Computer Science Journal of Moldova, vol.7, no.3(21), 1999

Complex Membership Grades with an Application to the Design of Adaptive Filters

D. Moses, H. Teodorescu, M. Friedman, A. Kandel

Abstract

In this paper, complex membership grades are introduced for the extension of fuzzy set theory to the complex domain. This model is based on the idea of viewing the complex domain in a linguistic manner, where two linguistic terms are required to define an object. Thus, as opposed to Buckley's model [13], after fuzzification the two-dimensionality of the universe of discourse is still apparent. One form for representing a complex fuzzy set is using the *Cartesian Complex Fuzzy Set* representation, which produces complex sets of the form $\tilde{Z}_c = \tilde{X} + j\tilde{Y}$. The motivation for this aberrant representation is oriented from the limitations in using a direct extension to Zadeh [17], that Buckley [13] introduced. These limitations pose the guidelines for *Complex Membership Grades* and, therefore, are initially discussed in this paper. *Complex Fuzzy Sets* are defined and a technique for converting between *Complex Fuzzy Sets* and *Fuzzy Relations* is developed based on *Cylindrical Extensions and Projections* defined by Zadeh [20]. Next, *linguistic coordinate transformations* are discussed and exemplified by a rule-base coordinate transformation between *Polar* and *Cartesian Complex Fuzzy Sets*. Arithmetic operations and defuzzification are demonstrated. The simplicity of these latter operations is crucial when considering implementation prospects. Finally, *Complex Membership Grades* are applied to the design of adaptive filters. It is shown that a logically derived rule-base can be described, using the linguistic complex domain, for the adaptation process. Emphasis, in this part, is put on the unique characteristics of the complex membership grades model.

1 Introduction

The idea of extending fuzzy set theory to the complex domain was explored as early as 1985. In their book, Kaufmann and Gupta [4] nourished their reader's motivation by presenting ideas and giving insights for possible fuzzification models of the complex domain. Although Kaufmann and Gupta's ideas were yet in their embryonic stages, it was those ideas that, nevertheless gave researchers directions and thoughts. The pioneering work in this field is accredited to Buckley [13] who initiated *fuzzy complex number* theory. In [13], [15], [14] and [24], those concepts and the affiliated formulations are developed.

Buckley [13] directly extends the fundamental concepts given by Zadeh [17], who suggested that membership in a set be governed by a measure of extent. Thus, treating the complex domain as any other universe of discourse and assigning a grade of membership that describes the degree of *inclusion* in that set. As Zadeh [17] permits the definition of a fuzzy membership on any universe of discourse, it seems a natural extension to view the complex domain as just another universe of discourse. This, however, fails to propagate to the linguistic model some features of the complex domain, which are evident in operations such as multiplication and coordinate transformations. It is those features that make complex analysis ubiquitous in engineering.

Referring back to the sequence of events that lead to the introduction of the complex numbers, there was a lot of turmoil among mathematicians with regrad to the need for complex numbers and their meaning. Although, well before their introduction, it was widely known that complex numbers could be used in several applications, many mathematicians were still uneasy with $\sqrt{-1}$ and did not have a cogent reason to prevail upon the future of mathematics complex number analysis. Inaguration came only when it was shown that, in some cases, a real solution to the cubic equation, using an existing formula, could be obtained by two complex numbers that canceled one another [12]. Thereafter, making the complex numbers accredited among the mathematical community.

This sequence of events, together with the fact that the complex numbers are application oriented and that fuzzy set theory is an engineering tool, urged the authors to evaluate fuzzification models for the complex domain based on their applicability prospects.

In this paper, complex membership grades are proposed as a model for the fuzzification of the complex domain. A complex fuzzy set is composed of two fuzzy sets [17] that, in the Cartesian representation corresponds to the real and imaginary components. Two primary complex membership grade representations are the Cartesian and the Polar. Once a linguistic representation (which is essentially complex fuzzy sets on the domain) is obtained, linguistic operations can be defined. One of the most important operations is the coordinate transformations. That is to take, for example, a Cartesian complex membership grade and transform it to a Polar complex membership grade. This paper proposes a *Rule-Base Coordinate Transformation(RBCT)* technique for that transformation. Further, as it is necessary to relate complex fuzzy set theory with the existing fuzzy set theory, a conversion scheme is developed. This conversion scheme converts complex fuzzy sets to fuzzy relations and vice versa.

Section 2 investigates the limitations of the existing model defined by Buckley [13]. These limitations are used as guidelines for the definition of the complex membership grades. A pertinent review of linguistic variables and fuzzy relations is given in section 3. Section 4 presents the definitions for complex fuzzy sets. Section 5 deals with conversions between complex fuzzy sets and fuzzy relations, thus permitting the application of existing results from fuzzy set theory. Section 6 investigates complex fuzzy set transformations, which are based on a rule-base system. The transformation model facilitates a change in the linguistic terminology that is used to describe a phenomenon in the complex domain. Methods for arithmetic operations and defuzzification are developed in section 7. An application to the design of a second order adaptive analog filter is discussed in section 8. Finally, section 9 discusses the motivation and potential of complex fuzzy sets.

Throughout this paper the term *fuzzy sets* is used to denote fuzzy sets as defined by Zadeh [17].

2 Limitations of The Existing Theory

The theory defined by Buckley [13] proposes to create fuzzy sets over the complex domain, therefore using the complex domain as the universe of discourse. This implies that, a point in the two-dimensional complex domain is evaluated as belonging to a fuzzy set using a membership grade. Essentially, in this method, a fuzzy set defined on the complex domain is a mapping from C to the interval $[0, 1]$. Buckley also defines the requirement from two-dimensional structures for the constitution of fuzzy complex numbers. Along with these definitions, mathematical operations are developed based on the extension principle. The most conspicuous feature of Buckley's direction is that the complex domain loses its two-dimensionality characteristic under fuzzification. As analysis in the complex domain is a mathematical tool and the components of a complex number relate to the physical world, it is necessary to preserve the two-dimensionality characteristic under fuzzification too. The following example can illuminate this requiremnet.

