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On approximate solution of non-elliptic singular integral equation systems in Lebesgue spaces

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Abstract

We investigate in this paper problems of theoretical foundation of collocations and mechanical quadratures methods for approximate solution of singular integral equation systems in the case, when their symbols have on the integration contour a finite set of integer degree zeroes.

We consider the case when equation digitalization points are taken in such a way that each point of the contour can be included in the set of points.

The proposed methods converge in Lebesgue spaces L_p, $1 < p < \infty$.

1 Algorithms

Let us consider a system of singular integral equations (SIE)

$$\begin{array}{c}(M\varphi)(t) \equiv C(t)\varphi(t) + D(t)(S\varphi)(t)+ \\ +\frac{1}{2\pi\ i}\int_\Gamma K(t,\tau)\varphi(\tau)d\tau = f(t), \quad t \in \Gamma,\end{array} \tag{1}$$

where $f(t)$ is a vector function of dimension m, $C(t), D(t)$ and $K(t,\tau)$ are matrix-valued functions of degree m, $\varphi(t)$ is an unknown vector function, and S is a singular operator with Cauchy kernel. Let us assume that Γ is a simple closed smooth contour, and let the Riemann function of this contour $z = \psi(w)$ have a continuous by Hölder second derivative $\psi^{''}(w) \in H_\nu(\Gamma_0)$, $\Gamma_0 = \{w : |w| = 1\}$. The class of such contours is denoted by $C(2,\nu)$ (see [1]).

Let us produce a digitization of equation (1) at points

$$t_j = \psi(w_j^{(\theta)}),\quad w_j^{(\theta)} = \exp(\tfrac{2\pi\ i}{2n+1}\cdot j + \theta\cdot i),\\ \theta\in[0;2\pi],\quad j=0,2n,\quad i^2=-1. \tag{2}$$

We note that the presence of parameter θ in equation (2) make it possible to take every preassigned point of the contour as a point of digitization.

In comparison with the investigations made earlier, we have extended in this case possibilities for applications, as well as for selection of the points of digitization.

We replace the unknown function $\varphi(\tau)$ in the digitizated equation by its interpolate Lagrange polynomial

$$(U_n\varphi)(\tau) = \textstyle\sum_{j=0}^{2n}\varphi_j\cdot l_j(\tau),\quad \tau\in\Gamma\\ l_j(\tau) = \textstyle\prod_{k=0,k\neq j}^{2n}\frac{\tau-t_k}{t_j-t_k} = \sum_{r=-n}^{n}\Lambda_r^{(j)}\cdot\tau^r,\quad j=0,2n. \tag{3}$$

As a result, we get the following system of linear algebraic equations (SLAE):

$$C(t_j)\varphi + D(t_j)\cdot\textstyle\sum_{k=0}^{2n}\varphi_k\cdot[\sum_{r=0}^{n}\Lambda_r^{(k)}\cdot t_j^r - \sum_{r=-n}^{-1}\Lambda_r^{(k)}\cdot t_j^r]+\\ +\textstyle\sum_{k=0}^{2n}\varphi_k\sum_{r=-n}^{n}\Lambda_r^{(k)}\cdot\frac{1}{2\pi\ i}\int_\Gamma k(t_j,\tau)\cdot\tau^r d\tau = f(t_j)\quad j=\overline{0,2n} \tag{4}$$

with respect to the unknowns $\varphi_j = \varphi(t_j),\quad j=\overline{0,2n}$.

The relations (2)-(4) define the algorithm of collocation method to solve SIE (1). We note that to calculate numbers $\Lambda_r^{(k)}\quad k=\overline{0,2n},\quad r=\overline{-n,n}$ introduced in (3) we can use the Vietta theorem.

If we substitute in SLAE (4) the contour integrals by some quadrature formula, then we obtain the algorithm of quadratures method. We can use the following interpolated quadrature formula:

$$\frac{1}{2\pi\ i}\int_\Gamma g(\tau)\cdot\tau^r d\tau \cong \frac{1}{2\pi\ i}\int_\Gamma U_n[\tau\cdot g(\tau)]\cdot\tau^{r-1}d\tau \equiv \sum_{k=0}^{2n} t_k\cdot g(t_k)\cdot\Lambda_{-r}^{(k)}. \tag{5}$$

