Computer Science Journal of Moldova, vol.**6**, no.3(18), 1998

# Games on digraphs and constructing maximin structures

V. Ungureanu

**Abstract**

Games and problems of constructing maximin structures on digraphs are investigated. Necessary and sufficient conditions for existence of equilibrium situation in matrix games with pure strategies are formulated and proved. Some particular games are considered and algorithms for their solving are sugested. $p$ - players games and dynamical games on digraph are introduced.

## 1 Introduction

Suppose that several companies manage the activity of a big network. They have their personal local frequently antagonistic interests. In this situation well-known extremal net problems [3, 7] and problems of constructing graph structures [1, 3, 7] become multi-criteria game problems.

Some problems of the hind were investigated in literature [5, 6]. They were arisen in the context of cyclic games solving. That approach determined a special type of strategies definition [5]. This paper introduces new types of games on digraphs by generalising the corresponding notions of pure strategy and cost function.

The paper is organized as follows. Section 2 introduces the notion of zero-sum matrix game on digraphs. Some properties giving a general tool for matrix games investigations are proved. Section 3 presents some particular solvable games. A special investigation is done for flow game. It is proved that the problem of maximin cost flow finding is NP-hard. Section 4 generalises the notion of matrix game on digraphs in the case of arbitrary finite number of players. Section 5 introduces the notion of dynamic games on digraphs using notions from previous sections.

# 2 Matrix games on digraphs and games with connected strategies

## 2.1 Notions

Let us consider a digraph $G = (V, E)$, $|V| = n$, $|E| = m$, further called simply graph. A length $c(e) \in Z$ is associated with every directed edge $e \in E$. Vertex set $V$ is partitioned into two disjoint subsets:

$$V_I, V_{II} \quad (V_I \cup V_{II} = V,\ V_I \cap V_{II} = \emptyset),$$

being positions of two players. The player edge sets are determined:

$$E_I = \{(u, v) \in E | u \in V_I\}; \quad E_{II} = \{(u, v) \in E | u \in V_{II}\}.$$

Every subset $S_I \subseteq E_I$, $S_{II} \subseteq E_{II}$ is called strategy of the corresponding player. It is obvious that every pair of strategies $(S_I, S_{II}) \in 2^{E_I} \times 2^{E_{II}}$, called game situation, generates a subgraph $G_S = (V, S_I \cup S_{II})$, called graph of the $(S_I, S_{II})$ situation.

Let:

- $2^G = \{\Gamma{=}(V, E') \mid E' \subseteq E\}$ be the set of all subgraphs of $G$;
- $\mathcal{D} = \{\Gamma \in 2^G \mid \mathcal{P}\}$ be the set of all subgraphs of $G$, verifying a set $\mathcal{P}$ of some properties;
- $M$: $2^G \to \mathcal{D}, M(S_I, S_{II}) = 2^{G_S} \cap \mathcal{D}$ be the choice function, that sets the correspondence of every situation $(S_I, S_{II}) \in 2^{E_I} \times 2^{E_{II}}$ and the set of subgraphs of $G_S$, verifying $\mathcal{P}$;
- $C$: $\mathcal{D} \to \mathcal{R}$ be the choice (quality) criterion.

Let $k = \max\limits_{(S_I, S_{II}) \in 2^{E_I} \times 2^{E_{II}}} |M(S_I, S_{II})|$ be the cardinality of a choice function. There are 3 possibilities:

$1^0$. $k = 1$;

$2^0$. $k > 1$ and $\forall\, \Gamma', \Gamma'' \in \forall\, M(S_I, S_{II}) \neq \emptyset$: $C(\Gamma') = C(\Gamma'')$;

$3^0$. $k > 1$ and $\exists\, M(S_I, S_{II}) \neq \emptyset, \Gamma', \Gamma'' \in M(S_I, S_{II})$: $C(\Gamma') \neq C(\Gamma'')$.

A matrix game can be defined if the choice function verify property $1^0$ or property $2^0$. Thus, we consider a normal univocal choice function

$$\overline{M}(S_I, S_{II}) = \underset{\Gamma \in M(S_I, S_{II})}{\operatorname{argmax}} \; C(\Gamma).$$

It sets the correspondence of every $(S_I, S_{II}) \in 2^{E_I} \times 2^{E_{II}}$ to an element of $M(S_I, S_{II})$, optimal by criterion $C$. $\overline{M}(S_I, S_{II}) \neq \emptyset$ is called feasible subgraph of situation $(S_I, S_{II})$. A function $\overline{M}$ is called the matrix of the feasible subgraphs. Thus, a cost function (payoff matrix) of the game may be defined as:

$$\overline{C}(S_I, S_{II}) = \begin{cases} C(\overline{M}(S_I, S_{II})), & \text{if } \overline{M}(S_I, S_{II}) \neq \emptyset, \\ -\infty, & \text{if } \overline{M}(S_I, S'_{II}) = \emptyset \;\; \forall\, S'_{II} \in 2^{E_{II}}, \\ +\infty, & \text{otherwise.} \end{cases}$$

**Remark 1.** *The set $\mathcal{P}$ of properties can define in a particular game: a tree; a path between two fixed vertex $v_s$ and $v_t$; a flow between output vertex $v_s$ and input vertex $v_t$; a matching; a median; a clique; a cycle; a Hamiltonian cycle, etc. Meanwhile a feasible subgraph $\overline{M}(S_I, S_{II})$ satisfies both $\mathcal{P}$ and the optimality property by $C$* choice criterion.

