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On Planar Mixed Hypergraphs *

Enzo Maria Li Marzi Maria Corinna Marino

Abstract

We consider the maximal planar graphs $\mathcal{G}_n = (\mathcal{X}, \mathcal{S})$, $|\mathcal{X}| = n$, and the set of the triangular faces $\mathcal{T}$ of $\mathcal{G}_n$.

In this paper, $\mathcal{H}_\mathcal{T}$ is a mixed hypergraph, each element of $\mathcal{T}$ is both an edge and a co-edge as in the terminology introduced by Voloshin.

We prove that the lower chromatic number of such hypergraphs is 2 and we determine the upper bound for the upper chromatic number, that is reached by some classes of these hypergraphs.

In 3. the chromatic spectrum is studied and it is proved that, in some cases, it is not broken.

1 Introduction

V.Voloshin defined (1993) [5] the mixed hypergraphs and colourings of such hypergraphs. A *mixed hypergraph* is a triple $\mathcal{H} = (\mathcal{X}, \mathcal{E}, \mathcal{A})$ where $\mathcal{X}$ is a vertex set, $\mathcal{A}$ and $\mathcal{E}$ are both edges set, the elements of $\mathcal{E}$ are called edge "or d-edge" and the elements of $\mathcal{A}$ are called anti-edge "or c-edge". If $\mathcal{A} = \mathcal{E}$, then each subset is called *bi-edge* and $\mathcal{H}$ is called *bi-hypergraph*.

Definition 1. *[5] A strict k-colouring of a mixed hypergraph $\mathcal{H}$ is a colouring of the vertices of $\mathcal{H}$, with colours $\{1, 2, .., k\}$, in such a way that the following conditions hold:*

1 each anti-edge has at least two vertices of the same colour;

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2 no edge is monochromatic;

3 the number of the used colours is exactly k;

4 all vertices are coloured.

Definition 2. *[5] The largest (smallest) k for which exists a strict colouring of $\mathcal{H}$ is called upper (lower) chromatic number, denoted $\bar{\chi}(\mathcal{H})$ ($\chi(\mathcal{H})$).*

Definition 3. *[5] A mixed hypergraph $\mathcal{H}$ is called uncolourable if it admits no strict colouring. Its chromatic numbers χ and $\bar{\chi}$ are both defined 0.*

We say that two strict k-colourings of $\mathcal{H}$ are different if there exist two vertices of $\mathcal{H}$ which have the same colour for one of these colourings and different colours for the other. Any strict k-colouring induces a partition $\{X_1, .., X_k\}$, whose elements are called *colour classes*. Two different strict colourings induce different partitions of $\mathcal{X}$. We associate with the hypergraph $\mathcal{H}$ the vector $R(\mathcal{H}) = (r_1, r_2, .., r_n) \in R^n$, $r_i = r_i(\mathcal{H})$ is the number of strict i-colourings of the hypergraph $\mathcal{H}$. We call $R(\mathcal{H})$ the *chromatic spectrum*; hence

$$R(\mathcal{H}) = (0, .., 0, r_{\chi}, ..., r_{\bar{\chi}}, 0, ..., 0).$$

We say that the chromatic spectrum is *broken* if there exists $\lambda \in N$, such that $\chi(\mathcal{H}) < \lambda < \bar{\chi}(\mathcal{H})$ and $r_{\lambda} = 0$. Recently it was demonstrated that the chromatic spectrum could be broken [3].

Definition 4. *A type of colouring with k colours on n vertices is a vector $(s_1, s_2, ..., s_k) \in N^k \subset R^k$; with $s_i \leq s_{i+1}$ and $s_i \neq 0$ for any $i \in \{1, 2.., k-1\}$, $s_1 + s_2 + ... + s_k = n$.*

A strict k-colouring of a mixed hypergraph $\mathcal{H}$, of order n, belongs to the type $(s_1, s_2, ..., s_k)$ if the colour classes have cardinality s_i, $i = 1, 2, .., k$.

We can give the lexicographic order between the types of colouring:

$$(s_1, s_2, ..., s_k) \leq (s'_1, s'_2, ..., s'_k) \text{ if } s_i = s'_i \text{ for } i \in \{1, 2, .., t\}, \; t < k,$$
$$\text{and } s_{t+1} \leq s'_{t+1}, \text{ and } (e_1, e_2, ..., e_h) < (s_1, s_2, ..., s_k) \text{ if } h < k.$$

We can indicate with t_p^k the type of colouring that occupies the place p-th in the lexicographic order between the types with k colours.

