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Stability of a majority efficient solution of a vector linear trajectorial problem*

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Abstract

The multicriteria problem of majority choice on a system of subsets of a finite set with linear partial criteria (MINSUM) is considered. Sufficient and necessary conditions of preserving majority efficiency by an efficient trajectory under "small" perturbations of vector criterion coefficients have been found. Lower and upper attainable estimates of the stability radius of a majority efficient trajectory have been obtained.

In the papers [1,2] stability of different types of efficient solutions (optimal by Pareto, Smale, and Slater) of a vector trajectorial problem was investigated. Sufficient and necessary conditions of local stability (the property of a trajectory to preserve appropriate efficiency under "small" independent perturbations of vector criterion parameters) of such solutions were presented. Lower attainable estimates of stability radii of such trajectories, and formulas in several cases, were obtained. A summary of these results can be found in [3].

In this work the parralel results have been obtained for a majority efficient solution being an element of the Pareto set.

Note that the papers [4–14] are devoted to various types of stability of the whole Pareto set of vector discrete optimization problems.

1 Definitions and properties

As in [1], we consider a class of vector discrete optimization problems described with the following model.

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Let $E = \{e_1, e_2, \ldots, e_m\}$, $m > 1$ be a set with a vector weight function $a : E \longrightarrow \mathbf{R}^n$, $n \geq 1$. We can thereby treat this function as a matrix $A = [a_{ij}]_{n\times m} \in \mathbf{R}^{nm}$, where $(a_{1j}, a_{2j}, ..., a_{nj}) = a(e_j)$, $j \in N_m = \{1, 2, ..., m\}$. Let $T = \{t\} \subseteq 2^E \setminus \{\emptyset\}$, $|T| > 1$ be a set of elements named trajectories and the vector criterion

$$f(t, A) = (f_1(t, A), f_2(t, A), ..., f_n(t, A))$$

be defined on T. The partial criteria of the vector criterion are linear:

$$f_i(t, A) = \sum_{j\in N(t)} a_{ij} \longrightarrow \min_{t\in T}, \quad i \in N_n,$$

where

$$N(t) = \{j \in N_m : e_j \in t\}.$$

By $Z^n(A)$ we denote the problem of finding the set of majority efficient trajectories [15–17]:

$$T^n_M(A) = \{t \in T : \mu(t) = \emptyset\},$$

where

$$\mu(t) = \{t' \in T : \sum_{i=1}^{n} \text{sign } \tau_i(t, t', A) > 0\},$$

$$\tau_i(t, t', A) = f_i(t, A) - f_i(t', A) = \sum_{j\in N(t\setminus t')} a_{ij} - \sum_{j\in N(t'\setminus t)} a_{ij}.$$

In the particular case, where the number of criteria $n = 1$ the set of majority efficient trajectories turns into the set of optimal solutions $T^1_M(A)$, $A = (a_{11}, a_{12}, ..., a_{1m})$, and our problem turns into the linear scalar (singlecriterion) trajectorial problem. Its stability radius is studied in detail by V.K.Leont'ev and E.N.Gordeyev (see, for example, refs [18–21] and bibliography there).

It is easy to see that many problems of combinatorial optimization with linear objectives can be treated as special cases of trajectorial problem: optimization problems on graphs, Boolean programming, and some sheduling problems.

It can easily be understood that any majority efficient trajectory is efficient, i.e. an element of the Pareto set [22]:

$$T_P^n(A) = \{t \in T : \pi(t) = \emptyset\},$$

where

$$\pi(t) = \{t' \in T : \tau(t,t',A) \geq \mathbf{0}, \quad \tau(t,t',A) \neq \mathbf{0}\},$$
$$\tau(t,t',A) = (\tau_1(t,t',A), \tau_2(t,t',A), ..., \tau_n(t,t',A)),$$
$$\mathbf{0} = (0,0,...,0) \in \mathbf{R}^n.$$

Indeed, if a majority efficient trajectory t does not belong to the Pareto set, then there exists a trajectory $t' \in T$ such that

$$\tau(t,t',A) \geq \mathbf{0}, \quad \tau(t,t',A) \neq \mathbf{0},$$

i.e. $\mu(t) \neq \emptyset$. This contradicts the definition of the majority efficient trajectory.

In the two-criteria case ($n = 2$) the following equality clearly holds:

$$T_M^2(A) = T_P^2(A).$$

Thus the sets $T_M^1(A)$ and $T_M^2(A)$ are always non-empty.