Let $J = < 2^{E_I}, 2^{E_{II}}, \overline{C}(S_I, S_{II}) >$ define a zero-sum matrix game in $G$. The first player has $2^{|E_I|}$ strategies, the second – $2^{|E_{II}|}$. Every situation $(S_I, S_{II}) \in 2^{E_I} \times 2^{E_{II}}$ has a numerical feature $\overline{C}(S_I, S_{II})$, that means the gain (loss if $\overline{C}(S_I, S_{II}) < 0$) of the first player, and the loss (gain if $\overline{C}(S_I, S_{II}) < 0$) of the second player in situation $(S_I, S_{II})$.

Players choose their strategies $(S_I, S_{II}) \in 2^{E_I} \times 2^{E_{II}}$ independently. The value $\overline{C}(S_I, S_{II})$ is determined and the first player receives the gain $\overline{C}(S_I, S_{II})$ (loss if $\overline{C}(S_I, S_{II}) < 0$) from the second player.

The sets

$$\mathcal{B}_I = \left\{ S_I \in 2^{E_I} | \exists\, S_{II} \in 2^{E_{II}} :\ \overline{M}(S_I, S_{II}) \neq \emptyset \right\}, \tag{1}$$

$$\mathcal{B}_{II} = \left\{ S_{II} \in 2^{E_{II}} | \exists\, S_I \in 2^{E_I} :\ \overline{M}(S_I, S_{II}) \neq \emptyset \right\}, \tag{2}$$

will be called the sets of admissible strategies. The sets

$$\mathcal{B}_I(S_{II}) = \left\{ S_I \in \mathcal{B}_I | \overline{M}(S_I, S_{II}) \neq \emptyset \right\},$$

$$\mathcal{B}_{II}(S_I) = \left\{ S_{II} \in \mathcal{B}_{II} | \overline{M}(S_I, S_{II}) \neq \emptyset \right\},$$

will be called the sets of admissible strategies connected with $S_I$ and $S_{II}$ correspondingly. In these notations, the game

$$J^+ = < \mathcal{B}_I, \mathcal{B}_{II}, \overline{C}(S_I, S_{II}) >$$

is defined and is called a matrix game in admissible strategies. The game $J^+(*)$, which is either game

$$J^+(I) = < \mathcal{B}_I, \mathcal{B}_{II}(S_I), C(\overline{M}(S_I, S_{II})) >,$$

or game

$$J^+(II) = < \mathcal{B}_{II}, \mathcal{B}_I(S_{II}), C(\overline{M}(S_I, S_{II})) >,$$

with the first step of the first player and of the second player correspondingly is called a matrix game with connected admissible strategies. All situations of the game $J^+$ are called admissible. All situations of the game $J^+(*)$ are called essentially admissible.

## 2.2 Properties

**Lemma 1.** $\max\limits_{S_I \in 2^{E_I}} \min\limits_{S_{II} \in 2^{E_{II}}} \overline{C}(S_I, S_{II}) \leq \min\limits_{S_{II} \in 2^{E_{II}}} \max\limits_{S_I \in 2^{E_I}} \overline{C}(S_I, S_{II}),$

$$\max_{S_I \in \mathcal{B}_I} \min_{S_{II} \in \mathcal{B}_{II}} \overline{C}(S_I, S_{II}) \leq \min_{S_{II} \in \mathcal{B}_{II}} \max_{S_I \in \mathcal{B}_I} \overline{C}(S_I, S_{II})$$

Lemma 1 is well-known property of matrix games. Because of lemma 1, right and left values of inequalities are named upper and lower costs of the games $J$ and $J^+$. A matrix game has a solution (is solvable) if its upper cost is equal to the its lower cost. The corresponding situation is called equilibrium situation of the game and its value is called value of the game.

In the game with connected strategies, the opposite correlation occurs for some cost functions $C$ : the cost of the game $J^+(II)$ $(\min\limits_{S_{II}\in\mathcal{B}_{II}} \max\limits_{S_I\in\mathcal{B}_I(S_{II})} \overline{C}(S_I, S_{II}))$ do not surpass the cost of the game $J^+(I)$ $(\max\limits_{S_I\in\mathcal{B}_I} \min\limits_{S_{II}\in\mathcal{B}_{II}} \overline{C}(S_I, S_{II}))$.

**Lemma 2.** *If* $C(\overline{M}(S_I, S_{II})) = C'(S_I) + C''(S_{II})$ *and* $\mathcal{B}_I \neq \emptyset$, *then*

$$\max_{S_I\in\mathcal{B}_I} \min_{S_{II}\in\mathcal{B}_{II}(S_I)} \overline{C}(S_I, S_{II}) \geq \min_{S_{II}\in\mathcal{B}_{II}} \max_{S_I\in\mathcal{B}_I(S_{II})} \overline{C}(S_I, S_{II})$$