A *walk* of a graph $\mathcal{G}$ is an alternating sequence of vertices and edges of the graph $\mathcal{G}$, beginning and ending with vertices, in which each edge is incident with two vertices immediately preceding and following it. It is a *path* if all the vertices are distinct, and it is a *cycle* if it is closed and all the vertices are distinct. The *degree* of a vertex of a graph $\mathcal{G}$ is the number of the edges incident with this vertex. A *wheel* is a graph that consists of a cycle and a vertex x that is incident to all vertices of the cycle. The vertex x is called the *center* of the wheel. A *planar graph* $\mathcal{G}$ is a graph which can be drawn on the plane so that no two edges intersect. A *plane embedding* of a planar graph $\mathcal{G}$ is a drawing of $\mathcal{G}$ on the plane.

A *maximal planar graph* of order n, denoted $\mathcal{G}_n = (\mathcal{X}, \mathcal{S})$, is a planar graph to which no edge can be added without losing planarity. It is known that in a maximal planar graph every face is a triangle and $|\mathcal{S}| = 3n - 6$, and from the Euler's formula it follows that the number of the faces in a maximal planar graph is $2n - 4$ [2].

We denote the neighbourhood of a vertex x by $\Gamma(x)$, *i. e.* the set formed by all the vertices adjacent to x is defined by $\Gamma(x) = \{y \in \mathcal{X}, \text{ \{x,y\}} \in \mathcal{S}\}$.

Let $X = \{x_1, ..., x_r\}$ be a subset of $\mathcal{X}$, the neighbourhood of X is defined by

$$\Gamma(X) = \Gamma(x_1, .., x_r) = \{y \in \mathcal{X}, y \notin X : \exists x_i \in X, \{x_i, y\} \in \mathcal{S}\}.$$

Let X be a subset of $\mathcal{X}$, we denote with $< X >$ the subgraph induced by X in $\mathcal{G}_n$. $< \Gamma(X) >$ is the subgraph induced by the neighbourhood of X. We denote with $\mathcal{T}$ the set of faces of $\mathcal{G}_n$. From the maximal planarity it follows that if we consider a set $\{x, y, z\} \subseteq \mathcal{X}$ of vertices mutually adjacent and we suppose the existence of another vertex $u \notin \{x, y, z\}$, then there exists a vertex $w \notin \{y, z\}$ adjacent to x. Therefore the maximal planarity implies that every vertex x of $\mathcal{G}_n$ has a degree $d(x) \geq 3$ [4], except the case $n = 3$. Given a plane embedding of $\mathcal{G}_n = (\mathcal{X}, \mathcal{S})$, we add a new vertex x to the vertex set $\mathcal{X}$ of $\mathcal{G}_n$ and we construct a maximal planar graph $\mathcal{G}_{n+1}$. Every vertex of $\mathcal{G}_n$ has the same degree in $\mathcal{G}_{n+1}$, except the vertices of the triangle in which x is added in the plane embedding. We construct a mixed hypergraph $\mathcal{H} = (\mathcal{X}, \mathcal{A}, \mathcal{E})$ where $\mathcal{A} = \mathcal{E} = \mathcal{T}$, denoted by $\mathcal{H}_{\mathcal{T}}$.

In the following $\mathcal{G}_n$ is a maximal planar graph of order n and $\mathcal{H}_{\mathcal{T}}$ the mixed hypergraph defined as above.

Let $E_1, ..., E_m$ be the edges of a hypergraph $\mathcal{H} = (\mathcal{X}, \mathcal{E})$, if $|E_i| = t$, for $i = 1, ..., m$, we say that $\mathcal{H}$ is a *t-uniform hypergraph* [1]. Since each edge of $\mathcal{H}_{\mathcal{T}}$ is a 3-subset of $\mathcal{X}$, we call it a *3-uniform bi-hypergraph*, and we observe that, in any strict colouring, each bi-edge has to be coloured with two colours.