The following example shows that the set of majority efficient trajectories of 3-criteria problem may be empty.

Example 1. Let $E = \{e_1, e_2, e_3\},\ t_1 = \{e_1\},\ t_2 = \{e_2\},\ t_3 = \{e_3\}$,

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{pmatrix}.$$

Then $f(t_1) = (1,2,3),\ f(t_2) = (2,3,1),\ f(t_3) = (3,1,2)$. It is obviously that any of trajectories t_1, t_2, t_3 is efficient but none of them are majority efficient.

The absence of majority efficient trajectories of the problem $Z^n(A)$ may be explained as follows. The binary relation

$$t \prec t' \ \Leftrightarrow\ \sum_{i=1}^{n} \text{sign}\ \tau_i(t,t',A) < 0,$$

specifying the set $T_M^n(A)$, is not always transitive for $n \geq 3$. In this example

$$t_1 \prec t_2 \prec t_3 \prec t_1.$$

The example characterizes the situation known as the Condorcet-Arrow's voting paradox [23,24] (see also [16,17]).

It is easy to see that the above binary relation can be intepreted as a group choice relation under that a trajectory t is "preferred" to a trajectory t', if t surpasses t' in more criteria than t' surpasses t.

For any vector $x \in \mathbf{R}^n$ we denote

$$N_n^+(x) = \{i \in N_n : x_i > 0\},$$

$$N_n^-(x) = \{i \in N_n : x_i < 0\},$$

$$N_n^0(x) = \{i \in N_n : x_i = 0\}.$$

Let

$$k^+ = k^+(t, t', A) =\mid N_n^+(\tau(t, t'A)) \mid,$$

$$k^- = k^-(t, t', A) =\mid N_n^-(\tau(t, t'A)) \mid,$$

$$k^0 = k^0(t, t', A) =\mid N_n^0(\tau(t, t'A)) \mid .$$

It is evident that $k^+ + k^- + k^0 = n$.

In this notation the following equivalent definition of the set $T_M^n(A)$ holds:

$$t \in T_M^n(A) \;\Leftrightarrow\; \forall t' \in T \quad (k^-(t, t', A) \geq k^+(t, t', A)). \tag{1}$$

Arising from the above definition, the following properties are valid for any natural number n.

Property 1 $\forall t \in T_M^n(A) \;\; \forall t' \in T \quad (k^-(t, t', A) \geq k^+(t, t', A))$.

Property 2 $\forall t \in T_M^n(A) \;\; \forall t' \in T_M^n(A) \quad (k^-(t, t', A) = k^+(t, t', A))$.

Property 3 $\forall t' \in T \setminus \{t\} \; (k^-(t, t', A) \geq \frac{n}{2} \;\Rightarrow t \in T_M^n(A))$.

Hereafter we investigate stability of a trajectory $t \in T^n_M(A)$ in the usual case [1–3, 4–11, 18–21], where all the vector criterion parameters are independently perturbed by adding of a perturbing matrix to the matrix A.

For an arbitrary number $\varepsilon > 0$ we define the set of perturbing matrices

$$\mathcal{B}(\varepsilon) = \{B \in \mathbf{R}^{nm} : \| B \| < \varepsilon\},$$

where $\| B \| = max\{| b_{ij} | : (i, j) \in N_n \times N_m\}$ is the Chebyshev norm (l_∞) of the matrix $B = [b_{ij}]_{n \times m}$.

The following property is valid by the continuity of functions $f_i(t, A)$ and, consequently, of their differences $\tau_i(t, t', A)$ on the set $\mathbf{R}^{nm}$ of all $n \times m-$ matrices.

Property 4

$$\exists \varepsilon > 0 \ \ \forall t, t' \in T \ \ \forall B \in \mathcal{B}(\varepsilon)$$

$$(k^-(t, t', A) \leq k^-(t, t', A + B)) \ \& \ (k^+(t, t', A) \leq k^+(t, t', A + B)).$$

2 Stability criterion

On the the analogy of [1,2], we say that a trajectory $t \in T^n_M(A)$ is stable if there exists a number $\varepsilon > 0$ such that the trajectory preserves majority efficiency in any perturbed problem $Z^n(A + B),\ B \in \mathcal{B}(\varepsilon)$, i.e.

$$\exists \varepsilon > 0 \ \ \forall B \in \mathcal{B}(\varepsilon) \qquad (t \in T^n_M(A + B)).$$

A trajectory $t \in T^n_M(A)$ is naturally called unstable if

$$\forall \varepsilon > 0 \ \ \exists B \in \mathcal{B}(\varepsilon) \qquad (t \notin T^n_M(A + B)).$$

To derive necessary and sufficient conditions of stability we need the following lemma. It is valid in particular for the vector trajectorial problems with linear partial criteria.