**Proof.** It is obvious that if $\mathcal{B}_I \neq \emptyset$, then $\mathcal{B}_{II} \neq \emptyset$ and vice versa. Thus, we have

$$\max_{S_I\in\mathcal{B}_I} \min_{S_{II}\in\mathcal{B}_{II}(S_I)} C(\overline{M}(S_I, S_{II})) = \max_{S_I\in\mathcal{B}_I} [C'(S_I) + \min_{S_{II}\in\mathcal{B}_{II}(S_I)} C''(S_{II})] \geq$$

$$\geq \max_{S_I\in\mathcal{B}_I} C'(S_I) + \min_{S_{II}\in\mathcal{B}_{II}} C''(S_{II}) \geq \min_{S_{II}\in\mathcal{B}_{II}} [C''(S_{II}) + \max_{S_I\in\mathcal{B}_I(S_{II})} C'(S_I)] =$$

$$= \min_{S_{II}\in\mathcal{B}_{II}} \max_{S_I\in\mathcal{B}_I(S_{II})} C(\overline{M}(S_I, S_{II})). \square$$

The game with connected strategies have the equilibrium situation if the costs of the games $J^+(I)$ and $J^+(II)$ are equal.

**Lemma 3.** *If* $\mathcal{B}_{\mathcal{I}} \neq \emptyset$, *then* $\max\limits_{S_I\in 2^{E_I}} \min\limits_{S_{II}\in 2^{E_{II}}} \overline{C}(S_I, S_{II}) =$

$$= \max_{S_I\in\mathcal{B}_I} \min_{S_{II}\in\mathcal{B}_{II}} \overline{C}(S_I, S_{II}) = \max_{S_I\in\mathcal{B}_I} \min_{S_{II}\in\mathcal{B}_{II}(S_I)} C(\overline{M}(S_I, S_{II})).$$

**Proof.** The cost function is so defined that $\min\limits_{S_{II}\in 2^{E_{II}}} \overline{C}(S_I, S_{II}) < \infty$ for every strategy $S_I \in 2^{E_I}$. As $\mathcal{B}_I \neq \emptyset$, then

$$-\infty < \max_{S_I\in 2^{E_I}} \min_{S_{II}\in 2^{E_{II}}} \overline{C}(S_I, S_{II}) < +\infty.$$

Therefore, the maximin situation is essentially admissible and

$$\max_{S_I\in 2^{E_I}} \min_{S_{II}\in 2^{E_{II}}} \overline{C}(S_I, S_{II}) = \max_{S_I\in \mathcal{B}_I} \min_{S_{II}\in \mathcal{B}_{II}} \overline{C}(S_I, S_{II}). \tag{3}$$

For some admissible strategy $S_I \in \mathcal{B}_I$ we have

$$\min_{S_{II}\in \mathcal{B}_{II}} \overline{C}(S_I, S_{II}) = \min_{S_{II}\in \mathcal{B}_{II}(S_I)} \overline{C}(S_I, S_{II}),$$

as $\overline{C}(S_I, S_{II}) = +\infty$ for every strategy $S_{II} \notin \mathcal{B}_{II}(S_I)$. Then

$$\max_{S_I\in \mathcal{B}_I} \min_{S_{II}\in \mathcal{B}_{II}} \overline{C}(S_I, S_{II}) = \max_{S_I\in \mathcal{B}_I} \min_{S_{II}\in \mathcal{B}_{II}(S_I)} \overline{C}(S_I, S_{II}). \tag{4}$$

(3) – (4) prove lemma. □

Considering lemma 3 and the equality $\max\limits_{S_I\in 2^{E_I}} \overline{C}(S_I, S_{II}) = +\infty$ for all $S_{II} \in \mathcal{B}_{II}$, the following theorem is evident

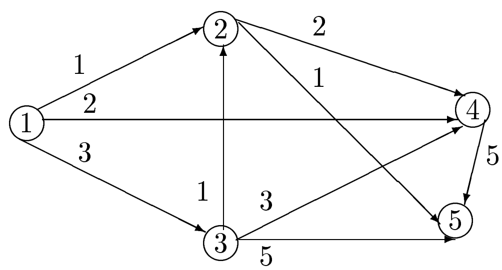
**Theorem 1.** *Let $\mathcal{B}_I \neq \emptyset$. Upper (lower) costs of the games $J$ and $J^+$ coincide.*

For sufficiently big set of edges $E$, an exhaustive search of the equilibrium situation in the games $J, J^+, J^+(*)$ is a hard problem. Theorem 1 indicates how to narrow the sets of the admissible strategies for players taking into account properties of $G$ and properties $\mathcal{P}$ of the feasible subgraphs.

**Assumption 1.** *Further on we assume that $\mathcal{B}_I \neq \emptyset$. We will take as sets of admissible strategies $\mathcal{B}_I$ and $\mathcal{B}_{II}$ also sets of strategies (from $2^{E_I}$ and $2^{E_{II}}$) less powerful than (1) – (2). Thus, we have for a fixed* *game $J$ several possible games $J^+$, and $J^+(*)$, defining $J^+$ from the condition $|M(S_I, S_{II})| \leq 1$, for all $(S_I, S_{II}) \in \mathcal{B}_I \times \mathcal{B}_{II}$.*

It is easy to observe that cost function is defined in such a way, that if only one essentially admissible situation exists in $J^+(*)$, then for both players it is advantageous the same situation, namely the equilibrium situation. However, in some games $J^+(*)$ several essential admissible situations exist, but the equilibrium situation itself does not exist in all games $J, J^+$ and $J^+(*)$.

$\bigtriangledown$**Example 1.** Let us consider the following acyclic digraph $G = (V, E)$.