Example 2.1 *Suppose a given universe of discourse for length and a fuzzy set of* long *objects with a membership function $\mu_{long}(x)$. Also, suppose a given universe of discourse of width and a fuzzy set of* wide *objects with a membership function $\mu_{wide}(y)$.*

In many practical cases, where complex numbers are applied, it is advantageous to represent two physical numbers as a single complex number for mathematical reasons. An aggregated representation could be useful to do operations such as coordinate transformations, which in the fuzzy case means a change in the linguistic terminology that is used to represent the object, or operations that cause interactions between the two components, such as the complex multiplication. In this example, the measure of length and the measure of width may be defined on the two independent axes that correspond to the real and the imaginary axes, respectively.

For an object having the complex value $z = x + jy$, it is clear that in order to evaluate it as being long *the first component in conjunction with $\mu_{long}(x)$ is used, and in a similar fashion the evaluation for being* wide. *Thus, it is essential to be able to extract the physical components, or implications, from the aggregated representation.*

Furthermore, the operations that are defined by Buckley [13] are based on the extension principle. As the universe of discourse for a fuzzy set is two-dimensional, these operations are computationally difficult to perform on the structures that are permitted and, henceforth, question the feasibility in an application designer's perspective. A simple multiplication, for example, on a Cartesian rectangular fuzzy number is computationally expensive. Key topics such as coordinate transformations and defuzzification are not discussed in [13] or any subsequent work. Moreover, considering the set of structures that are permitted, or that can be generated by applying simple operations to simple fuzzy structures, defuzzification seems a serious drawback in the model developed in [13]. Finally, up-to-date, there has not been any application based on the theory presented by Buckley that would unveil it's potential as an engineering tool. In this respect, it should be emphasized that both complex numbers and fuzzy sets are tools used in innumerable applications.

3 Linguistic Variables and Fuzzy Relations

3.1 Linguistic Variables

The values of linguistic variables [19] [20] [21] are words rather than numbers. The purpose of having variables of this type is to provide a means to describe phenomena that are too ill-defined to be suitable for characterization using quantitative models. Fuzzy sets are essentially summaries of sub-classes of elements in a universe of discourse.

Definition 3.1 *(Zadeh [19] [20] [21]) A linguistic variable is defined by a quintuple* $(\chi, T(\chi), U, G, M)$ *in which* χ *is the name of the variable;* $T(\chi)$ *(or* T*) is the set of* linguistic values *of* χ*, where each value is a fuzzy variable and is denoted generically by* X *that ranges over a universe of discourse* U *which is associated with the* base variable *u; G is a* syntactic rule *(usually a grammer) for generating the names,* *X, of values* χ*; and M is a* semantic rule *for relating with each X its* meaning, *M(X) which is a fuzzy subset of U.*

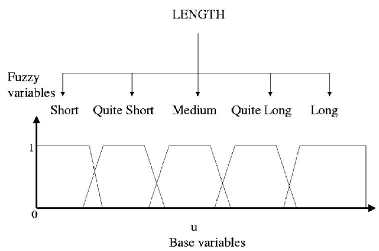

Figure 1. Linguistic Variables

As a complex number comprises two components, usually, representing two physical phenomena, and implications regarding a complex number are projected back to the physical domain, it seems essential to define the linguistic variables of the fuzzy model using the physical universes of discourse as basis. Thus, as there are two distinct physical universes of discourse, there should be two linguistic variables to describe a single object in the linguistic complex domain.

3.2 Fuzzy Relations

In addition to the linguistic pairs that are used to describe phenomena in the complex domain, it is also necessary, in many cases, to describe relations between the two components of the complex number. Thus, as opposed to the linguistic pair, constructing fuzzy sets with a base universe of discourse, which is the complex domain. Fuzzy relations are used to describe interactions [19] [20] [21] between the two components of the complex number. The motivation to describe interactions is given in the following example.

Example 3.1 *Given the case in example 2.1, it is clear that as the width of the object changes, the subjective classification of the object as being* long *would change too. Thus, the membership function of the fuzzy set* long*, which evaluates the length of the object, would have to be dependent on the width of the object. In other words, a typical object evaluated on the width dimension as being* wide *and on the length dimension as being* long*, would be, in measurement, longer than the a typical object evaluated on the width dimension as being* narrow *and on the length dimension as being* long.

The fundamental concepts of fuzzy relations are given in Zadeh's series of papers [19] [20] [21] and the mathematical foundations are extensively discussed in Di Nola, Sessa, Pedrycz and Sanchez's book [1]. A brief summary of the concepts for the two-dimensional case used in this paper is presented in the following.

If U is the Cartesian product of two universes of discourse U_1, U_2, then a binary *fuzzy relation*, R, in U is a fuzzy subset of U. Using the standard notation, R may be expressed through $\mu_R(u_1, u_2)/(u_1, u_2)$ as,

$$R = \int_{U_1 \times U_2} \mu_R(u_1, u_2)/(u_1, u_2) \tag{1}$$

μ_R is the membership function of R. Common examples of binary fuzzy relations are: *much smaller than*, *quite bigger than*, etc.

If R is a binary fuzzy relation in U_1, U_2, then its *projection* on U_i (where i may equal 1 or 2) is a fuzzy set F_q.

$$F_q = \mathrm{ProjRon}U_i$$

$$= \int_{U_i} [\vee_{u_{(i')}} \mu_R(u_1, u_2)]/(u_1, u_2)$$

where i' is the complement index of i, for example, if $i = 1$ then $i' = 2$. Distinct fuzzy relations in U_1, U_2 can have identical projections on U_i. However, given a fuzzy set F_q in U_i, there exists a unique *largest* relation $\bar{R}_q$ in U_1, U_2 whose projection on U_i is F_q. The membership function of $\bar{R}_q$ is given by:

$$\mu_{\bar{R}_q}(u_1, u_2) = \mu_{F_q}(u_i) \tag{2}$$

$\bar{R}_q$ is referred to as the *Cylindrical extension* of F_q, with F_q constituting the *base* of $\bar{R}_q$.

Projections and cylindrical extensions, for two-dimensional fuzzy relations, are extensively discussed and graphically illustrated in [11].

4 Definitions

Using linguistic terms to describe phenomena in the complex domain is actuated from the need to view the complex domain in a more coarse resolution. Thus, in order to avoid dealing with unnecessary precision in the description of objects in the complex domain, the complex fuzzy set is introduced. As discussed in the previous sections, it is, nevertheless, necessary to preserve the two-dimensionality characteristic of the fuzzified object. The complex membership grade, essentially, contains two membership grades, each of which is related to a specific dimension. As an example, the complex fuzzy set $\tilde{Z}_c = \tilde{X} + j\tilde{Y}$, contains information on the x-axis and y-axis fuzzy sets. It is obvious that the features of the two fuzzy sets and, henceforth, the complex fuzzy set are dependent on the coordinate system being used such as the *Cartesian coordinate system* used for $\tilde{Z}_c$. This leads to the requirement of a scheme for coordinate transformations that is described in the next section.

