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About some classes of periodic orbits in a problem of two fixed centres

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Abstract

Five classes of periodic motions for the plane restricted problem of two fixed centres constructed on the base of more than a thousand of periodic orbits, detected owing to the conducted numerical experiments are described in the article.

1 Equation of motion in a problem of two fixed centres

As it is known [1], the restricted problem of two fixed centres consists in the study of motion of the point M_3 which has vanishingly small mass m_3, attracted by two fixed points M_1 and M_2 with finite masses m_1 and m_2. Let initial conditions of motion of point M_3 are such that it all the time moves in the same plane. The value m_1+m_2 as a unit of mass and mutual distance of points M_1 and M_2 as a length one are accepted. In Cartesian coordinates with origin O at the centre of gravity and with axis Ox, passing through fixed points M_1 and M_2, attracting centres have coordinates $(-\mu, 0)$ and $(1-\mu, 0)$, and equations of motion for point M_3 are:

$$\ddot{x} = -(1-\mu)\frac{x+\mu}{r_1^3} - \mu\frac{x-1+\mu}{r_2^3}$$
$$\ddot{y} = -(1-\mu)\frac{y}{r_1^3} - \mu\frac{y}{r_2^3} \qquad (1)$$

where

$$r_1 = \sqrt{(x+\mu)^2 + y^2},$$
$$r_2 = \sqrt{(x-1+\mu)^2 + y^2},$$
$$\mu = m_1/(m_1+m_2).$$

The equations of motion (1) have the first integral – integral of energy:

$$C = \frac{2(1-\mu)}{r_1} + \frac{2\mu}{r_2} - \dot{x}^2 - \dot{y}^2. \qquad (2)$$

It is necessary to mark that the equations of motion (1) as well as the first integral (2) have a certain symmetry, namely they are still the same after the replacement:

$$x \to x, \qquad y \to -y, \qquad C \to C \qquad (3)$$

and it is possible to be convinced in this fact by a direct substitution. By virtue of this property among periodic orbits of a dynamic system with necessity exist symmetrical ones relatively to the axis of abscissas [2].

As the detection and study of all types of the symmetrical periodic orbits of the plane restricted problem of two fixed centres is hampered in view of their large number, especially those which close after a great number of axis Ox intersections, we limited the search by symmetrical ones relatively to the axis of abscissas of periodic orbits, which become closed after two or four intersections of the axis Ox.

2 About search of periodic orbits

Generally the solutions of the differential equations (1) can be represented by trajectories in a four-dimensional phase space $(x, y, \dot{x}, \dot{y})$. For the fixed value of a constant of integration C and known values of three phase coordinates the fourth one is determined by the integral of energy (2) and the phase trajectories of a system can be represented in a three-dimensional phase space, for example, $(x, y, \dot{x})$ instead of four-dimensional one. The following step is reduced to the analysis not of the entire phase trajectory, but only of its sequential intersections with some transversal surface, and the intersections themselves – the consequents of the Poincare's map.

The analysis of distribution of the dynamic system consequents totality on the transversal surface is easier and more obvious than study of the trajectory of a system in configuration or phase spaces. In particular, closed after the $2n$ intersections with the axis Ox periodic orbit will be represented on the transversal surface by a set of n invariant points. Periodic orbit represented on the transversal surface by one invariant point of the Poincare map is named simpler periodic one [4].

Proceeding from problem symmetries (3) it is expedient to select as a transversal surface the plane $(x, \dot{x})$ and to consider only those intersections, which satisfy the conditions $y = 0, \dot{y} > 0$.

In configuration space (x, y) of a problem these consequents correspond to cross points of the mass point M_3 orbit with the axis Ox for motion of the last one in a positive direction of the axis Oy. Symmetrical periodic orbits intersect the axis Ox at the right angle not less than two times. For want of their search the distribution of consequents is analyzed depending on choice of the initial point x_0, for fixed value of the constant C.