If $V_I = \{1\}$ and $V_{II} = \{2; 3; 4; 5\}$, then

$$\begin{aligned} E_I &= \{(1,2);\ (1,3);\ (1,4)\}, \\ E_{II} &= \{(2,4);\ (2,5);\ (3,2);\ (3,4);\ (3,5);\ (4,5)\}. \end{aligned}$$

The set $\mathcal{D}$ of feasible graphs is defined as the set of all paths from $v_s = 1$ to $v_t = 5$, containing exactly 3 edges. It is easy to calculate that the first player has $2^3 = 8$ strategies in $J_3$ and the second has $2^6 = 64$. Finding by the exhaustive search the equilibrium situation is a hard problem because the game has in total 512 situations. Therefore, because $G$ is acyclic and feasible graphs are specific, admissible strategies are defined as the sets of the following cardinality: $|E_I| = |V_I| = 1$ for the first player, and $|E_{II}| = |V_{II}| - 1 = 3$ for the second, and exactly one edge is going out from every vertex except $v_t = 5$. As $G$ is acyclic, for every situation $S$ its $G_S$ is directed tree, going to vertex $v_t = 5$, and it has a path from $v_s = 1$ to $v_t = 5$, not obligatory having exactly 3 edges. Observe, that for $S_I = \{(1,4)\}$ a 3 – path from $v_s = 1$ to $v_t = 5$ does not exist for every strategy of the second player. Besides that, for strategy $S_{II} = \{(2,5); (3,5); (4,5)\}$ a 3 – path from $v_s = 1$ to $v_t = 5$

does not exist for every strategy of the adversary. If the cost function $C$ is defined as the length of the path from $v_s = 1$ to $v_t = 5$ and the first player has the purpose to maximise the length of the path (the second trying to minimise), then it is easy to find the following payoff matrix of the game $J_3^+$, containing the cost function of the game $J_3^+(*)$ :

| $\mathcal{B}_I$ | $\mathcal{B}_{II}$ | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | (3,2) (2,4) (4,5) | (2,4) (3,4) (4,5) | (2,4) (4,5) (3,5) | (3,2) (2,5) (4,5) | (2,5) (3,4) (4,5) | (2,5) (3,5) (4,5) | $J_3^+$: $\min\limits_{S_{II}\in\mathcal{B}_{II}}$ | $J_3^+(I)$: $\min\limits_{S_{II}\in\mathcal{B}_{II}(S_I)}$ |
| (1,2) | 8 | 8 | 8 | $+\infty$ | $+\infty$ | $+\infty$ | 8 | 8 |
| (1,3) | $+\infty$ | 11 | $+\infty$ | 5 | 11 | $+\infty$ | 5 | 5 |
| (1,4) | $-\infty$ | $-\infty$ | $-\infty$ | $-\infty$ | $-\infty$ | $-\infty$ | $-\infty$ | |
| $J_3^+$: $\max\limits_{S_I\in\mathcal{B}_I}$ | $+\infty$ | 11 | $+\infty$ | $+\infty$ | $+\infty$ | $+\infty$ | 11\ 8 | |
| $J_3^+(II)$: $\max\limits_{S_I\in\mathcal{B}_I(II)}$ | 8 | 11 | 8 | 5 | 11 | | | 5\ 8 |

In the $J_3^+$ (consequently in $J_3$)

$$\max_{S_I\in\mathcal{B}_I}\min_{S_{II}\in\mathcal{B}_{II}}\overline{C}(S_I,S_{II}) = 8 \le \min_{S_{II}\in\mathcal{B}_{II}}\max_{S_I\in\mathcal{B}_I}\overline{C}(S_I,S_{II}) = 11,$$

but in $J_3^+(*)$ we have

$$\max_{S_I\in\mathcal{B}_I}\min_{S_{II}\in\mathcal{B}_{II}(S_I)}\overline{C}(S_I,S_{II}) = 8 \ge \min_{S_{II}\in\mathcal{B}_{II}}\max_{S_I\in\mathcal{B}_I(S_{II})}\overline{C}(S_I,S_{II}) = 5,$$

Let us consider the set $\mathcal{D}_2$ of all paths from $v_s = 1$ to $v_t = 5$, having exactly 2 edges, as the set of feasible graphs. Analogically, $J_2, J_2^+$ and $J_2^+(*)$, are defined, and the payoff matrix of the game $J_2^+$ and the cost function of the game $J_2^+(*)$ are determined.

| $\mathcal{B}_I$ | $\mathcal{B}_{II}$ | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | (3,2) (2,4) (4,5) | (2,4) (3,4) (4,5) | (2,4) (4,5) (3,5) | (3,2) (2,5) (4,5) | (2,5) (3,4) (4,5) | (2,5) (3,5) (4,5) | $J_3^+$: $\min\limits_{S_{II}\in\mathcal{B}_{II}}$ | $J_3^+(I)$: $\min\limits_{S_{II}\in\mathcal{B}_{II}(S_{II})}$ |
| (1,2) | $+\infty$ | $+\infty$ | $+\infty$ | 2 | 2 | 2 | 2 | 2 |
| (1,3) | $+\infty$ | $+\infty$ | 8 | $+\infty$ | $+\infty$ | $\boxed{8}$ | 8 | 8 |
| (1,4) | 7 | 7 | 7 | 7 | 7 | 7 | 7 | 7 |
| $J_3^+$: $\max\limits_{S_I\in\mathcal{B}_I}$ | $+\infty$ | $+\infty$ | $+\infty$ | $+\infty$ | 11 | 8 | 8\ 8 | |
| $J_3^+(II)$: $\max\limits_{S_I\in\mathcal{B}_I(II)}$ | 7 | 7 | 7 | 7 | 11 | 8 | | 7\ 8 |