Lemma 1 [1] Let $t, t' \in T,\ t \neq t',\ B(\alpha) = [b_{ij}]_{n\times m}$ be a perturbing matrix with elements

$$b_{ij} = \begin{cases} \alpha, & \text{if } \ i \in N_n,\ j \in N(t \setminus t'); \\ -\alpha, & \text{if } \ i \in N_n,\ j \in N(t' \setminus t); \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

Then setting

$$\Delta(t, t') =\mid t \setminus t' \ \cup \ t' \setminus t \mid,$$

we have

$$\forall i \in N_n \ \forall \alpha \in \mathbf{R} \ \ (\tau_i(t, t', A + B(\alpha)) = \tau_i(t, t', A) + \alpha\Delta(t, t')).$$

Note that $\Delta(t, t') > 0$ for $t \neq t'$.

Theorem 1 *A majority efficient trajectory* t *of the problem* $Z^n(A), n \geq 1$ *is stable if and only if*

$$\forall t' \in T \setminus \{t\} \qquad \left(k^-(t, t', A) \geq \frac{n}{2}\right). \tag{3}$$

Proof. Sufficiency. From (3) and property 4, we obtain the following

$$\exists \varepsilon > 0 \quad \forall t' \in T \setminus \{t\} \ \ \forall B \in \mathcal{B}(\varepsilon) \quad \left(k^-(t, t', A + B) \geq \frac{n}{2}\right).$$

Thus by property 3 we have

$$\exists \varepsilon > 0 \quad \forall B \in \mathcal{B}(\varepsilon) \qquad (t \in T^n_M(A + B)),$$

i. e. the trajectory t is stable.

Necessity. Suppose the opposite: the condition (3) is not valid for a stable majority efficient trajectory t of the problem $Z^n(A)$, i. e. there exists a trajectory $t' \in T$ such that

$$k^-(t, t', A) < \frac{n}{2}.$$

Then in view of property 1, we have

$$k^+(t, t', A) + k^0(t, t', A) > k^-(t, t', A) \geq k^+(t, t', A). \tag{4}$$

Let $\varepsilon > 0$, $B(\alpha) = [b_{ij}]_{n \times m}$ be the perturbing matrix with elements defined by rule (2), where $0 < \alpha < \varepsilon$. Then using lemma 1, we get

$$\forall i \in N_n \ \ (\tau_i(t, t', A + B(\alpha)) = \tau_i(t, t', A) + \alpha\Delta(t, t')).$$

This implies the following (by virtue of $\alpha > 0, \ \Delta(t, t') > 0$):

$$k^+(t, t', A + B(\alpha)) \geq k^+(t, t', A) + k^0(t, t', A),$$

$$k^-(t, t', A + B(\alpha)) \leq k^-(t, t', A).$$

Therefore due to (4) we have

$$k^+(t, t', A + B(\alpha)) > k^-(t, t', A) \geq k^-(t, t', A + B(\alpha)).$$

Now, according to definition (1) of the majority efficient trajectory we conclude that $t \notin T^n_M(A + B(\alpha))$.

Thus under our assumption we can find a perturbing matrix $B \in \mathcal{B}(\varepsilon)$ for any $\varepsilon > 0$ such that the majority efficient trajectory t of the problem $Z^n(A)$ loses its efficiency in the perturbed problem $Z^n(A + B)$. Hence the trajectory t is unstable. The contradiction proves the necessity of the theorem.

Theorem 1 has been proved.

If a majority efficient trajectory t of the problem $Z^n(A)$ is not strictly efficient, i. e. does not belong to the traditional Smale set [22]

$$T^n_S(A) = \{t' \in T : \sigma(t') = \emptyset\},$$

where $\sigma(t') = \{t'' \in T \setminus \{t'\} : f(t') \geq f(t'')\}$, then it is easy to understand that

$$\exists t^* \in T \ \ (k^0(t, t^*, A) = n),$$

i. e. $k^-(t, t^*, A) = 0 < \frac{n}{2}$. Thus from theorem 1, we derive

Corollary 1 *Any stable majority efficient trajectory is strictly efficient.*

The next statement follows directly from theorem 1.