In this discussion the following notational conventions are used:

- C – Denotes the complex numbers domain.
- R – Denotes the real numbers domain.
- z – Denotes a complex number.
- $\tilde{Z}_g$ – Is used to denote a complex fuzzy set in a given representation g.
- $\tilde{Z}_c$ – Denotes a complex fuzzy set in the Cartesian representation.

- $\tilde{Z}_p$ – Denotes a complex fuzzy set in the Polar representation.
- U_c – $[0, 1] \times [0, 1]$.

Definition 4.1 $\tilde{Z}_g : C \rightarrow U_c$ *is defined as the* complex fuzzy set *on the complex domain in a given representation* g*, if the complex fuzzy set is composed of a pair of fuzzy set* $(\tilde{U}, \tilde{V})_g$*. The measure of extent to which* $z \in C$*, where* $z = (u, v)_g$*, belongs to the complex fuzzy set* $\tilde{Z}_g$ *is called the* complex membership grade *and is defined to equal the pair value* $(\tilde{U}(u), \tilde{V}(v))_g$

In contrast to fuzzy sets [17], which would map a domain to $[0, 1]$, in complex membership grades the mapping is to a two-dimensional range. Before we delve into the essence of this definition, two useful representations are given.

Definition 4.2 $\tilde{Z}_c$ *is defined to be the* Cartesian complex fuzzy set *composed of the two fuzzy sets* $\tilde{X}(x)$ *and* $\tilde{Y}(y)$*, if every element* $z \in C$ *such that* $x = Re(z)$ *and* $y = Im(z)$ *has the complex membership grade* $(\tilde{X}(x), \tilde{Y}(y))_c$*, which in this case is called the* Cartesian complex membership grade*. This complex fuzzy set is written* $\tilde{Z}_c = \tilde{X}(x) + j\tilde{Y}(y)$ *or simply* $\tilde{Z}_c = \tilde{X} + j\tilde{Y}$*.*

Definition 4.3 $\tilde{Z}_p$ *is defined to be the* Polar complex fuzzy set *composed of the two fuzzy sets* $\tilde{R}(r)$ *and* $\tilde{\Theta}(\theta)$*, if every element* $z \in C$ *such that* $r = |z|$ *and* $\theta = \angle(z)$ *has the complex membership grade* $(\tilde{R}(r), \tilde{\Theta}(\theta))_p$*, which in this case is called the* Polar complex membership grade*. This complex fuzzy set is written* $\tilde{Z}_p = \tilde{R}(r) \cdot e^{j\tilde{\Theta}(\theta)}$ *or simply* $\tilde{Z}_p = \tilde{R} \cdot e^{j\tilde{\Theta}(\theta)}$

The *complex membership function* has the following forms for the general case:

$$\mu_{\tilde{Z},g}(z) = (\mu_{\tilde{U}}(u), \mu_{\tilde{V}}(v))_g \tag{3}$$

The Cartesian and Polar can be written more intuitively as:

$$\mu_{\tilde{Z},c}(z) = \mu_{\tilde{X}}(x) + j\mu_{\tilde{Y}}(y) \tag{4}$$

and

$$\mu_{\tilde{Z},p}(z) = \mu_{\tilde{R}}(r) \cdot e^{j\mu_{\tilde{\Theta}}(\theta)} \tag{5}$$

As opposed to fuzzy sets defined in [17], where the membership function has a physical interpretation, the complex membership function, as a whole, does not. Only by inspecting the two components that constitute it, individually, can a physical interpretation be drawn. To illuminate this point, suppose a given *Cartesian complex fuzzy set* $\tilde{Z}_c = \tilde{X} + j\tilde{Y}$. The compatibility of a Cartesian represented complex number $z = x + jy$ to this complex fuzzy set is given by the Cartesian complex membership function $\mu_{\tilde{Z},c}(z) = \mu_{\tilde{X}}(x) + j\mu_{\tilde{Y}}(y)$. This means that the extent to which z belongs to the complex fuzzy set $\tilde{Z}_c$ is given by a pair of numbers. Although, each of the latter pair of numbers has a physical interpretation, the pairing as a unit does not constitute a physical interpretation. The compatibility of x to $\tilde{X}$ is given by $\mu_{\tilde{X}}(x)$ and is interpreted as defined by Zadeh [17]. The compatibility of y is measured and interpreted in a similar way. This idiosyncratic method of interpretation is also encountered in complex analysis.

Definition 4.4 $\tilde{Z}_g = (\tilde{U}, \tilde{V})_g$ *is a* Complex Fuzzy Number *under representation* g *if and only if* $\tilde{U}$ *and* $\tilde{V}$ *are Fuzzy Numbers.*

Definition 4.5 *Let* $\tilde{Z}_{1,g}$ *and* $\tilde{Z}_{2,g}$ *be two complex fuzzy sets with a general pair representation of* $(\tilde{U}_1, \tilde{V}_1)_g$ *and* $(\tilde{U}_2, \tilde{V}_2)_g$*, respectively.*

1. $\tilde{Z}^c_{1,g} = (\tilde{U}^c_1, \tilde{V}^c_1)_g$ *- where superscript c denotes a complement operation.*
2. $\tilde{Z}_{1,g} \cup \tilde{Z}_{2,g} = (\tilde{U}_1 \cup \tilde{U}_2, \tilde{V}_1 \cup \tilde{V}_2)_g$
3. $\tilde{Z}_{1,g} \cap \tilde{Z}_{2,g} = (\tilde{U}_1 \cap \tilde{U}_2, \tilde{V}_1 \cap \tilde{V}_2)_g$