In the game $J_2^+$ (consequently $J_2$)

$$\max_{S_I \in \mathcal{B}_I} \min_{S_{II} \in \mathcal{B}_{II}} \overline{C}(S_I, S_{II}) = 8 = \min_{S_{II} \in \mathcal{B}_{II}} \max_{S_I \in \mathcal{B}_I} \overline{C}(S_I, S_{II}) = 8,$$

but in $J_2^+(*)$ we have

$$\max_{S_I \in \mathcal{B}_I} \min_{S_{II} \in \mathcal{B}_{II}(S_I)} \overline{C}(S_I, S_{II}) = 8 \geq \min_{S_{II} \in \mathcal{B}_{II}} \max_{S_I \in \mathcal{B}_I(S_{II}} \overline{C}(S_I, S_{II}) = 7,$$

Remark, that in the games $J_2, J_2^+$ there is the equilibrium situation, but in $J_2^+(*)$ there is none.

Let us consider the set $\mathcal{D}$ of all paths from $v_s = 1$ to $v_t = 5$ as the set of feasible graphs. Analogically, the games $J, J^+, J^+(*)$ are defined and the payoff matrix and the cost function are determined.

| $\mathcal{B}_I$ | $\mathcal{B}_{II}$ | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | (3,2) (2,4) (4,5) | (2,4) (3,4) (4,5) | (2,4) (4,5) (3,5) | (3,2) (2,5) (4,5) | (2,5) (3,4) (4,5) | (2,5) (3,5) (4,5) | $J^+$: $\min_{S_{II} \in \mathcal{B}_{II}}$ | $J^+(I)$: $\min_{S_{II} \in \mathcal{B}_{II}(S_I)}$ |
| (1,2) | 8 | 8 | 8 | 2 | 2 | 2 | 2 | 2 |
| (1,3) | 11 | 11 | 8 | 5 | 11 | 8 | 5 | 5 |
| (1,4) | 7 | 7 | 7 | 7 | 7 | 7 | 7 | 7 |
| $J^+$: $\max_{S_I \in \mathcal{B}_I}$ | 11 | 11 | 8 | 7 | 11 | 8 | 7\ 7 | |
| $J^+(II)$: $\max_{S_I \in \mathcal{B}_I(S_{II})}$ | 11 | 11 | 8 | 7 | 11 | 8 | | 7\ 7 |

Let us remark, that all the games $J, J^+, J^+(*)$ have the equilibrium situation $(S_I = \{(1,4)\},\ S_{II} = \{(3,2);(2,5);(4,5)\})$ with feasible path $P = \{(1,4);(4,5)\}$ of the length $C(P) = 7$, having exactly 2 edges. $\triangle$

**Theorem 2.** *If $C(\overline{M}(S_I, S_{II})) = C'(S_I) + C''(S_{II})$ for all $\overline{M}(S_I, S_{II}) \neq \emptyset$ and if all situations in the game $J^+$ are essentially feasible, then $J, J^+, J^+(*)$ have the equilibrium situation, moreover, it is the same in all three cases.*

**Proof.** All situations in the game $J^+$ are essentially feasible. Then $\mathcal{B}_{II}(S_I) = \mathcal{B}_{II}$ for all $S_I \in \mathcal{B}_I$, and $\mathcal{B}_I(S_{II}) = \mathcal{B}_I$ for all $S_{II} \in \mathcal{B}_{II}$. Then, taking into account lemma 1, we have

$$\max_{S_I \in \mathcal{B}_I} \min_{S_{II} \in \mathcal{B}_{II}(S_I)} C(\overline{M}(S_I, S_{II})) = \max_{S_I \in \mathcal{B}_I} \min_{S_{II} \in \mathcal{B}_{II}} C(\overline{M}(S_I, S_{II})) \leq$$

$$\leq \min_{S_{II}\in\mathcal{B}_{II}} \max_{S_I\in\mathcal{B}_I} C(\overline{M}(S_I, S_{II})) = \min_{S_{II}\in\mathcal{B}_{II}} \max_{S_I\in\mathcal{B}_I(S_{II})} C(\overline{M}(S_I, S_{II})),$$

and, taking into account lemma 2, we have

$$\max_{S_I\in\mathcal{B}_I} \min_{S_{II}\in\mathcal{B}_{II}} C(\overline{M}(S_I, S_{II})) = \max_{S_I\in\mathcal{B}_I} \min_{S_{II}\in\mathcal{B}_{II}(S_I)} C(\overline{M}(S_I, S_{II})) \geq$$

$$\geq \min_{S_{II}\in\mathcal{B}_{II}} \max_{S_I\in\mathcal{B}_I(S_{II})} C(\overline{M}(S_I, S_{II})) = \min_{S_{II}\in\mathcal{B}_{II}} \max_{S_I\in\mathcal{B}_I} C(\overline{M}(S_I, S_{II})).$$