A *bipartite graph* $\mathcal{G} = (\mathcal{V}, \mathcal{S})$ is a graph whose vertex set $\mathcal{V}$ can be partitioned into two subsets $\mathcal{V}_1$ and $\mathcal{V}_2$ such that every edge of $\mathcal{G}$ joins $\mathcal{V}_1$ with $\mathcal{V}_2$ [2].

Any $\mathcal{H}_{\mathcal{T}}$ is a planar hypergraph in accordance with the following definition of Zykov [7].

Definition 5. *[6, 7] Let $\mathcal{H} = (\mathcal{X}, \mathcal{E})$ be a hypergraph. We define the bipartite graph $\mathcal{G} = (\mathcal{X} \cup \mathcal{E}, \mathcal{S})$ where the edge set $\mathcal{S}$ conserves the incidence between the vertices of $\mathcal{X}$ and the edges of $\mathcal{E}$. If $\mathcal{G}$ is planar, we say that $\mathcal{H}$ is a planar hypergraph.*

In [6] Voloshin first suggested to investigate colourings of planar hypergraphs in context of mixed hypergraphs. In 2. we determine the lower chromatic number and the upper bound for the upper chromatic number, that is reached by some classes of these particular kinds of hypergraphs.

In 3. we investigate on the chromatic spectrum and we prove that in some cases it is continuous.

2 Lower and upper chromatic numbers.

Theorem 6. *Let $\mathcal{G}_n = (\mathcal{X}, \mathcal{S})$ be a maximal planar graph of order n, and $\mathcal{H}_\mathcal{T}$ the mixed hypergraph associated to $\mathcal{G}_n$. Then $\mathcal{H}_\mathcal{T}$ is 2-colourable.*

Proof. It is trivial that the lower chromatic number of $\mathcal{H}_\mathcal{T}$ can not be less than 2. The four colour problem says that any planar graph is 4-colourable, so we can colour $\mathcal{G}_n$ with 4 colours labelled $0_4, 1_4, 2_4, 3_4$ as the classes of $\mathbf{Z}_4$. Then to colour $\mathcal{G}_n$ we change each class of $\mathbf{Z}_4$ with the correspondent class of $\mathbf{Z}_2$; so the vertices of $\mathcal{G}_n$ are coloured with the colour 0_2 and 1_2, so that any face is coloured with 2 colours; it follows that every hypergraph $\mathcal{H}_\mathcal{T}$ is 2-colourable. □

Lemma 7. *Let $\mathcal{G}_n = (\mathcal{X}, \mathcal{S})$ be a maximal planar graph of order n, $\mathcal{H}_\mathcal{T}$ the mixed hypergraph associated to $\mathcal{G}_n$ and we suppose that there exists a strict k-colouring of $\mathcal{H}_\mathcal{T}$. Then the following assertions hold:*

1. If $X_j = \{x\}$ is a colour class ,then $< \Gamma(x) >$ has at least one monochromatic cycle.

2. If $X_j = \{x, y\} \in \mathcal{S}$, $x \neq y$, $\Gamma(x) \setminus \{y\} \neq \Gamma(y) \setminus \{x\}$, is a colour class, then $< \Gamma(x, y) >$ has at least one monochromatic cycle.

3. If $X_j = \{x, y\} \notin \mathcal{S}$, $x \neq y$, is a colour class, then $< \Gamma(x, y) >$ has at least one monochromatic cycle.

4. If there exists a colour class $X_j = \{x, y\} \in \mathcal{S}$, $x \neq y$, and $\Gamma(x) \setminus \{y\} = \Gamma(y) \setminus \{x\}$, then $< \Gamma(x, y) >$ is a monochromatic path, $k = 2$ and $\mathcal{X} = \Gamma(x, y) \cup \{x\} \cup \{y\}$.

Proof. 1. For every $x \in \mathcal{X}$, $d(x) \geq 3$. We consider the wheel with center x, then $< \Gamma(x) >$ has a cycle. This cycle is monochromatic because there are no triangles 3-coloured.

2. We consider the vertices $u, z \in \Gamma(y) \setminus \{x\}$, $u, z \notin \Gamma(x) \setminus \{y\}$, the wheel with center x and the wheel with center y as in figure 1 .

3. We consider the wheel with center x (or y) and the proof is the same of 1.

4. The wheel with center x and the wheel with center y determine a monochromatic path (figure 2).

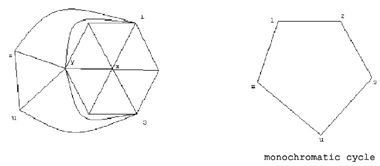

Figure 1.