Taking into account the circumstance, that the indicated trajectories proceed from the axis Ox at the right angle, for some initial value of abscissa x_0 of the mass point M_3 suppose $y_0 = 0$ and $\dot{x}_0 = 0$ and from the equation (2) determine $\dot{y}_0$. Under these initial conditions numerically integrating equations (1) up to n^{th} intersection of the axis of abscissas, or equally the transversal plane, the sign of the phase coordinate is determined. The operations are repeated for the other initial point x_0 on the axis Ox, neighbouring to the first. If the signs of coordinates $\dot{x}_1$ are different for these points, then this section with necessity contains such point x_0, for which $\dot{x}_1 = 0$ and the orbit, corresponding to this, is periodic. If the section with different signs is chosen, the position of the invariant point, belonging to it, can be determined with a predesigned exactness by one of the methods of specification of the non-linear equation root.

It is necessary to take into consideration the circumstance, that for variation of the value of the integration constant C by virtue of continuous dependence of system (1) solutions on the initial conditions, periodic orbits of a dynamic system are not situated isolate. They are grouped in classes represented by some curves on a plane (C, x_0), which are named the classes characteristics [3, 4].

3 Classes of periodic orbits in a problem of two fixed centres

The numerical experiments on search of symmetrical periodic trajectories were carried out for a dynamic system with the values of a parameter $\mu = 0.5$ and $\mu = 0.1$.

In case $\mu = 0.5$ the investigated system has an additional symmetry:

$$x \to -x, \qquad y \to y, \qquad C \to C, \tag{4}$$

the account of which enables to exclude from examination a half-plane $x < 0$ generalizing for it the results of numerical experiments conducted for $x > 0$.

Found periodic orbits were refereed to this or that class, following the orbits classification method proposed by E.Stromgren in study of the Copenhagen variant of the restricted problem of three bodies [2], based on the analysis of a position of orbits in relation to finite masses m_1 and m_2 and to the libration points. It is necessary to mark, that unlike the Copenhagen problem, at which there are five libration points, the problem of two fixed centres has only one libration point which coincides with the system masses centre. This fact explains why the classes of its symmetrical periodic orbits are relatively not numerous.

Another classification criterion can be that circumstance, that for initial $\dot{y}_0 > 0$ the periodic orbits can be circumscribed by a mass point in two directions: direct or converse.

During numerical experiments it was revealed more than a thousand of symmetrical simpler periodic orbits which were grouped in two classes and the symmetrical orbits closed after two revolutions which were grouped in three classes.

3.1 Class (a)

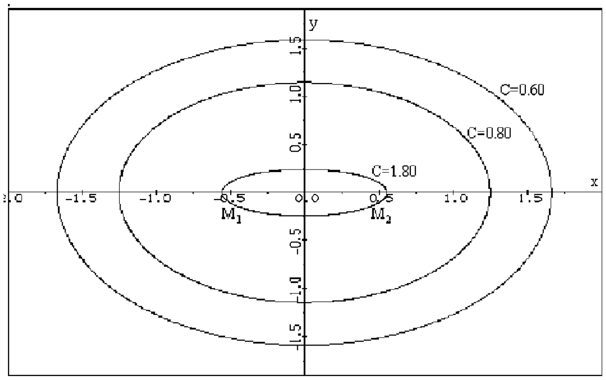

Fig. 1. Class (a)

The class (a) includes direct simpler periodic orbits of the oval form circumscribed by the mass point M_3 in a plane (x, y) round attracting centres M_1 and M_2. In accordance with magnification of the constant C value orbits of the class (a) do not change the configuration, remaining enclosed in each other in such a way that smaller one is completely enclosed in areas limited by trajectories of large sizes.

For $\mu = 0.5$ the orbits of the class (a) are symmetric relatively to axes Ox and Oy (fig. 1). The class (a) begins with simpler periodic orbit going out from a point $x_0 = 4.7619$, which corresponds to value of the energy constant $C = 0.21$, and the halfcycle of motion on it is equal $T/2 = 64.7534$. At magnification of the parameter C, sizes of orbits of the class (a) and period of motion decrease. The class (a) evolves up to value $C = 1.95$, to which there corresponds the value $x_0 = 0.5128$ and $T/2 = 1.0939$.