Thus the algorithm of quadrature is defined by relations (2), (3) and SLAE

$$\begin{aligned} C(t_j)\varphi_j + D(t_j) \cdot \textstyle\sum_{k=0}^{2n} \{\cdot[\sum_{r=0}^{n} \Lambda_r^{(k)} \cdot t_j^r - \sum_{r=-n}^{-1} \Lambda_r^{(k)} \cdot t_j^r] + \\ + \textstyle\sum_{r=-n}^{n} \Lambda_r^{(k)} \cdot t_k \cdot K(t_j, t_k) \Lambda_{-r}^{(k)}\}\varphi_k = f(t_j) \quad j = \overline{0, 2n}. \end{aligned} \tag{6}$$

2 Theoretical foundation of the method

We shall get the theoretical foundation of proposed computational schemes for nonelliptic system SIE, that is for SIE systems whose symbols $A(t) = C(t) + D(t)$ and $B(t) = C(t) - D(t)$ have zeroes on the contour Γ. Note that in the case of elliptic system SIE, the theoretical foundation of these methods was obtained in [3]. For a standard contour $\Gamma = \Gamma_0$, the problems of foundation of these methods were investigated in [5].

Let us assume that the function A(t) has a system of zeroes $\{\alpha_1, \alpha_2, \ldots, \alpha_r\}$ of integer degrees $\{m_1, m_2, \ldots, m_r\}$ and the function $\det B(t)$ has a system of zeroes $\{\beta_1, \beta_2, \ldots, \beta_s\}$ of integer degrees $\{m_1, m_2, \ldots, m_s\}$.

In this case the matrix-valued functions $A(t)$ and $B(t)$ permit the following representation

$$\begin{aligned} A(t) &= A_1(t) D_-(t) S_-(t), \\ B(t) &= B_1(t) D_+(t) S_+(t). \end{aligned} \tag{7}$$

Here $D_-(t)$ and $D_+(t)$ are diagonal matrix-valued functions of the form

$$D_-(t) = \{\prod_{k=1}^{r} (t^{-1} - \alpha_k^{-1})^{\mu_j^{(k)}} \delta_{jl}\}_{j,l=1}^{m},$$

$$D_+(t) = \{\prod_{k=1}^{r} (t - \beta_s)^{\nu_j^{(k)}} \delta_{jl}\}_{j,l=1}^{m},$$

$\mu_1^{(k)} \geq \mu_2^{(k)} \geq \ldots \geq \mu_m^{(k)} \geq 0, \quad k = \overline{1, r}$ and $\nu_1^{(k)} \geq \nu_2^{(k)} \geq \ldots \geq \nu_m^{(k)}, \quad k = \overline{1, s}$ are integer numbers, $S_\pm(t)$ are polinomials with respect to t and t^{-1} matrixes with constant and non-zero determinant, and $A_1(t)$, $B_1(t)$ are nonsingular matrix-valued functions.

Let us introduce the following notation

$$k = \max\{\mu_1^{(1)}, \dots, \mu_1^{(r)}, \nu_1^{(1)}, \dots, \nu_1^{(s)}\}. \tag{8}$$

Theorem 1. *Let the following conditions fulfill*

1) *the contour* Γ *belongs to the class* $C(2, \nu)$*;*

2) *the representations (7) are true, in which* $A_1(t)$ *and* $B_1(t)$ *are nonsingular matrix-valued function of the class* $H_{\alpha,m\times m}^{(k+1)}(\Gamma)$, $0 < \alpha \leq 1$*, and the number* k *is defined by formula (8);*

3) *the left partial indexes of matrix-valued function* $B_1^{-1}(t)A_1(t)$ *are equal to zero;*

4) *the kernel of the system SIE (1) is contained in the class* $C_{m\times m}^{(k)}(\Gamma \times \Gamma)$*;*

5) *the homogeneous equations corresponding to SIE (1) have only trivial solutions.*

Then for all sufficiently large n *and for any* $f \in C_m^{(k)}(\Gamma)$*, SLAE (4) has a unique solution* φ_j, $j = \overline{0, 2n}$*. The vector functions (3) converge for* $n \to \infty$ *by the norm of space* $L_{p,m}(\Gamma)$ *to solution* $\varphi \in L_{p,m}(\Gamma)$*. Moreover, the following estimation holds*

$$\|\varphi - U_n\varphi\|_{L_p} = O(n^{\sigma(\alpha)}) + O(\omega(f^{(k)}, \frac{1}{n})) + O(\omega^t(K^{(k)}; \frac{1}{n})) \tag{9}$$

$\sigma(\alpha) = \alpha$ *if* $0 < \alpha < 1$ *and* $\sigma(1) = 1 - \varepsilon$, $\varepsilon(> 0)$ *is an arbitrary small number.*

Theorem 2. *Let the conditions 1)-3) and 5) of the Theorem 1 fulfill, and let instead of condition 4) of the Theorem 1 the following condition holds*

4') *the kernel of SIE (1) belongs to* $H_{\alpha,m\times m}^{(k)}(\Gamma)$ *with respect to* τ *and* $C_{m\times m}^{(k)}(\Gamma)$ *with respect to* t.