Corollary 2 *Let $t, t' \in T_M^n(A)$, $n \geq 1$. If $k^0(t, t', A) > 0$, then both trajectories t and t' are unstable.*

If we combine this with property 2, we obtain

Corollary 3 *If the number of criteria $n \geq 1$ is odd, then for stability of a trajectory $t \in T_M^n(A)$ it is necessary the trajectory to be a unique majority efficient trajectory, i. e. $\mid T_M^n(A) \mid = 1$.*

In particular, the next well-known result [18] follows from the above: *An optimal trajectory of scalar problem $Z^1(A)$ is stable only if the problem has a unique optimal trajectory.*

Following examples attest the converses of corollaries 1–3 to be not always valid.

Example 2. Let $E = \{e_1, e_2, e_3, e_4\}$, $t_1 = \{e_1\}$, $t_2 = \{e_2, e_3\}$, $t_3 = \{e_2, e_4\}$,

$$A = \begin{pmatrix} 1 & 2 & 1 & 0 \\ 2 & 2 & 1 & 1 \\ 3 & 1 & 1 & 1 \\ 3 & 1 & 0 & 2 \end{pmatrix}.$$

Then $f(t_1) = (1, 2, 3, 3)$, $f(t_2) = (3, 3, 2, 1)$, $f(t_3) = (2, 3, 2, 3)$.

The both majority efficient trajectories t_1 and t_2 in this example are strictly efficient but the trajectory t_1 is stable, whereas t_2 is unstable $(k^-(t_2, t_3, A) = 1 < 2)$.

This means that the converse of corollary 1 is not valid in the instant case.

Example 3. Let $E = \{e_1, e_2, e_3\}$, $t_1 = \{e_1\}$, $t_2 = \{e_2\}$, $t_3 = \{e_1, e_3\}$, $t_4 = \{e_2, e_3\}$,

$$A = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 2 & 1 & 1 \\ 1 & 2 & 0 \end{pmatrix}.$$

Then $f(t_1) = (2,1,2,1)$, $f(t_2) = (1,2,1,2)$, $f(t_3) = (2,1,3,1)$, $f(t_4) = (1,2,2,2)$.

The majority efficient trajectories t_1 and t_2 are both unstable since $k^-(t_1, t_3, A) = k^-(t_2, t_4, A) = 1 < 2$. Nevertheless $k^0(t_1, t_2, A) = 0$. Thus the converse of corollary 2 does not hold.

Example 4. Let $E = \{e_1, e_2, e_3, e_4\}$, $t_1 = \{e_1, e_2\}$, $t_2 = \{e_1, e_3\}$, $t_3 = \{e_4\}$,

$$A = \begin{pmatrix} 1 & 2 & 0 & 2 \\ 1 & 0 & 1 & 2 \\ 1 & 1 & 2 & 1 \end{pmatrix}.$$

Then we have $f(t_1) = (3,1,2)$, $f(t_2) = (1,2,3)$, $f(t_3) = (2,2,1)$. It is obviously that the strictly efficient trajectory t_3 is a unique majority efficient trajectory of this problem. But it is unstable because $k^-(t_3, t_2, A) = 1 < \frac{3}{2}$.

This means that uniqueness of a majority efficient trajectory is not a sufficient condition of its stability (see corollary 3).

3 Stability radius

Let us remind (see [1,2]) that the stability radius of an efficient trajectory t of the problem $Z^n(A)$ is the value

$$\rho_P^n(t, A) = \begin{cases} \sup \Omega(A), & \text{if } \Omega(A) \neq \emptyset, \\ 0, & \text{if } \Omega(A) = \emptyset, \end{cases}$$

where

$$\Omega(A) = \{\varepsilon > 0 : \ \forall B \in \mathcal{B}(\varepsilon) \quad (t \in T_P^n(A + B))\}.$$

The equality $\rho_P^n(t, A) = 0$ shows unstability of the efficient trajectory t. The following formula of the stability radius of a stable efficient (optimal by Pareto) trajectory t of the vector linear trajectorial problem $Z^n(A)$ has been derived in [1].

$$\rho_P^n(t, A) = \varphi^n(t, A), \quad n \geq 1, \tag{5}$$

where

$$\varphi^n(t,A) = \min_{t' \in T \setminus \{t\}} \ \max_{i \in N_n} \ \gamma_i(t,t',A),$$

$$\gamma_i(t,t',A) = -\frac{\tau_i(t,t',A)}{\Delta(t,t')}.$$

Here as before we denote

$$\Delta(t,t') = \mid t \setminus t' \ \cup \ t' \setminus t \mid .$$

On the analogy of [1,2], the value

$$\rho_M^n(t,A) = \sup\{\varepsilon > 0 : \ \ \forall B \in \mathcal{B}(\varepsilon) \ \ (t \in T_M^n(A+B))\}$$

is called the stability radius of the stable majority efficient trajectory $t \in T_M^n(A)$.