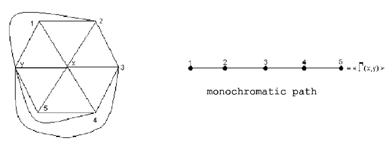

Figure 2.

We consider a plane embedding of $\mathcal{G}_n$ and we suppose that there exists a vertex $z \notin \Gamma(x,y) \cup \{x\} \cup \{y\}$. In this plane embedding z is inside a triangle that has a vertex x (or y). Then x (or y) has another adjacent vertex not considered in the wheel with center x (or y). Therefore $\mathcal{X} = \Gamma(x,y) \cup \{y\} \cup \{x\}$ and $k = 2$. $\square$

Suppose that $\mathcal{H}_{\mathcal{T}}$ has a strict k-colouring, we say that a *monochromatic cycle of $\mathcal{G}_n$ is generated by a colour class X_j, such that* $|X_j| \leq 2$, if it is constructed as in the proof of the Lemma 7.

Lemma 8. *Let $\mathcal{H}_{\mathcal{T}}$ be the mixed hypergraph associated to $\mathcal{G}_n$. Suppose that there exists a strict k-colouring, $k \geq 3$, of $\mathcal{H}_{\mathcal{T}}$, then the following assertions hold:*

1. Every colour class X_j, $|X_j| \le 2$, generates at least one monochromatic cycle of $\mathcal{G}_n$.

2. Let i, c be respectively the number of colour classes X_j, $|X_j| \le 2$, and the number of monochromatic cycles of $\mathcal{G}_n$ generated by them, then $(i-1) \le c$ and $(i-1) = c$ implies $k = 3$.

Proof. 1. Since $k \ge 3$, we can not have the hypothesis of the lemma 7 (part 4), then every colour class X_j, $|X_j| \le 2$, generates at least one monochromatic cycle.

2. We suppose that $X_1, \ldots.., X_i$ are all the colour classes, such that $|X_j| \le 2$, $1 \le j \le i$, and we suppose that $X_1, \ldots, X_h$ $(h < i)$ generate the monochromatic cycles $c_1, \ldots, c_h$, ($c_r \neq c_s$, $\forall\ r, s \le h$, $r \neq s$). If $X_{h+1} = \{x\}$ generates cycles that are already obtained from $X_1, \ldots, X_h$, then $< \Gamma(x) >= c_r$ $(r \le h)$ and there are no other vertices. Then $k = 3$, $c = 1$, $(i-1) = c$.

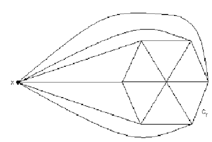

Figure 3.

If there is another vertex in $\mathcal{G}_n$, it is inside a triangle of vertex x in the plane embedding (figure 3). Then there is another vertex adjacent to x, which is not in c_r and $< \Gamma(x) > \neq c_r$. If $X_{h+1} = \{x, y\} \in \mathcal{S}$, then similarly $k = 3$, $c = 1$, $i = 2$, $(i-1) = c$.

If $X_{h+1} = \{x, y\} \notin \mathcal{S}$, then $< \Gamma(x) >= c_r$, $(r \le h)$ and there are no other vertices. Therefore $y \notin \mathcal{X}$. □

Given a strict k-colouring of $\mathcal{H}_{\mathcal{T}}$, $k \ge 3$, we indicate with $\mathcal{C}$, $c = |\mathcal{C}|$, the set of monochromatic cycles of $\mathcal{G}_n$, $n > 3$, generated by colour classes of cardinality $l \leq 2$.

We consider the colour classes of cardinality $l \geq 3$. On the vertices of each of these classes we construct the maximal planar graph without the external triangle. We indicate with $\mathcal{T}'$ the triangular face set obtained. If the strict k -colouring belongs to the type $\mathbf{t}_p^k = (1, .., 1, 2, ..., 2, s_{i+1}, ..., s_k)$, then from the Euler's formula we have

$$|\mathcal{T}'| \ = 2\Sigma_{j=i+1}^k s_j - 5(k-i).$$

For $k \geq 3$, we define also $\Delta_p^k = i - |\mathcal{T}'| - 1$. It is easy to observe that $|\mathcal{C}| \leq |\mathcal{T}'| + 1$, and if $i < k-1$ $|\mathcal{C}| \leq |\mathcal{T}'|$.