From these inequalities follows that

$$\max_{S_I\in\mathcal{B}_I} \min_{S_{II}\in\mathcal{B}_{II}} C(\overline{M}(S_I, S_{II})) = \max_{S_I\in\mathcal{B}_I} \min_{S_{II}\in\mathcal{B}_{II}(S_I)} C(\overline{M}(S_I, S_{II})) =$$

$$= \min_{S_{II}\in\mathcal{B}_{II}} \max_{S_I\in\mathcal{B}_I(S_{II})} C(\overline{M}(S_I, S_{II})) = \min_{S_{II}\in\mathcal{B}_{II}} \max_{S_I\in\mathcal{B}_I} C(\overline{M}(S_I, S_{II})).$$

Therefore, $J^+$ and $J^+(*)$ have the equilibrium situation. Finally, it follows from theorem 1 that all games $J, J^+, J^+(*)$ have the same equilibrium situation. □

**Remark 2.** *Lemma 2 and theorem 2 are true for other functions. For example:* $C(\overline{M}(S_I, S_{II})) = C'(S_I) \cdot C''(S_{II})$, *where* $C'$: $\mathcal{B}_I \to N^*$, $C''$: $\mathcal{B}_{II} \to N^*$.

**Remark 3.** *Example 1 shows that if in* $J^+$ *there is situation that is not essentially admissible, then the equilibrium situation may be absent in any game* $J, J^+, J^+(*)$.

**Remark 4.** *Example 1 also illustrates that the equilibrium situation may exist in the games* $J, J^+$, *but in the corresponding game* $J^+(*)$ *it may be absent. Inverse is possible: in* $J^+(*)$ *there is an equilibrium situation, but in* $J, J^+$ *it is absent.*

Theorem 2 formulates only sufficient condition for the existence of the equilibrium situation in games $J, J^+, J^+(*)$. The following theorem formulates necessary and sufficient condition.

**Theorem 3.** *Let* $C(\overline{M}(S_I, S_{II})) = C'(S_I) + C''(S_{II})$, *for all* $\overline{M}(S_I, S_{II}) \neq \emptyset$. $(S_I^*, S_{II}^*)$ *is equilibrium situation in games* $J, J^+, J^+(*)$

*if and only if* $(S_I^*, S_{II}^*)$ *is equilibrium situation in the game* $J^+$ *and*

$$\max_{S_I \in \mathcal{B}_I(S_{II})} C(\overline{M}(S_I, S_{II})) \geq C(\overline{M}(S_I^*, S_{II}^*)), \quad \forall\, S_{II} \in \mathcal{B}_{II}.$$

**Proof.** Necessity is obvious. Sufficiency follows from the relations:

$$\min\max J^+ \overset{\text{L. 1}}{\geq} \max\min J^+ \overset{\text{L. 3}}{=} \max\min J^+(I) \overset{\text{L. 2}}{\geq} \min\max J^+(II), \quad (5)$$

and theorem 1. □

If $J^+$ has an equilibrium situation $(S_I^*, S_{II}^*)$, then, taking into account lemma 3 and theorem 3, we deduce that $\min\max J^+ < +\infty$. This means that, for strategy $S_{II}^* \in \mathcal{B}_{II}$, $\overline{C}(S_I, S_{II}^*) = C(\overline{M}(S_I, S_{II}^*)) < +\infty$ for all $S_I \in \mathcal{B}_I$. The strategy, which may have only essentially admissible situations ($\mathcal{B}_I(S_{II}) = \mathcal{B}_I$ for the first player and $\mathcal{B}_{II}(S_I) = \mathcal{B}_{II}$ for the second), will be called essentially admissible. From theorem 3 follows:

**Corollary 1.** *If the second player does not have at least one essentially admissible strategy in the game* $J^+$*, then both* $J$ *and* $J^+$ *do not have equilibrium situations.*

## 3 Solvable matrix games on digraphs

The investigation of the matrix games includes solving of three main problems:

- determining of the maximin and minimax situations;
- determining of the feasible graphs in maximin and minimax situations graphs;
- determining of the equilibrium situation.

From (5) follows that for equilibrium situation determined in $J, J^+, J^+(*)$, it is sufficiently to determine minimax situation in $J^+$ and $J^+(II)$. If $\min\max J^+ = \min\max J^+(II)$, then all three games $J, J^+, J^+(*)$ are solvable and have the same equilibrium situation. Otherwise, it is necessary to find $\max\min J^+$ which is equal to $\max\min J^+(II)$ and to compare it with $\min\max J^+$ and $\min\max J^+(I)$ :

- if $\max\min J^+ = \min\max J^+$, then $J^+$ is solvable;
- if $\max\min J^+ = \min\max J^+(I)$, then $J^+(*)$ is solvable;
- if $\max\min J^+ \neq \min\max J^+$ and $\max\min J^+ \neq \min\max J^+(I)$, then all games $J, J^+, J^+(*)$ are not solvable.

Therefore, to investigate games $J^+, J^+(*)$, the problem of determining maximin and minimax situations with corresponding maximin and minimax feasible graphs become the principal. As the games have limited number of situations, maximin and minimax situations hypothetically may be found by the exhaustive search, which, with a big $m$, is impracticable even on modern computers. It is obvious, that the game has polynomial complexity if both maximin and minimax situations may be found in polynomial time by $n$ and $m$. If the problem of a feasible graph $\overline{M}(S_I, S_{II}) \neq \emptyset$ construction in $G_S$ for some situation $(S_I, S_{II})$ is NP – complete, then the game is at least NP – hard.

