\Gamma_i(t,t',A_i,B_i) > 0).$$

Further, taking account of $\dot{T}_k^n(A,B) \subseteq T_k^n(A,B)$, on the basis of the theorem, we derive

$$\rho_k^n(A,B) = \max_{t \in T_k^n(A,B)} \ \min_{t' \in W_k^n(t,B)} \ \max_{i \in I^n(t,t',B)} \Gamma_i(t,t',A_i,B_i) > 0. \ \square$$

In the partial case ($n = 1$), Corollary 7 is as follows.

Corollary 8 . *A scalar nontrivial problem $Z_2^1(A,B)$ is strongly quasistable if the condition*

$$\forall t', t'' \in T_2^1(A,B)\ (\delta_1(t',t'',B)=0) \tag{8}$$

holds.

Proof. Necessity. We give the proof by contradiction. Let there exist optimal substitutions t' and t'' of a problem $Z_2^1(A,B)$ such that

$$\delta_1(t',t'',B) > 0. \tag{9}$$

Since the problem $Z_2^1(A,B)$ is strongly quasistable, it follows from Corollary 7 that $\dot{T}_2^1(A,B) \neq \emptyset$. Let $t \in \dot{T}_2^1(A,B)$, i.e. we have

$$\forall \hat{t} \in W_2^1(t,B)\ (q(t,\hat{t},A,B) < 0).$$

This points that $W_2^1(t,B) \cap T_2^1(A,B) = \emptyset$. It follows that $t', t'' \notin W_2^1(t,B)$. Therefore, we have

$$I^1(t,t',B) = I^1(t,t'',B) = \emptyset.$$

Hence, on the basis of Property 5, we get $I^1(t',t'',B) = \emptyset$, i.e. $\delta_1(t',t'',B) = 0$. This contradicts (9).

Sufficiency. Suppose a problem $Z_2^1(A,B)$ is not strongly quasistable. By Corollary 7, $\dot{T}_2^1(A,B) = \emptyset$, so that

$$\forall t' \in T_2^1(A,B)\ \exists t'' \in W_2^1(t',B)\ (q(t',t'',A,B)=0). \tag{10}$$

Hence $t'' \in T_2^1(A,B)$. Therefore, by (8), we see that $\delta_1(t',t'',B)=0$, i.e. $I^1(t',t'',B)=\emptyset$, and finally that $t'' \notin W_2^1(t',B)$. This contradicts (10). $\square$

Aside from Corollary 7, we obtain the following concomitant result (cf. [1], Corollary 2).

Corollary 9 . *Suppose $n \in \mathbb{N}$, $|T| > 1$, and all elements of each row in a matrix B are pairwise different. A necessary and sufficient condition that the problems $Z_2^n(A,B)$ and $Z_3^n(A,B)$ be strongly quasistable is that the Smale set $T_1^n(A,B)$ be nonempty.*

Proof. Necessity. If all elements of each row in a matrix B are pairwise different, then the problem $Z_k^n(A,B)$, $k = 2, 3$, is nontrivial according to Property 7. Therefore it follows from the strong quasistability of this problem and by virtue of Corollary 7 that $\dot{T}_k^n(A,B) \neq \emptyset$. Suppose $t \in \dot{T}_k^n(A,B)$. Then (7) holds and on account of Property 7, it takes the form

$$\forall t' \in T \setminus \{t\} \; \exists i \in N_n \; (q_i(t,t',A_i,B_i) < 0). \tag{11}$$

Hence the substitution t is strongly efficient, i.e. $T_1^n(A,B) \neq \emptyset$.

Sufficiency. Suppose $T_1^n(A,B) \neq \emptyset$ and $t \in T_1^n(A,B)$. Therefore (1) gives $t \in T_k^n(A,B)$, $k = 2, 3$. Further, according to the definition of the set $T_1^n(A,B)$, we have (11). On account of Property 7, it takes the form (7). Consequently,

$$t \in \dot{T}_2^n(A,B) \cap \dot{T}_3^n(A,B),$$

i.e. the sets $\dot{T}_2^n(A,B)$ and $\dot{T}_3^n(A,B)$ are nonempty. To complete the proof, it remains to apply Corollary 7 stating that the problems $Z_2^n(A,B)$ and $Z_3^n(A,B)$ are strongly quasistable. □

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