A finite complex fuzzy set may be expressed using the following notation:

$$\begin{aligned} A_g &= [(\mu_{u_1}, \mu_{v_1})/(u_1, v_1)]_g + [(\mu_{u_2}, \mu_{v_2})/(u_2, v_2)]_g + \\ &\quad + \ldots + [(\mu_{u_n}, \mu_{v_n})/(u_n, v_n)]_g \\ &= (\mu_{u_1}/u_1 + \mu_{u_2}/u_2 + \ldots + \mu_{u_n}/u_n, \\ &\quad \mu_{v_1}/v_1 + \mu_{v_2}/v_2 + \ldots + \mu_{u_n}/u_n)_g \end{aligned} \tag{6}$$

or

$$A_g = \sum_{i=1}^{n} [(\mu_u, \mu_v)/(u, v)]_g = \left(\sum_{i=1}^{n} \mu_u/u, \sum_{i=1}^{n} \mu_v/v \right)_g \tag{7}$$

In a similar manner, the notation for the representation of uncountably infinite complex fuzzy sets is also extended, i.e.,

$$A_g = \int ((\mu_u, \mu_v)/(u, v))_g = (\int \mu_u/u, \int \mu_v/v)_g \tag{8}$$

In equations 6, 7 and 8, the second equality indicates a slight deviation from the usual notation interpretation. Thus $A_g = ([\mu_{u_1}, \mu_{v_1})/(u_1, v_1)]_g + [(\mu_{u_2}, \mu_{v_2})/(u_2, v_2)]_g$ implies that (u_1, v_2) and (u_2, v_1) belong to A_g, under representation g, with complex grades of memberships (μ_{u_1}, μ_{v_2}) and (μ_{u_2}, μ_{v_1}), respectively. This deviation is consistent with definition 4.5.

As indicated in the notation, elements of the set must be in the same coordinate system representation.

5 Fuzzy Relations on the Complex Domain

The complex fuzzy set does not relate points in the complex domain in a fashion described by Zadeh [17]. In many cases, however, it is necessary to have C as the universe of discourse and define fuzzy sets [17] over them. A model for the definition of fuzzy sets on the complex domain of this form is described by Buckley [13]. These types of fuzzy sets are essentially a mapping from $C \to [0, 1]$. The advantage of these sets over complex fuzzy sets is their physical interpretation.

In this section a conversion scheme, based on *cylindrical extensions* and *projections* [19] [20] [21] is introduced. The conversion from complex fuzzy sets to fuzzy relations is done using *cylindrical extensions*, while the conversion from fuzzy relations to complex fuzzy sets is done using *projections*. The scheme for these conversion is given in figure 2.

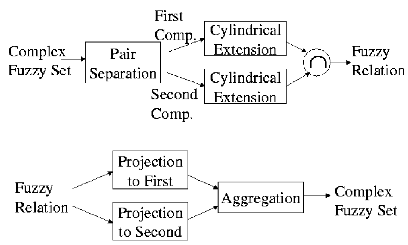

Figure 2. Conversion Schemes

Suppose a given *Cartesian complex fuzzy set* $\tilde{Z}_c = \tilde{X} + j\tilde{Y}$. The top diagram in figure 2, presents the steps to convert this complex fuzzy set into a fuzzy relation. Thus, each of the fuzzy set components, $\tilde{X}(x)$ and $\tilde{Y}(y)$, is *cylindrically extended* to obtain two-dimensional fuzzy relations with membership functions $\mu_{\tilde{X}}(x, y)$ and $\mu_{\tilde{Y}}(x, y)$, respectively. Using the relationships

$$\mu_{\tilde{X}}(x, y) = \mu_{\tilde{X}}(x)$$
$$\mu_{\tilde{Y}}(x, y) = \mu_{\tilde{Y}}(y)$$

Next, the two obtained fuzzy relations are intersected to derive the *Induced Fuzzy Relation (IFR).*

$$\mu_R(x, y) = \mu_{\tilde{X}}(x, y) \wedge \mu_{\tilde{Y}}(x, y) \tag{9}$$

For converting to the *Cartesian Complex Fuzzy Set*, the reverse operation starts out with a fuzzy relation $\mu_R(x,y)$ on the complex domain, which is not necessarily the *IFR*. Two projections are done on this relation - one to U_x and the other to U_y, where U_x and U_y are the universes of discourse of x and y, respectively. These two projections constitute $\tilde{X}(x)$ and $\tilde{Y}(y)$, repectively. The equations for the conversions are given by:

$$\tilde{X} = \int_{U_x} [\subset U_y \bigwedge \mu_R(x,y)]/x$$

$$\tilde{Y} = \int_{U_y} [\subset U_x \bigwedge \mu_R(x,y)]/y$$

where,

$$\tilde{Z}_c = \tilde{X} + j\tilde{Y} \tag{10}$$

This conversion template can be used for any representation.

6 Coordinate Tranformations

The linguistic complex domain analysis proposed, suggests that the two-dimensionality characteristic of the complex domain be preserved. Hence, a phenomenon in the linguistic complex domain is described using two linguistic variables. A representation of an object in the complex domain is dependent on the values of the linguistic variables and the choice of the linguistic variable pair. The latter, essentially, determines the terminology used to describe the object. Although this is consistent with the theory of complex numbers by which one can represent a point in the complex domain as a pair of real and imaginary components or, alternatively, a pair of magnitude and angle components, the theory of complex numbers permits, in addition, the transformation from one representation to the other.

Two issues are discussed in this section. The first is, given complex fuzzy sets for each of the two coordinate systems, the establishment of a mechanism to infer the complex grade of membership to a second complex fuzzy set based on the known complex grade of membership to the first complex fuzzy set. This is done using a rule based coordinate transformation knowledge base of the desired transformation.

The second topic to be discussed is a technique to determine the complex fuzzy sets in some representation given the complex fuzzy sets of another and the crisp transformation.

6.1 Rule Based Coordinate Transformations

The model proposed here is based on a set of rules that transform one representation to another. Returning back to example 2.1, in the linguistic Cartesian representation the two axes are length and width for real and imaginary, respectively. In a simplified model, where each of the component sets has crisp boundaries, a Cartesian linguistic partition of this linguistic complex domain can have the form shown in figure 3. In figure 3, the linguistic variable *length* may take a linguistic value from the set $\{\text{VeryShort}, \text{Short}, \text{QuiteShort}, \text{Medium}, \ldots\}$, while the linguistic variable *width* may take a linguistic value from the set $\{\text{VeryNarrow}, \text{Narrow}, \text{QuiteNarrow}, \text{Medium}, \text{QuiteWide}, \ldots\}$.