Method of the exhaustive search in $J$ has the exponential complexity, supposing $2^{|E|}$ situations to examine. If the algorithm for construction feasible subgraph in $G$ has the polynomial complexity $O(n^{k_0}m^{l_0})$ and $|\mathcal{B}_I| = O(n^{k_1}m^{l_1}), |\mathcal{B}_{II}| = O(n^{k_2}m^{l_2})$, where $k_0, k_1, k_2, l_0, l_1, l_2$ are numbers independent of $m$ and $n$, then the same straightforward method in $J^+$ and $J^+(*)$ has the polynomial complexity $O(n^{k_0+k_1+k_2}m^{l_0+l_1+l_2})$.

Thus depending on of properties of $G$ and elements of $\mathcal{D}$, in particular games this problems may be essentially simplified. Further on, we illustrate this for three particular games.

### 3.1 Maximin directed tree

Let $G$ be an acyclic digraph and let paths exist in $G$ from every vertex $v \in V$ to vertex $v_0$. Let $\mathcal{D}$ be the set of all directed trees of $G$ going to

$v_0$; $C : \mathcal{D} \to \mathcal{R}$ be the length of tree (sum of edge lengths). The first player has the aim to maximize the length of tree, the second tries to minimize it.

Take into consideration that feasible graphs are directed trees, we will define admissible strategies so that from every vertex except $v_0$ exactly one edge is going out. In this case every element belonging to $\mathcal{D}$ at least once is feasible subgraph in $J^+$. Remark, that inadmissible strategies are not advantageous to players because they either not ensure tree construction or they lead to adversary possibility to choice from several alternatives. Therefore, either $\mathcal{B}_I$ and $\mathcal{B}_{II}$ contain the optimal strategies of the players and the game $J^+$ is right defined, that ensure equality of costs of the games $J$ and $J^+$.

Remark further, that for all $(S_I, S_{II}) \in \mathcal{B}_I \times \mathcal{B}_{II}$ we have $\overline{M}(S_I, S_{II}) = (V, S_I \cup S_{II})$. This mean, that all situations of the game $J^+$ are essentially admissible. From theorem 2 follows that $J, J^+, J^+(*)$ have the same equilibrium situation.

To determine the maximin input in $v_0$ tree we propose the following application of the dynamical programming method [9]:

$M = \{v_0\},\ T = \emptyset$
while $|M| < n$ do
begin $(u^*, v^*) = \operatorname*{argmax}_{(u,v)\in((V\setminus M)\times M)\cap E} C(u,v)$
If $(u^* \in V_I)$ or $((u^* \in V_{II})$ and $|((V\setminus M)\times M)\cap E| = 1)$
then $M = M \cup u^*$, $T = T \cup (u^*, v^*)$
else $E = E\setminus(u^*, v^*)$.
end
$T$ – maximin tree.

**Remark 5.** *Algorithm determines maximin tree in arbitrary digraph. But, because in general case maximin may be not equal to minimax, for determining of minimax tree, we must found on every iteration*

$$(u^*, v^*) = \operatorname*{arg\,min}_{(u,v)\in((V-M)\times M)\cap E} C(u,v).$$

### 3.2 Maximin directed path

Let $G$ be an acyclic digraph and let a path exists from every vertex $v \in V$ to vertex $v_t$. Let $\mathcal{D}$ be the set of all directed paths from $v_0$ to $v_t$; $C : \mathcal{D} \to \mathcal{R}$ be the length of path (sum of edge lengths). The first player has the aim to maximize the length of path, the second has the aim to minimize it.

We will define admissible strategies so that from every vertex except $v_0$ exactly one edge is going out. In this case every $G_S$ is an input in $v_0$ tree, containing a path from $v_s$ to $v_t$. The set of all essentially admissible situations of maximin path game $J^+$ is equivalent to the set of all essentially admissible situations of the maximin tree game $J^+$. Therefore all three games $J$, $J^+$, $J^+(*)$ have the same equilibrium situation.

To determine maximin path we may use an adaptation of Dijkstra algorithm. An example of such adaptation is presented in [2].

### 3.3 Maximin cost flow

Let the net has a structure of digraph $G = (V, E)$, $|V| = n$, $|E| = m$ and let us distinguish in $G$ the output vertex $v_s \in V$ and input vertex $v_t \in V$, $v_s \neq v_t$. Let $b(u, v) \in Z^+$ be the capacity of edge $(u, v) \in E$ and let $c(u, v) \in Z$ be the transportation cost of the unit flow through the same edge. The set $V$ is partiotioned into two disjoint sets of player positions:

$$V_I,\ V_{II},\ (V_I \cup V_{II} = V,\ V_I \cap V_{II} = \emptyset),$$

and without loss of generality let $v_s \in V_I$. Thus, we have the edge sets of players:

$$E_I = \{(u, v) \in E \,|\, u \in V_I\}, \quad E_{II} = \{(u, v) \in E \,|\, u \in V_{II}\}.$$