At $\mu = 0.1$ orbits of a class (a) were detected in the range of values from $C = 0.36$ up to $C = 1.95$. To the value $C = 0.36$ there corresponds the orbit with initial value $x_0 = 3.1778$, final value $x_1 = -2.3778$ and value of a halfcycle $T/2 = 14.4132$. To maximum value $C = 1.95$ there corresponds the orbit, defined by values $x_0 = 0.9128$, $x_1 = -0.1128$ and $T/2 = 0.6455$.

3.2 Class (k)

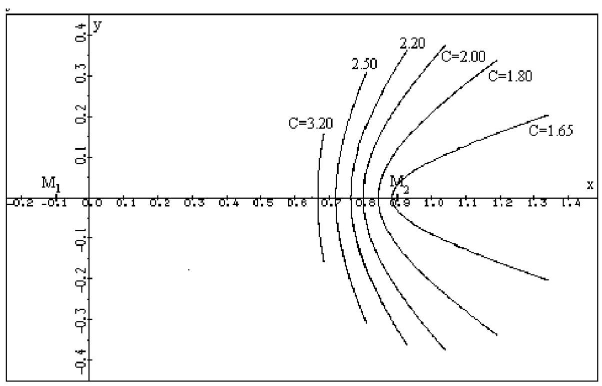

Fig. 2. Class (k)

The class (k) is formed from symmetrical simpler periodic orbits circumscribed by a system in a neighbourhood of the problem libration point.

At $\mu = 0.5$ orbits of this class represent the sections of straight lines passing through the origin of coordinates, coinciding with the axis Oy. They were detected in the range of values from $C = 0.20$ up to $C = 0.50$. For value $C = 0.2$ the magnitude of oscillation is maximum $y_{max} = 9.9875$, and the halfcycle of motion is $T/2 = 141.3505$. For magnification of value of a constant C amplitude and period of motion on orbits decrease. For value $C = 0.50$ amplitude of oscillation is equal to 3.9686, and the halfcycle of motion is equal $T/2 = 36.3753$.

At $\mu = 0.1$ orbits of a class (k) are no more the sections of straight lines, and are similar to some parabolical curves (fig.2). They were detected in the range of values from $C = 1.65$ up to $C = 3.20$. For $C = 1.65$ motion of the mass point M_3 begins from the point $x_0 = 0.8848$, and the halfcycle of motion on it is equal $T/2 = 1.8304$. At increase of C the class (k) orbits move to the left along the axis Ox and are rectified a little. The greatest value of energy constant at which the orbits of this class were detected is $C = 3.2$, with corresponding $x_0 = 0.6500$ and $T/2 = 1.0750$.

Let's pass to the description of the detected classes of symmetrical periodic orbits, closing after four intersections of the axis Ox. Let's remark, that such orbits were found only for system with equal attracting masses $\mu = 0.5$. For value $\mu = 0.1$ such orbits were not revealed.

3.3 Class (b)

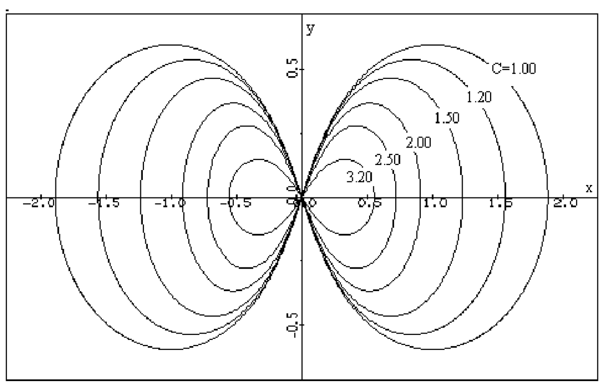

Fig. 3. Class (b)

The class (b) is formed from direct periodic orbits, which are circumscribed around M_1 and M_2 (fig.3). These orbits form each two closed loops, so that every closed loop is described round one of the attracting centres, and the intermediate intersection of the axis Ox happens in the libration point. The class (b) begins in the point

$C = 0.78$, $x_0 = 2.4422$. The value of a halfcycle for this orbit is equal $T/2 = 8.8552$. At the increase of the constant C the class (b) orbit do not change their configuration, and only decrease in sizes, coming nearer to the attracting centres.