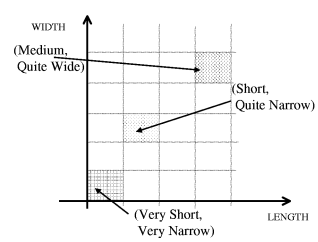

Figure 3. An Example of a Cartesian Linguisic Complex Domain for Length and Width

A transformation to the linguistic polar representation of the domain corresponds to a change in terminology from the pair *length* and *width* to the pair *size* and *compatibility*. Figure 4 gives a sample partition for a crisp linguistic domain with the liguistic variable *size* taking the linguistic values {VerySmall, Small, QuiteSmall, Medium, ...} and the linguistic variable *compatibility* taking the linguistic values {VeryGood, Good, QuiteGood, ...}. Compatibility, which connotes the angle, describes the extent to which the length and the width match.

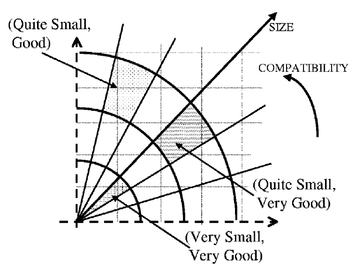

Figure 4. An Example of a Polar Linguistic Complex Domain for Size and Compatibility

A table may be generated to describe the transformation from the *Cartesian* to the *Polar* linguistic representations. Or, in other words, from one pair of linguistic variables to the other. Table 1 presents this rule base coordinate transformation.

As can be seen from figure 4, the linguistic regions do not match in size or formation. In other words, there does not exist a one-to-one mapping from one linguistic representation to another. In the general

Table 1. Example of a Rule Base Coordinate Transformation Table

	LENGTH			
WIDTH	Very Short	Short	Quite Short	...
Very Narrow	(Very Small, Very Good)	(Small, Good)	(Small, Quite Good)	
Narrow	(Very Small, Good)	(Small, Very Good)	(Quite Small, Good)	
Quite Narrow	(Small, Good)	(Small, Very Good)	(Quite Small, Very Good)	
⋮				

case, a phenomenon would be described using many complex lingustic values in some representation and would transform to many complex linguistic values in another. The crisp boundary case as in figure 4 does not permit to describe which region is more significant in the representation of the object. However, the fuzzy case, with a complex grade of membership would.

6.2 Transformation Generated Complex Fuzzy Sets

A transformation generated complex fuzzy set in a target representation is defined by two parameters: the complex fuzzy set in some given representation and a crisp transformation to the desired representation. The most commonly used transformation will be developed, as a model, in this section - the Polar/Cartesian transformations.

The crisp Polar to Cartesian Transformation is given by the following two equations:

$$x = r \cdot \cos(\vartheta)$$

$$y = r \cdot \sin(\vartheta)$$

Using these two equations as a starting point, a method for the computation of the complex fuzzy set in the Cartesian coordinate system, given the Polar complex fuzzy set, may be derived. This computation is based on fuzzy arithmetic presented in [2] and [4]. Thus, the fuzzy case would yield:

$$\tilde{X} = \tilde{R} \cdot \cos(\tilde{\Theta})$$

$$\tilde{Y} = \tilde{R} \cdot \sin(\tilde{\Theta})$$

where,

$$\tilde{Z}_p = \tilde{R} \cdot \exp^{j\tilde{\Theta}} \Longrightarrow \tilde{Z}_c = \tilde{X} + j\tilde{Y} \tag{11}$$

The problem in computing these two equations arises from the cosine and sine terms. Given that $\tilde{\Theta}$ is a fuzzy number, the cosine and sine functions applied to the fuzzy number according to fuzzy arithmetic [4], in general, would not yield a fuzzy number. The few cases, however, that guarantee the result being a fuzzy number pose restrictions on the membership function of $\tilde{\Theta}$. One of the simplest cases occurs when $\tilde{\Theta}$ is a fuzzy number with a support spanning only a monotonous section of the cosine and sine functions. Note that in this case, the restriction must hold for both the cosine and sine functions together, in order to have fuzzy numbers for both $\tilde{X}$ and $\tilde{Y}$.

An important result that extends significantly the set of membership functions that would yield fuzzy numbers for $\tilde{X}$ and $\tilde{Y}$ is presneted by in [9]. In that work it was found that when a gaussian function is applied to a fuzzy number, a fuzzy number is produced. The implications of this result are demonstrated in [3]. Since the cosine and sine functions are periodic, it is possible to select an interval that would conform to the model described in [9] and apply the same concept to get a fuzzy number.

Applying this concept, the following example is given. Working with $\tilde{\Theta}$ that is a fuzzy number under the interval $[-\pi, \pi)$, would yield a fuzzy number from the cosine function (figure 5).

Several other intervals exist. For the sine function we may pick $[-\frac{\pi}{2}, \frac{3\pi}{2})$.

The Cartesian to Polar transformation is simpler to handle. The

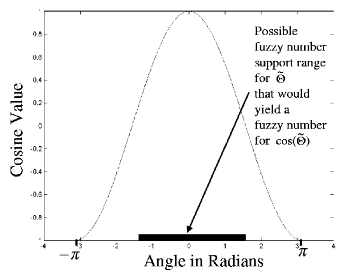

Figure 5. The range of angles over which $\tilde{\theta}$ should a fuzzy number

crisp transformation is given by:

$$r = \sqrt{x^2 + y^2}$$

$$\theta = \arctan\left(\frac{y}{x}\right)$$

Computation of these functions is discussed in existing fuzzy set literature. Thus, the fuzzy case can easily be computed as:

$$\tilde{R} = \sqrt{\tilde{X}^2 + \tilde{Y}^2}$$

$$\tilde{\Theta} = arctan\left(\frac{\tilde{Y}}{\tilde{X}}\right)$$

where

$$\tilde{Z}_c = \tilde{X} + j\tilde{Y} \Longrightarrow \tilde{Z}_p = \tilde{R} \cdot \exp^{j\tilde{\Theta}} \tag{12}$$

7 Operations

As stated in section 2, the simplicity of some operations is crucial for the feasibilty of implementing applications based on this model. Two classes of operations are considered here - arithmetic operations and defuzzification.

In addition to a simple model for computing arithmetic operations, in order for the arithmetic operations to remain realtively easy to compute, the property of closure must be preserved. Thus, operating on complex fuzzy numbers should generate complex fuzzy numbers.