Lemma 9. *Let $\mathbf{t}_p^k$ be a type of colouring of $\mathcal{H}_\mathcal{T}$ such that $\Delta_p^k = \ i - |\mathcal{T}'| - 1 > 0$, then there does not exist a strict k-colouring belonging to $\mathbf{t}_p^k$.*

Proof. From the lemma 8, if $(i-1) = c$, then $k = 3$, $i = 2$, $c = 1$ and $i - |\mathcal{T}'| - 1 = 1 - |\mathcal{T}'| \leq 0$. If $i \leq c$, because $c \leq |\mathcal{T}'| + 1$, we have $0 < i - |\mathcal{T}'| - 1 \leq c - |\mathcal{T}'| - 1$, which is a contradiction.

□

Lemma 10.

$$\begin{aligned} a)\ If\ \Delta_p^k &> 0,\ then\ \Delta_j^k > 0,\ for\ every\ j > p; \\ b)\ If\ \Delta_1^\lambda &> 0,\ then\ \Delta_p^k > 0, \forall k \geq \lambda,\ \forall p > 1. \end{aligned}$$

Proof. a) If $\mathbf{t}_p^k = (s_1, ..., s_k)$, then

$$\begin{aligned} \mathbf{t}_{p+1}^k &= (s'_1, .., s'_k),\ \text{where}\ s'_l = s_l + 1, s'_m = s_m - 1,\ l < m \\ \text{and}\ s'_j &= s_j,\ \forall\ \ j\ \in\ \{1, .., l-1, l+1, ..., m-1, m+1, ...k\}\,. \end{aligned}$$

By the definition of $\mathcal{T}'$ it follows that $\Delta_j^k \geq \Delta_p^k > 0$.

b) From a) it follows that $\Delta_j^\lambda > 0$, $\forall\ j > 1$.

From $\Delta_1^\lambda > 0$ it follows that $\Delta_j^{\lambda+1} > 0$ and then from a) $\Delta_p^k > 0$, $\forall\ k \geq \lambda$, $\forall\ p \geq 1$. □

Theorem 11. *Let* $\mathcal{H}_{\mathcal{T}}$ *be of order* n, $n > 3$.

$$a)\ \textit{If}\ n \equiv 0(mod\ 3)\ \textit{then}\ \bar{\chi}(\mathcal{H}_{\mathcal{T}}) \leq \frac{2n}{3} - 1;$$
$$b)\ \textit{If}\ n \equiv 1(mod\ 3)\ \textit{then}\ \bar{\chi}(\mathcal{H}_{\mathcal{T}}) \leq 2(\frac{n-1}{3});$$
$$c)\ \textit{If}\ n \equiv 2(mod\ 3)\ \textit{then}\ \bar{\chi}(\mathcal{H}_{\mathcal{T}}) \leq \frac{2n-1}{3}.$$

Proof. a) If $\lambda = \frac{2n}{3}$, there is no colouring that belongs to the type $\mathbf{t}_1^\lambda = (1,..,1,\frac{n}{3}+1)$. In fact $\Delta_1^\lambda = 1 > 0$ then, by the lemmas, $\Delta_p^k > 0$, $\forall\ k \geq \lambda$ and $\forall p \geq 1$.

b) If $\lambda = \frac{2n+1}{3}$, $\mathbf{t}_1^\lambda = (1,...,1,\frac{n+2}{3})$, then $\Delta_1^\lambda = 2 > 0$ that implies $\Delta_p^k > 0$, $\forall k \geq \lambda$ and $\forall p \geq 1$.

c) If $\lambda = 2\frac{n+1}{3}$, $\mathbf{t}_1^\lambda = (1,..,1,\frac{n+1}{3})$, then $\Delta_1^\lambda = 3 > 0$ that implies $\Delta_p^k > 0$, $\forall k > \lambda$ and $\forall p \geq 1$. □

Theorem 12. *Let* $\mathcal{H}_{\mathcal{T}}$ *be of order* $n > 3$*:*

a) If $n \equiv 0(mod\ 3)$ *and there exist strict* λ*-colourings with* $\lambda = \frac{2n}{3} - 1$, *then these colourings belong to the types* t_1^λ, t_2^λ.

b) If $n \equiv 1(mod\ 3)$ *and there exist strict* λ*-colourings with* $\lambda = 2(\frac{n-1}{3})$, *then these colourings belong to the type* t_1^λ.

c) If $n \equiv 2(mod\ 3)$ *and there exist strict* λ*-colourings, with* $\lambda = \frac{2n-1}{3}$, *then these colourings belong to the type* t_1^λ.