The greatest value C at which orbits of this class is $C = 3.47$ were still detected to which there corresponds value $x_0 = 0.5121$ and halfcycle $T/2 = 0.8239$.

3.4 Class (f)

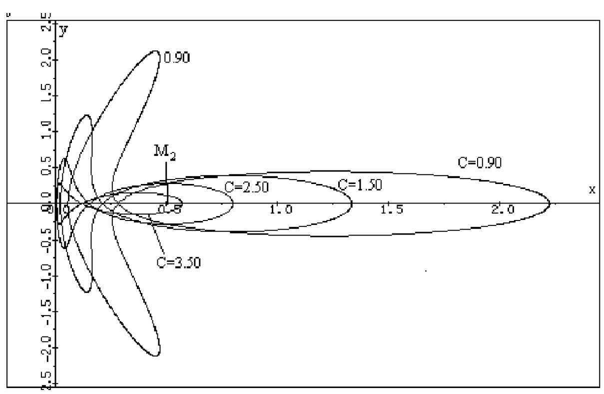

Fig. 4. Class (f)

Direct periodic orbits (fig. 4) circumscribed by the mass point M_3 in a neighbourhood of the point M_2 were included in the class (f). To the minimum value $C = 0.81$, for which orbits of a class (f) were detected, there correspond the values $x_0 = 2.4710$, $x_1 = 0.6838$ and $T/2 = 12.9664$. Orbits are autointersected and form each two closed loops. At the increase of the value C the orbits of this class reduce their size, moving simultaneously to the left. The maximum value C, at which orbits of this class were detected, is $C = 3.90$ for which $x_0 = 0.5128$, $x_1 = 0.0027$ and $T/2 = 1.4612$.

3.5 Class (g)

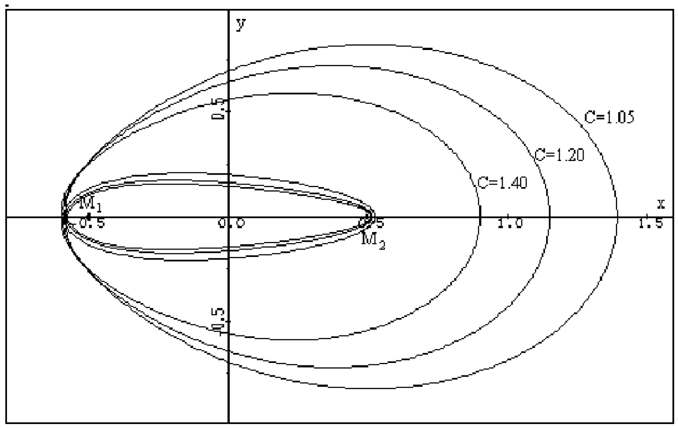

Fig. 5. Class (g)

The class (g) is composed from direct periodic orbits circumscribed by the mass point M_3 round the points M_1 and M_2 (fig. 5). The orbits of the class (g) are detected for the minimum value of the parameter $C = 1.05$, to which corresponds the greatest orbit with initial value $x_0 = 0.5112$, $x_1 = 1.3936$ and halfcycle $T/2 = 3.2350$. Orbits of the class (g) have each one internal closed loop. At the increase of value C the sizes of orbits decrease, practically not changing sizes of internal closed loops. For the maximum value $C = 1.63$ the orbit sizes are rather close to sizes of an internal closed loop. The motion on the orbit begins in the point $x_0 = 0.5809$ and the second intersection of the axis Ox at the right angle happens in the point $x_1 = 0.6459$, and it the halfcycle is $T/2 = 1.8079$.

By virtue of a symmetry (4), to each of the above described classes there corresponds a symmetrical class of periodic orbits circumscribed by a dynamic system in a direction opposite to described motion. The periods of motion on these orbits also coincide with the periods of motion on orbits of main classes. These classes were called (c), (d), (h), (i) and they are identical to classes (c) = (a), (d) = (b), (h) = (f), (i)=(g), correspondingly. In the configuration space (x, y) their orbits coincide with the orbits of main classes for their mirror map relatively the axis Oy.

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