Defuzzification has permeated into fuzzy sets due to its demand in the application arena. Due to its importance, it will be discussed too.

7.1 Arithmetic Operations

The basic operations of addition, subtraction, multiplication and division are discussed in this section. These four operations fall into two categories. The first consisting of addition and subtraction while the second includes multiplication and division. Addition and subtraction are computed with ease when the linguistic terminology is Cartesian. On the other hand, multiplcation and division require a Polar representation. Coordinate transformations may be used to transform from one representation to the other. Both the models are extensions of the fuzzy set arithmetic case.

Addition is computed using the following equation:

$$\tilde{Z}_{1,c} + \tilde{Z}_{2,c} = (\tilde{X}_1 + \tilde{X}_2) + j(\tilde{Y}_1 + \tilde{Y}_2) \tag{13}$$

subtraction is done in a similar fashion.

Multiplcation follows the model in equation.

$$\tilde{Z}_{1,p} \cdot \tilde{Z}_{2,p} = (\tilde{R}_1 \cdot \tilde{R}_2)e^{j(\tilde{\Theta}_1 + \tilde{\Theta}_2)} \tag{14}$$

Theorem 7.1 *If $\tilde{Z}_{1,c}$ and $\tilde{Z}_{2,c}$ are Cartesian Complex Fuzzy Numbers then $\tilde{Z}_{1,c} + \tilde{Z}_{2,c}$ and $\tilde{Z}_{1,c} - \tilde{Z}_{2,c}$ are Cartesian Complex Fuzzy Numbers.*

Theorem 7.2 *If $\tilde{Z}_{1,p}$ and $\tilde{Z}_{2,p}$ are Polar Complex Fuzzy Numbers then $\tilde{Z}_{1,p} \cdot \tilde{Z}_{2,p}$ and $\tilde{Z}_{1,p}/\tilde{Z}_{2,p}$ are Polar Complex Fuzzy Numbers.*

The proofs follow directly from the definitions given and arithmetic operations of fuzzy sets [4].

7.2 Defuzzification

As each of the components of a complex fuzzy set is dependent on a single variable, defuzzification remains a simple operation. Leading to the idea that in order to defuzzify the complex fuzzy set, defuzzification of each of its components must be done. As the components are fuzzy sets, the defuzzification is an extension to the standard models such as the ones discussed in [16].

$$Dfz\{\tilde{Z}_g\} = (Dfz\{\tilde{U}\}, Dfz\{\tilde{V}\}) \tag{15}$$

where *Dfz* is the defuzzification operation. The above equation holds in any given representation.

8 An Application to The Design of Adaptive Filters

In this section, complex membership grades are applied to the design of adaptive filters. As part of this example, it is explained how the concepts are used in applications that include theory based on complex numbers.

First, it is shown that complex fuzzy sets are easily constructed for the adaptive filter problem by pair aggregations of fuzzy sets in a given representation system. Both the Cartesian and Polar linguistic complex fuzzy set representations are required to solve the problem of the adaptive filter design. For each representation, the underlaying fuzzy sets are derived by assigning linguistic attributes to physical phenomena. This makes the task, of constructing the underlaying fuzzy sets, simple. In addition, the extraction of physical interpretations from linguistic complex variables is done by component inspections.

The second goal of this example is to emphasize the difference between a regular two-dimensional aggregation of linguistic variables and a complex linguistic variable. This is achieved by applying the linguistic coordinate transformations, which are an extension to the coordinate transformations frequently used when the application has theory based on complex analysis. These linguistic coordinate transformations are unique to complex fuzzy sets.

As opposed to fuzzy set models for adaptive filters that are based on adaptation algorithms in the time domain such as in [10], complex membership grades establishes a framework for frequency domain based adaptation algorithms. The frequency domain derived rule-base motivates future research in this area as it has the potential to overcome problems extant in conventional adaptive algorithms such as LMS.

8.1 Second-Order Analog Filter Preliminaries

For simplicity, the filter model described is based on a second order analog filter. Nevertheless, the techniques presented here form the foundation from which high-order adaptive analog filters and digital adaptive filters, based on complex membership grades, may be designed.

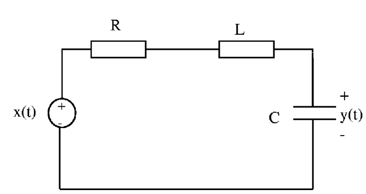

Figure 6. Second-order Analog Filter

When the output voltage is measured on the capacitor, the circuit shown in figure 6 is a low-pass filter [22] [5] and the corresponding differential equation can be written as:

$$LC\frac{d^2y(t)}{dt^2} + RC\frac{dy(t)}{dt} + y(t) = x(t) \tag{16}$$

A standard representation of the Laplace transform for a second order system is given by:

$$H(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{\omega_n^2}{(s-B)(s-D)} \tag{17}$$

where,

$$2\zeta\omega_n = \frac{R}{L} \tag{18}$$

$$\omega_n^2 = \frac{1}{LC} \tag{19}$$

From the Laplace transform, the zero-pole plot may be obtained, where ω_n and ζ in the above equation have an important role as can be seen in figure 7.

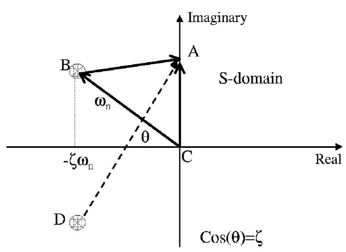

Figure 7. Key Components in The Poles and Zeros Plot of a Second Order Filter

In figure 7, the frequency response, and as part of it the magnitude response, which is given by $|H(s)|$, at coordinate A is mainly affected by pole B. This is due to the fact that vector AB is smaller than vector AD. When AB is much smaller than AD, the effects of pole D on the frequency response may be neglected. It is assumed that this is the case here. Thus, in the adaptation process described, the magnitude response at the frequencies around which the noise varies will be almost solely determined by pole B, and, therefore, only adjustments to the placement of pole B will be considered. Obviously, as the poles are conjugate, the location of pole D would also be changed.

8.2 The Adaptive Filter Model

The contamination of a desired signal by unwanted, often larger, signals or noise is a problem often encountered in many applications [7]. Linear filters are selected as a tool to obtain an estimation of the desired signal when the signal and noise occupy fixed frequency bands.