Proof. a) $\Delta_1^\lambda = -2 < 0$, $\Delta_2^\lambda = 0$. If $\Delta_j^\lambda > 0$, $(j > 2)$, from the lemma 9 there is no strict λ- colouring belonging to the type $\mathbf{t}_j^\lambda$ and from the lemma 10 there is no strict λ-colouring belonging to the type $\mathbf{t}_p^\lambda$, $(p > j)$. Suppose $\Delta_j^\lambda = 0$, $(j > 2)$. For $\lambda > 3$, from the lemma 8, it follows that $i \leq c$. The monochromatic triangles, generate by the type of colouring $\mathbf{t}_j^\lambda$ are $(i-1)$, $(i - |\mathcal{T}'| - 1 = 0\ )$, then we have no colourings of this type. If $\lambda = 3$ there are monocoloured triangles in $\mathcal{H}_{\mathcal{T}}$, that exclud strict λ-colorings belonging to the type $\mathbf{t}_j^\lambda$.

b) $\Delta_1^\lambda = -1 < 0$, $\Delta_2^\lambda = 1 > 0$ then b) follows from the lemma 10.
c) $\Delta_1^\lambda = 0$, $\Delta_2^\lambda = 2 > 0$ analogously □

Example 13. Let $\mathcal{G}_{12}$ be a maximal planar graph of order 12, and $\mathcal{H}_\mathcal{T}$ its associated mixed hypergraph.

From the theorem 11 $\bar{\chi}(\mathcal{H}_\mathcal{T}) \leq 7$ and from the theorem 12 if $\bar{\chi}(\mathcal{H}_\mathcal{T}) = 7$ then a strict 7-colouring belongs to the types $\mathbf{t}_1^7$, $\mathbf{t}_2^7$. In fact $\mathbf{t}_1^8 = (1,1,1,1,1,1,1,5)$, $\Delta = 1 > 0$, then there are no strict 8-colourings.

$\mathbf{t}_1^7 = (1,1,1,1,1,1,6)$, $\Delta = -2 < 0$; $\mathbf{t}_2^7 = (1,1,1,1,1,2,5)$, $\Delta = 0$. $\mathbf{t}_3^7 = (1,1,1,1,1,3,4)$, $\Delta = 0$, from the lemma 8 $i \leq c$ and from $\Delta = 0$ $i = |\mathcal{T}'| + 1$; but $i = 5$, $|\mathcal{T}'| = 4$, then there is no strict 7-colourings of this type.

$\mathbf{t}_4^7 = (1,1,1,1,2,2,4)$, $\Delta = 2$ and we exclude strict 7-colourings of the types:

$(1,1,1,1,2,2,4), \ldots\ldots\ldots\ldots\ldots\ldots, (1,1,2,2,2,2,2)$.

$\mathbf{t}_1^6 = (1,1,1,1,1,7), \ldots\ldots\ldots\ldots, \mathbf{t}_6^6 = (1,1,2,3,4)$, $\Delta < 0$.

$\mathbf{t}_7^6 = (1,1,1,3,3,3)$, $\Delta = 0$, every colour class of cardinality 1 generates a monochromatic cycle (a triangle), then we have a 3-coloured triangle. It follows that there is no strict 6-colouring of this type.

$\mathbf{t}_8^6 = (1,1,2,2,2,4)$, $\Delta > 1$, then we exclude all the other types:

$(1,1,2.2,2,4), \ldots\ldots\ldots\ldots, (2,2,2,2,2,2)$.

$\mathbf{t}_1^5 = (1,1,1,1,8), \ldots\ldots\ldots \mathbf{t}_{11}^5 = (2,2,2,2,4)$, $\Delta \leq 0$.

$\mathbf{t}_{12}^5 = (2,2,2,3,3)$, $\Delta = 0$; from the lemma 8 it follows $i \leq c$, then there is no strict 5-colouring of this type.