Then the statements of Theorem 1 are valid, if we replace SLAE (4) by SLAE (6) and add in the right-hand side of the estimation (9) the term $O(\omega^\tau(K;\frac{1}{n}))$

3 The proof of Theorem 1

By conditions 2) and 3), the matrix-valued function $B_1^{(-1)}A_1$ admits the left canonical factorization

$$B_1^{-1}(t)\cdot A_1(t) = V_+(t)\cdot V_-(t); \tag{10}$$

$V_+^{\pm 1}(t)\in PH^{(k+1)}_{\sigma(\alpha),m}$, $V_-^{\pm 1}(t)\in QH^{(k+1)}_{\sigma(\alpha),m}\oplus\{const\}$, $P=\frac{1}{2}(I+S)$, $Q=I-P$.

Denoting $G=D_-S_-$, $G_+=D_+S_+$ from the relation (7) and (10), we get

$$M = AP+BQ+K = B_1V_+\{(PV_-+QV_+^{-1})(PG_-+QG_+)+ \\ QV_-G_-P+PV_+^{-1}G_+Q+V_+^{-1}B_1^{-1}K\};$$

To deduce this equality, it was taken into account that $PV_-GQ = QV_+^{-1}G_+P=0$.

Then SLAE of collocation method is equivalent to the following operator equation

$$\begin{gathered}(V_n\equiv)\\ U_n[(PV_-+QV_+^{-1})(PG_-+QG_+)+K_1+K_2]U_n\varphi = U_nf,\end{gathered} \tag{11}$$

where $K_1 = QV_-G_-P+PV_+^{-1}G_+Q$, $\;K_2=V_+^{-1}B_1^{-1}K$.

Next we shall prove that the operator V_n is inversible for sufficiently large n. It is easy to verify that $ImK_1|_{L_{p,m}(\Gamma)}\subset H^{(k)}_{\alpha,m}(\Gamma)$ $\forall g(t)\in L_{p,m}(\Gamma)$,

$$\|K_1g\|_{H^{(k)}_{\alpha,m}}\le d_1\cdot\|g\|_p. \tag{12}$$

Moreover, $ImK_2|_{L_{p,m}(\Gamma)}\subset C^{(k)}_m(\Gamma)$ and

$$\forall g\in L_{p,m}(\Gamma), \|K_2g\|_{C^{(k)}_m(\Gamma)}\le d_2\cdot\|g\|_{L_{p,m}}. \tag{13}$$

Let us denote by Y the Banach space

$$Y = \{g(t) \in L_{p,m}(\Gamma) : (Rg)(t) \in W^{(k)}_{p,m}(\Gamma)\},$$

where $R = PG_- + QG_+, W^{(k)}_{p,m} = W$ is the Sobolev space of vector functions $g(t) \in L_{p,m}(\Gamma)$ having the generalized derivatives $g^{(r)}(t) \in L_{p,m}(\Gamma), \quad r = 0, 1, \ldots, k$. The norm in W is defined by formula

$$\|g\|_W = \sum_{r=0}^{k} \|g^{(r)}\|_{L_{p,m}}.$$

A space Y is the Banach space if we introduce the norm by the rule

$$|g| = \|Rg\|_W.$$

The operator $PV_- + QV_+^{-1}$ is inversible in W. Therefore the operator $M_0 = (PV_- + QV_+^{-1})R$ is inversible as an operator acting from Y into W.

Let $\bar{U}_n$ be the contraction of U_n onto Y. It is clear that $ImR\bar{U}_n \subset W$ and for each function $g(t) \in Y$, we have

$$R\bar{U}_n g \in ImU_n. \tag{14}$$

Let us show now that the operator

$$U_n M_0 \bar{U}_n = U_n(PV_- + QV_+^{-1})R\bar{U}_n$$

considered as the operator acting from $\bar{U}_n Y$ into $U_n W (= Z_n)$ is inversible for sufficiently large $n (\geq n_0)$. Let us assume that $V_-^{(n)}(t)$ and $V_+^{(n)}(t)$ are polynomial matrix-valued functions of degree n in powers of t and $\frac{1}{t}$ respectively of the best uniform approximation for matrix-valued functions $V_-(t)$ and $V_+^{-1}(t)$. Then we obtain by [2]

$$\|V_- - V_-^{(k)}\|_c = O(\frac{1}{n^{\sigma(\alpha)+k}}) \; and \; \|V_+^{-1} - V_+^{(n)}\|_c = O(\frac{1}{n^{\sigma(\alpha)+k}}). \tag{15}$$