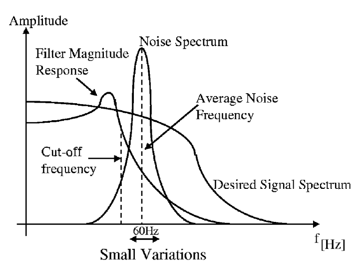

Figure 8. Adaptive filtering under a spectral overlap between a signal and a strong interference

When the spectrum of the desired signal and noise overlap, as in figure 8, and when these signals are not known in advance or vary, an adaptive filter must be used. In such cases, the coefficients of the filter cannot be specified in advance and must vary.

An adaptive filter has the property that its frequency response adjusts automatically to improve its performance with respect to some criterion. This characteristic allows the filter to adapt to changes in the input signal characteristics. A filter in this category consists of two distinct parts (figure 9); a filter with adjustable coefficients and an adaptive algorithm which is used to adjust or modify the coefficients of the filter. Other variants are discussed in [7].

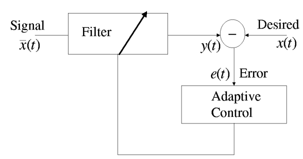

Figure 9. The Adaptive Filter Model

The block diagram in figure 9 is a simplified architecture. In most practical cases, $X(t)$ will not be given, but rather a signal that is correlated to $X(t)$ would be available. The only affect of this change would be in the phase of constructing the fuzzy sets for the error signal. In addition, the error would be integrated over a short period of time.

In medical applications such as in EEG, the noise is a strong interference with a spectral distribution concentrated around some time varying frequency (figure 8). Further, the desired signal is measured in the scale of milli-volts, while the interference signal is in the order of tens of volts and subtly varies around $60Hz$. Figure 8 does not depict to scale the ratio between the signal to noise. When the ratio beween the signal and noise is that large, small changes, over time, in the ratio between the two signals may be overlooked. This case leads to a simplified adaptation algorithm that, virtually, needs to adjust the location of the pole in parallel to the frequency axis only.

Initial filter parameters and, thus, the pole locations may be computed using the average parameters of the signals and once these initial locations of the poles are found, adjustments are made in parallel to the frequency axis only.

When the adaptive filter is implemented using a second order filter, the goal is to trim the input signal and adjust the cut-off frequency of the filter so as to keep the *signal-to-noise ratio* close to some constant SNR_{ct}. The simple, and well-known, adaptation procedure, which will be translated to rules, increases the cut-off frequency, when the average frequency of the noise increases, and decreases the cut-off frequency when the average frequency of the noise decreases. In terms of SNR, given the measured SNR of the filter (SNR_m), which in this case is the ratio between the signal and the error integrated over a given period of time, the adaptation algorithm increases the cut-off frequency when ΔSNR is negative and decreases the cut-off frequency when ΔSNR is positive, where,

$$\Delta SNR = SNR_m - SNR_{ct} \tag{20}$$

8.3 Representation using complex membership grades

In the Cartesian representation, the pole would be described by a complex linguistic variable, i.e.,

$$(X\_Location, Y\_Location)_c \tag{21}$$

As explained in 8.2, this example does not vary the *X_Location*. In addition, along with the Cartesian representation for the pole, a polar representation is also used.

$$\tilde{Z}_p = (R\_Location, \Theta\_Location)_p \tag{22}$$

A rule-base transformation is used to transform between the two representations. It should be emphasized that the fuzzy sets for the

X_Location and *Y_Location* are optimized with the intention that the adaptation would be satisfactory. On the other hand, the fuzzy sets for the linguistic Polar components are designed with the intention that the parameters for the physical components would be optimal, i.e.,

$$(\omega_n, \zeta) = Dfz\{\tilde{Z}_p\} \tag{23}$$

where (ω_n, ζ) are used to determine the values of R, L and C as can be seen from equations 18 and 19.

Apart from the optimization issue, it may seem that the pole may be first defuzzified and then transformed to the polar coordinates to obtain ω_n and ζ. Doing this operation is conceptually flawed, as if the data represented in polar coordinates were an input to a second linguistic controller, the defuzzification followed by crisp transformation and a fuzzification would be inconsistent with fuzzy set theory that advocates computation using words.

8.4 A Linguistic Adaptation Architecture

The linguistic model has the architecture shown in figure 10. The main two components are the fuzzy controller and the rule-base coordinate transformation modules. As stated in the previous section, the adaptation is performed using the Cartesian linguistic complex number representation, while the updates to the filter are derived from the Polar linguistic complex number representation. Hence, requiring a coordinate transformation module.

q As part of this architecture the fuzzy controller is used to adjust the *Y_Location* of the pole. The inputs to the fuzzy controller are the fuzzified SNR and the linguistic *Y_Location* of the pole. Its output is the new linguistic *Y_Location* of the pole. As can be seen, the architecture of the fuzzy controller used in this example is simple and common to many fuzzy controller implementations.

Suppose the linguistic values of *Y_Location* are from the set $\{\ldots, \text{LittleBelowAverage}, \text{Average}, \text{LittleAboveAverage}, \ldots\}$, and the terms for the fuzzified SNR are from $\{\ldots, \text{NegativeSmall}, \text{Zero}, \text{PositiveSmall}, \ldots\}$. Rules from the rule-base have the form,

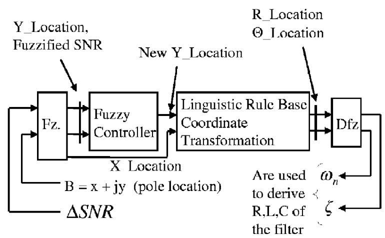

Figure 10. The Linguistic Adaptation Architecture

§1. **if** Y_Location is Average **and** SNR is Zero **then** New_Y_Location is Average

§2. **if** Y_Location is Average **and** SNR is NegativeSmall **then** New_Y_Location is PositiveSmall

§3. **if** Y_Location is LittleAboveAverage **and** SNR is PositiveSmall **then** New_Y_Location is Average

After the adaptation step, the linguistic rule-base coordinate transformation computes the linguitic Polar representation. Although, the fuzzy controller could have been used to generate the Polar representation directly, it would have been a much more complex controller. Without the transformation module, the fuzzy controller would have to compute two linguistic variables as opposed to one. Further, specific to this problem, moving the pole in one dimension is easier and more intuitive to the designer than moving it in two dimensions. Thus, making the rules easier to derive.