We exclude at last the type $\mathbf{t}_1^2 = (1,11)$, because it determines monocoloured triangles.

Let $\mathcal{G}_m$ be a maximal planar graph of order m, we consider the following constructions:

a. We contruct a maximal planar graph $\mathcal{G}_n$, $n = 3m - 3$ obtained by $\mathcal{G}_m$ adding a new vertex in every face of $\mathcal{G}_m$, excluding in a face in which we add two new vertices.

b. We construct the maximal planar graph $\mathcal{G}_n$, $n = 3m - 2$ adding a new vertex in every face of $\mathcal{G}_m$, excluding in a face where we add three new vertices.

c. We construct the maximal planar graph $\mathcal{G}_n$, $n = 3m - 4$, adding a new vertex in every face of $\mathcal{G}_m$.

Corollary 14. *Let $\mathcal{G}_m$ be a maximal planar graph, $\mathcal{G}_n$ a maximal planar graph which follows by the constructions* $\mathbf{a}, \mathbf{b}, \mathbf{c}$ *and $\mathcal{H}_\mathcal{T}$ its associated mixed hypergraph. Then:*

$$
\begin{aligned}
a)\ \bar{\chi}(\mathcal{H}_\mathcal{T}) &= \frac{2n}{3} - 1,\ if\ n = 3m - 3;\\
b)\ \bar{\chi}(\mathcal{H}_\mathcal{T}) &= \frac{2(n-1)}{3},\ if\ n = 3m - 2;\\
c)\ \bar{\chi}(\mathcal{H}_\mathcal{T}) &= \frac{2n-1}{3},\ \ if\ n = 3m - 4.
\end{aligned}
$$

Proof. a) We colour all vertices of $\mathcal{G}_m$ with the same colour α, the two new vertices added in the same face of $\mathcal{G}_m$ with the colour β and every other vertex with colours different from α and β and different from each other.

b) We colour all vertices of $\mathcal{G}_m$ with α and we colour also with α the vertex indicated in the figure 4.

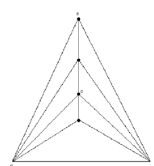

Figure 4.

We colour all the other vertices with colours different from α and different from each other.

c) We colour all vertices of $\mathcal{G}_m$ with α and the other vertices with colours different from α and from each other.

□

3 Chromatic spectrum.

Theorem 15. *If there exists a strict k-colouring of $\mathcal{H}_\mathcal{T}$, with $k \geq 3$, such that at least a colour class has cardinality $l \leq 2$, then there exists a strict $(k-1)$-colouring of $\mathcal{H}_\mathcal{T}$.*

Proof. Let X_j be a colour class such that $|X_j| \leq 2$ and α, β, γ colours of the strict k-colouring. If $X_j = \{x\}$ (or $X_j = \{x, y\} \in \mathcal{S}$) and colour of X_j is α, colour of $\Gamma(X_j)$ is β, we recolour X_j with the colour γ, eliminating in this way one colour. The case $X_j = \{x, y\} \notin \mathcal{S}$ is resolved in the same way.

□

Corollary 16. *If there exists a strict $\bar{\chi}$-colouring such that $(\bar{\chi} - 1)$ classes have cardinality $l \leq 2$, then the chromatic spectrum of $\mathcal{H}_\mathcal{T}$ is not broken.*

Corollary 17. *Let $\mathcal{G}_m$ be a maximal planar graph and $\mathcal{G}_n$ a maximal planar graph which follows by the constructions* $\mathbf{a}, \mathbf{b}, \mathbf{c}$. *Then the mixed hypergraph $\mathcal{H}_\mathcal{T}$, associated to $\mathcal{G}_n$, has the chromatic spectrum not broken.*

Proof. It follows from the corollary 16, if we consider the $\bar{\chi}(\mathcal{H}_\mathcal{T})$-colouring given in the proof of corollary 14.

□

Open problem:

Determine a lower bound for the upper chromatic number of the mixed hypergraphs $\mathcal{H}_\mathcal{T}$, reached by elements of this class of hypergraphs.

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E.M. Li Marzi, M.C. Marino,
Department of Mathematics, University of Messina,
Contrada Papardo, Salita Sperone 31,
98166 - Sant'Agata - Messina
e-mail: *limarzi@dipmat.unime.it*
marino@ingegneria.unime.it

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