If a trajectory $t \in T_M^n(A)$ is unstable, then it is naturally to assume its stability radius to be equal to zero.

Thus the stability radius of a trajectory $t \in T_M^n(A)$ is defined as the limit of perturbations of elements of the matrix A that preserve majority efficiency of the trajectory. Since $T_M^n(A) \subseteq T_P^n(A)$, in virtue of (5) for any majority efficient trajectory t of the problem $Z^n(A)$ we have

$$\rho_M^n(t,A) \leq \rho_P^n(t,A) = \varphi^n(t,A). \qquad (6)$$

The next lemma follows directly from the definition of number $\rho_M^n(t,A)$.

Lemma 2 *Let $t \in T_M^n(A)$, $\psi > 0$. If*

$$\forall B \in \mathcal{B}(\psi) \ \ (t \in T_M^n(A+B)),$$

then

$$\rho_M^n(t,A) \geq \psi.$$

By definition, put

$$\psi^n(t,A) := \min_{t' \in T \setminus \{t\}} \ \min_{i \in N_n^-(\tau(t,t',A))} \ \gamma_i(t,t',A).$$

We need the following well-known statement (see lemma 3.1 [1]).

Lemma 3 *If* $\gamma_i(t,t',A) > 0$, *then the inequalities*

$$\forall B \in \mathcal{B}(\psi) \quad (\tau_i(t,t',A+B) < 0)$$

are true for any number ψ *satisfying the inequalities*

$$0 < \psi \leq \gamma_i(t,t',A).$$

Theorem 2 *The following estimates*

$$\psi^n(t,A) \leq \rho^n_M(t,A) \leq \varphi^n(t,A). \tag{7}$$

are valid for any stable majority efficient trajectory t *of the problem* $Z^n(A), n \geq 1$.

Proof. Since under the conditions of the theorem the majority efficient trajectory t is stable, then by theorem 1 inequalities (3) hold. It follows that

$$\forall t' \neq t \quad (N^-_n(\tau(t,t',A)) \neq \emptyset).$$

Therefore taking into account the obvious inequality $\Delta(t,t') > 0$ (in virtue of $t \neq t'$), we obtain

$$\varphi^n(t,A) \geq \psi^n(t,A) > 0.$$

At first, the upper estimate for the stability radius

$$\rho^n_M(t,A) \leq \varphi^n(t,A)$$

is valid due to inequality (6).

Now let us prove inequality

$$\rho^n_M(t,A) \geq \psi^n(t,A). \tag{8}$$

By the definition of the number $\psi := \psi^n(t,A)$ we have

$$\forall t' \in T \setminus \{t\} \quad \forall i \in N^-_n(\tau(t,t',A)) \quad (\gamma_i(t,t',A) \geq \psi > 0).$$

Thus by lemma 3 we obtain

$$\forall t' \in T \setminus \{t\} \quad \forall i \in N^-_n(\tau(t,t',A)) \quad \forall B \in \mathcal{B}(\psi) \quad (\tau_i(t,t',A+B) < 0).$$

From this, in virtue of stability of the majority efficient trajectory t, by theorem 1 we have

$$\forall t' \in T \setminus \{t\} \ \ \forall B \in \mathcal{B}(\psi) \ \ \left(k^-(t, t', A+B) \geq k^-(t, t', A) \geq \frac{n}{2} \right).$$

The above propositions and (1) imply

$$\forall B \in \mathcal{B}(\psi) \quad (t \in T_M^n(A+B)).$$

Hence applying lemma 2, we derive (8).
Theorem 2 has been proved.

Since $T_M^1(A) = T_P^1(A)$ and $T_M^2(A) = T_P^2(A)$, in virtue of (5) we obtain the next corollary of theorem 2 for $n = 1, 2$.

Property 5 *The stability radius of a majority efficient trajectory t of the problems $Z_M^1(A)$ and $Z_M^2(A)$ is respectively expressed by the formula*

$$\rho_M^n(t, A) = \varphi^n(t, A), \quad n = 1, 2.$$

Therefore a trajectory $t \in T_M^n(A), n = 1, 2$ is stable if and only if $\varphi^n(t, A) > 0$.