The architecture used in this problem is a simplified one. As the control is done through the frequency domain, additional linguistic variables may be added to galvanize a pole when a local minimum in the error is reached.

9 Discussion

Based on the success of fuzzy set theory in the real domain, it seems that the introduction of the concepts and techniques from fuzzy set theory to the complex domain would unleash a framework that could be applied to the design of systems described in the complex domain. More significantly, the design of systems that, using conventional mathematical methods, were difficult to describe.

The guidelines behind which complex membership grades were introduced included the need to project the features of the complex domain to the linguistic domain. Primarily, propagating the two-dimensionality of the universe of discourse through the linguistic spectacle. This, subsequently, gave birth to other essential operations that are extensions to crisp complex domain operations, such as coordinate transformations. Also, as opposed to two-dimensional linguistic variables, complex membership grades defines operations that cause an interaction between the two components.

The application played an important role in testing and tuning the proposed model, but by no means is the model confined to filter design. In this context, it should be emphasized that fuzzy set theory is an attractive engineering tool and fuzzification models for the complex domain must consider application design prospects.

In order to facilitate the feasibility of applications, simple and essential operations, such as arithmetic operations and defuzzification, must remain computationally feasible for a subset of operands, such as the complex fuzzy number subset. Without this property, the theory would have remained in the realm of computation with symbols. This point is probably the most conspicuous limitation of the theory presented in [13].

The filter application presented establishes a new framework for the design of adaptive filters. This paradigm is based on a direct frequency domain rule-base inference that gives the designer a platform for a simple and more intuitive filter design process.

References

[1] Di Nola A., Sessa S., and Sanchez E. *Fuzzy Relation Equations and Their Applications to Knowledge Engineering.* Kulmar Academic Publishing, Dordrecht, 1989.

[2] Kandel A. *Fuzzy Mathematical Techniques with Applications.* Wesley, Massachusetts, 1986.

[3] Kandel A., Teodorescu H.N., and Arotaritei D. *Analytic Fuzzy RBF Neural Network.* In Bezdek J. and Hall L.O., editors, *1998 Conference of the North American Fuzzy Information Processing Society*, pages 281–285, Pesacola Beach,Florida, 1998.

[4] Kaufmann A. and Gupta M. *Introduction to Fuzzy Arithmetic: Theory and Application.* Van Nostrand Reinhold, New York, 1985.

[5] Oppenheim A.V., Willsky A.S., and Nawab S.H. *Signals and Systems.* Prentice Hall, New Jersey, second edition, 1997.

[6] Dubois D. and Prade H. *Fuzzy Sets and Systems: Theory and Applications.* Academic Press, New York, 1980.

[7] Ifeachor E.C. and Jervis B.W. *Digital Signal Processing - A Practical Approach.* Addison-Wesley, Wokingham,England, 1993.

[8] Klir G.J. and Yuan B. *Fuzzy Sets and Fuzzy Logic: Theory and Applications.* Prentice HAll, New Jersey, 1995.

[9] Teodorescu H.N. *Generalized Analytic Monotonous and RBF Fuzzy NN - submitted.* 1998.

[10] Teodorescu H.N., Tacu A.P., and Aluja J.H. *Fuzzy Systems in Economy and Engineering.* Publishing House of the Romanian Academy, Bucharest, 1994.

[11] Jang J.-S., R. Sun C.-T, and Mitzutani E. *Neuro-Fuzzy and Soft Computing: A Computational Approach to Learning and Machine Intelligence.* Prentice Hall, New Jersey, 1997.

[12] Mathews J.H. and Howell R.W. *Complex Analysis for Mathematics and Engineering.* Wm.C.Brown, Dubuque,IA, third edition, 1996.

[13] Buckley J.J. Fuzzy Complex Numbers. *Fuzzy Sets and Systems*, 33:333–345, 1989.

[14] Buckley J.J. Fuzzy complex analysis II: Integration. *Fuzzy Sets and Systems*, 49:171–179, 1992.

[15] Buckley J.J. and Yunxia Q. Fuzzy complex analysis I: Differentiation. *Fuzzy Sets and Systems*, 41:269–284, 1991.

[16] Passino K.M. and Yurkovich S. *Fuzzy Control.* Addison-Wesley, California, 1998.

[17] Zadeh L.A. Fuzzy Sets. *Information and Control*, 8:338–353, 1965.

[18] Zadeh L.A. Calculus of Fuzzy Restrictions. In Tanaka K. Zadeh L.A., Fu K.S. and Shimura M., editors, *Fuzzy Sets and Their Applications to Cognitive and Decision Processes*, pages 1–39. Academic Press, New York, 1975.

[19] Zadeh L.A. The concept of a linguistic variable and its application to approximate reasoning - Part 1. *Information Sciences*, 8:199–249, 1975.

[20] Zadeh L.A. The concept of a linguistic variable and its application to approximate reasoning - Part 2. *Information Sciences*, 8:301–357, 1975.

[21] Zadeh L.A. The concept of a linguistic variable and its application to approximate reasoning - Part 3. *Information Sciences*, 9:43–80, 1975.

[22] Huelsman L.P. *Active and Passive Analog Filter Design: An Introduction.* McGraw-Hill, 1993.

[23] Haykin S. *Adaptive Filter Theory.* Prentice Hall, New-Jersey, third edition, 1996.

[24] Guang-Quan Z. Fuzzy limit theory of fuzzy complex numbers. *Fuzzy Sets and Systems*, 46:227–235, 1992.

[25] Yue Z. and Guangyuan W. Frequency domain methods for the solution of N-order fuzzy differential equations. *Fuzzy Sets and Systems*, 94:45–59, 1998.

[26] Yue Z. and Guangyuan W. Time domain methods for the solution of N-order fuzzy differential equations. *Fuzzy Sets and Systems*, 94:77–92, 1998.

D.Moses, H.Teodorescu, M.Friedman, A.Kandel, Received November 10, 1999

Daniel Moses, Abraham Kandel
Department of Computer Science and Engineering,
University of South Florida, Tampa, FL

Horia-Nicolai Teodorescu
Department of Computer Science and Engineering,
University of South Florida, Tampa, FL;

On leave from The Technical University of Iasi,
Iasi, Romania
e-mail: *teo@tuiasi.ro*

Menahem Friedman
NRCN, Physics department,
Beer-Sheva, P.O. Box 9001, Israel