Taking into account the stability of the trivial particular indexes [6], we conclude that beginning with numbers n for which hold the inequalities

$$\|V_-^{-1}(V_- - V_-^{(n)})\|_c \le q_1 < 1 \; and \; \|V_+(V_+^{-1} - V_+^{(n)})\| \le q_2 < 1, \quad (16)$$

the left partial indexes of the matrix-valued functions $V_-^{(n)}(t)$ and $V_+^{(n)}(t)$ all are equal to zero and

$$\det V_-^{(n)}(t) \neq 0, \quad \det V_+^{(n)}(t) \neq 0, \quad t \in \Gamma.$$

It is easy to deduce from this that the operator $R_n = U_n[(PV_-^{(n)} + QV_+^{(n)})R]\bar{U}_n : \bar{U}_n Y \to Z_n$ is inversible, and also $R_n^{-1} = \bar{U}_n R^{-1}[P(V_-^{(n)})^{-1} + Q(V_+^{(n)})^{-1}]U_n$.

The form of R^{-1} is brought in [5, page 269]. Estimate the norm $R_n^{-1} : Z_n \to \bar{U}_n Y$. Let $g_n(t) \in Z_n$. Then

$$\|R_n^{-1} g_n\|_{\bar{U}_n Y} = |R_n^{-1} g_n| = \|R \cdot R_n^{-1} g_n\|_W =$$

$$= \|P(V_-^{(n)})^{-1} + Q(V_+^{(n)})^{-1}\|_W \le$$

$$\le \sum_{j=0}^{k} \{\|[P(V_-^{(n)})^{-1} g_n]^{(j)}\|_{L_{p,m}} + \|[Q(V_+^{(n)})^{-1} g_n]^{(j)}\|_{L_{p,m}}\}.$$

Taking into consideration that $P(V_-^{(n)})^{-1} g_n$ and $Q(V_+^{(n)})^{-1} g_n$ are polynomials of the form $\sum_{k=0}^{n} r_k \cdot t^k$ and $\sum_{k=-n}^{-1} r_k \cdot t^k$ respectively and applying the analog of the Marcov inequality for the norm of derivative of polynominal deduced in [4], we obtain

$$\|R_n^{-1} g_n\|_{\bar{U}_n Y}\| \le \sum_{j=0}^{k} \{c_j n^j \|P(V_-^{(n)})^{-1} g_n\|_{L_{p,m}} +$$

$$+ d_j n^j \|Q(V_+^{(n)})^{-1} g_n\|_{L_{p,m}}\} \le \; const \cdot n^k \cdot \|g\|_{L_{p,m}}.$$

Hence

$$\|R_n^{-1}\|_{\bar{U}_n Y \to Z_n} = O(n^k) \quad (17)$$

Using relations (15), it is easy to find that

$$\|U_n M_0 \bar{U}_n - R_n\|_{\bar{U}_n \to Z_n} = O(\frac{1}{n^{\sigma(\alpha)}}).$$

It follows from this and from (17) that for sufficiently large values of $n(\geq N_1)$, namely for all n for which (16) is valid and

$$\|R_n^{-1} \cdot (U_n M_0 \bar{U}_n - R_n)\|_{Z_n} \leq q_3 < 1,$$

the operator $U_n M_0 \bar{U}_n : \bar{U}_n Y \to Z_n$ is inversible and

$$\|(V_n M_0 \bar{V}_n)^{-1}\| = O(n^k).$$

Now the inversibility of the operator V_n defined in (11) follows from relations (12) and (13), and from the inversibility of the operator $\mu_0 + k_1 + k_2$; moreover

$$\|U_n(M_0 + K_1 + K_2)\bar{U}_n\| = O(n^k), \quad n \geq (n_1 \geq N_1). \tag{18}$$

Hence the inversibility of SLAE (4) of colocations method is established for all $n \geq n_1(\geq N_1)$. The estimation for the rate of convergence (9) can be obtained by well-known method (see, e.g., [7]) using the following equality from [4]:

$$\forall g \in H_\alpha^{(k)}, \|U_n g - g\|_{L_{p,m}} = O(\frac{1}{n^{k+\alpha}}),$$

which holds because of selection of the interpolation points and of smoothness of the contour $\Gamma : \Gamma \in C(2;\mu)$. Theorem 1 is proved.

The proof of Theorem 2 is performed following the same scheme as for Theorem 1, using the sharp estimation of quadrature formula (5) obtained in monography [2]. Theorem 2 is proved.

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