The following examples illustrate that upper and lower bounds of the stability radius $\rho_M^n(t, A)$ of a majority efficient trajectory, stated by theorem 2, are attainable in the case of $n = 3$.

Example 5. Let $E = \{e_1, e_2, e_3\}, t_1 = \{e_1\}, t_2 = \{e_2\}, t_3 = \{e_3\}$,

$$A = \begin{pmatrix} 2 & 1 & 4 \\ 3 & 1 & 2 \\ 1 & 2 & 2 \end{pmatrix}.$$

Then $f(t_1) = (2, 3, 1), \quad f(t_2) = (1, 1, 2), \quad f(t_3) = (4, 2, 2)$.

Now if we calculate the bounds for the stability radius of the majority efficient trajectory t_2, using theorem 2, we obtain

$$\frac{1}{2} \leq \rho_M^3(t_2, A) \leq 1.$$

Let us show that the radius is equal to its lower estimate. Really, for any $\varepsilon > \frac{1}{2}$ there exists a matrix $B \in \mathcal{B}(\varepsilon)$, for example,

$$B = \begin{pmatrix} -\frac{1}{2} - \alpha & \frac{1}{2} + \alpha & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

where $0 < \alpha < \varepsilon - \frac{1}{2}$ such that the majority efficient trajectory t_2 loses its efficiency in the perturbed problem $Z^3(A+B)$. Hence, $\rho^3_M(t_2, A) \leq \frac{1}{2}$. Taking into account the lower bound, we conclude that the radius is equal to $\frac{1}{2}$.

Example 6. Let $E = \{e_1, e_2, e_3\}, t_1 = \{e_1\}, t_2 = \{e_2\}, t_3 = \{e_3\}$,

$$A = \begin{pmatrix} 1 & 3 & 3 \\ 0 & 2 & 1 \\ 1 & 1 & 3 \end{pmatrix}.$$

Then $f(t_1) = (1, 0, 1)$, $\quad f(t_2) = (3, 2, 1)$, $\quad f(t_3) = (3, 1, 3)$. Therefore, $t_1 \in T^3_M(A)$ and by theorem 2,

$$\frac{1}{2} \leq \rho^3_M(t_1, A) \leq 1.$$

It is easy to see that the relations $t_1 \prec t_2, \quad t_1 \prec t_3$ are true for any matrix $B \in \mathcal{B}(\varepsilon),\ 0 < \varepsilon \leq 1$. This means that the trajectory t_1 preserves majority efficiency in the perturbed problem $Z^3(A+B)$ for any matrix $B \in \mathcal{B}(\varepsilon)$. From this, by lemma 2, we obtain $\rho^3_M(t_1, A) \geq 1$. Hence in virtue of the upper bound, we derive $\rho^3_M(t_1, A) = 1$.

Now let us illustrate that the stability radius of a majority efficient trajectory can differ from the bounds stated by theorem 2.

Example 7. Let $E = \{e_1, e_2, e_3\}, t_1 = \{e_1\}, t_2 = \{e_2\}, t_3 = \{e_3\}$,

$$A = \begin{pmatrix} 3 & 0 & 3 \\ 3 & 1 & 2 \\ 0 & 1 & 3 \end{pmatrix}.$$

Then by theorem 2 the following inequalities are valid for the majority efficient trajectory t_2.

$$\frac{1}{2} \leq \rho_M^3(t_2, A) \leq \frac{3}{2}.$$

Let us show that the stability radius of the trajectory t_2 is equal to 1.

It is obvious that for any $\varepsilon > 1$ there exists a matrix $B \in \mathcal{B}(\varepsilon)$, for example,

$$B = \begin{pmatrix} 0 & 0 & 0 \\ -1-\alpha & 1+\alpha & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

where $0 < \alpha < \varepsilon - 1$ such that $t_2 \notin T_M^3(A+B)$. Therefore,

$$\rho_M^3(t_2, A) \leq 1.$$

On the other hand, it is easy to see that t_2 is the majority efficient trajectory of the problem $Z^3(A+B)$ for any number $0 < \varepsilon \leq 1$ and for any perturbing matrix $B \in \mathcal{B}(\varepsilon)$. Therefore by lemma 2 we have $\rho_M^3(t_2, A) \geq 1$.

Hence we obtain $\rho_M^3(t_2, A) = 1$.

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