Every subset $S_I \subseteq E_I$, $S_{II} \subseteq E_{II}$ we will call a strategy of a corresponding player. For every situation $(S_I, S_{II}) \in 2^{E_I} \times 2^{E_{II}}$ we have a situation graph $G_S = (V, S_I \cup S_{II})$. In the net $G_S$, the flow $f$ of fixed value $\varphi_0$ is defined as the vector $f \in R^{|S_I \cup S_{II}|}$ (one component for every edge), such that:

$1^0$. $\quad 0 \le f(u,\, v) \le b(u,v), \qquad (u,\, v) \in S_I \cup S_{II},$

$2^0$. $$\sum_{(u,\,v)\in S_I\cup S_{II}} f(u,\,v) - \sum_{(v,\,u)\in S_I\cup S_{II}} f(v,\,u) = \begin{cases} 0, & u \notin \{v_s,\, v_t\},\\ \varphi_0, & u = v_s,\\ -\varphi_0, & u = v_t. \end{cases}$$

Cost of the flow $f$ is equal to $\sum\limits_{(u,\,v)\in S_I\cup S_{II}} c(u,\,v) f(u,\,v)$.

Let us suppose that there is at least one flow with value $\varphi_0$ in $G = (V,\, E)$.

For every pair of strategies $(S_I,\, S_{II})$ there are a polyhedron of solutions of the system $1^0 - 2^0$, denoted by $F_s = F(G_s)$. In general, $F_S$ may be empty for some pair of strategies if the system $1^0 - 2^0$ does not have solutions, but, due to our supposition, there are at least one pair of strategies for which $F_s \neq \emptyset$. It is known, that if all $b(u,\, v),\ (u,\, v) \in E$ are integer, then all the vertex of polyhedron $F_S$ are integer, and as $F_S$ is limited, then the set of all flows, corresponding to $(S_I, S_{II})$, is a linear convex combination of the finite number of integer flows. The cardinality of the set $F_S$ may be equal to 0, when the system $1^0 - 2^0$ does not have solutions, may be equal to 1, when $1^0 - 2^0$ has one solution, or may be equal to $\aleph$, when $1^0 - 2^0$ has an infinite number of solutions.

Let $\mathcal{D}$ be the set of all subgraph of $G$ which has the flow of the value $\varphi_0$ from $v_s$ to $v_t$. Let us examine the choice function $M$: $2^G \to \mathcal{D}$, $M(S_I,\, S_{II}) = G_s \cap \mathcal{D}$ and the choice criterion $C$: $\mathcal{D} \to R$, $C(\Gamma) = \max\limits_{f\in F(\Gamma)} \sum c(e) f(e)$. Let us define the univocal choice function

$$\overline{M}(S_I,\, S_{II}) = \begin{cases} \operatorname*{argmax}\limits_{\Gamma\in M(S_I,S_{II})} C(\Gamma), & \text{if } M(S_I,\, S_{II}) \neq \emptyset,\\ \emptyset, & \text{otherwise}, \end{cases}$$

which choose the flow of the value $\varphi_0$ with minimal cost in the net $G_S$.

Then, we have the following cost function (payoff matrix):

$$\overline{C}(S_I,\, S_{II}) = \begin{cases} C(\overline{M}(S_I,\, S_{II})), & \text{if } \overline{M}(S_I,\, S_{II}) \neq \emptyset, \\ -\infty, & \text{if } \overline{M}(S_I,\, S'_{II}) = \emptyset,\ \forall\, S'_{II} \in 2^{E_{II}}, \\ +\infty, & \text{otherwise.} \end{cases}$$

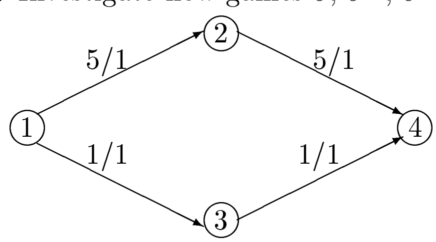
Therefore, the matrix game $J$ is defined. By analogy with general case, we define the games $J^+$, $J^+(*)$. Strategies are called admissible if there exist at least one such strategy of adversary, that corresponding situation is essentially admissible (has the $\varphi_0$ flow).

**Lemma 4.** If in the net $G$ at least one flow of value $\varphi_0$ exists, then in the game $J^+$ both players have at least one essentially admissible strategy.

**Proof.** Let us order rows and columns of the matrix of the game $J^+$ so that cardinalities of the corresponding strategies are not decreased. It is obvious, that the strategies equal to the union of all admissible strategies of the corresponding player are essentially admissible. □

The following example shows that for the same solvable game $J$ there are several different possibilities to define games $J^+$, $J^+(*)$ so that the game $J^+(*)$ will be solvable or not.

▽**Example 2.** Investigate flow games $J$, $J^+$, $J^+(*)$ in the net:




where $\varphi_0 = 1;\ v_s = 1;\ v_t = 4;\ V_I = \{1\};\ V_{II} = \{2;\, 3;\, 4\};$

$$E_I = \{(1,\, 2);\ (1,\, 3)\};\quad E_{II} = \{(2,\, 4);\ (3,\, 4)\}.$$

We have the following payoff matrix:

| I | II: (2, 4) | (3, 4) | (2, 4) (3, 4) | $\min_{\mathcal{B}_{II}}$ | $\min_{\mathcal{B}_{II}(S_I)}$ |
|---|---|---|---|---|---|
| (1, 2) | 10 | $+\infty$ | **10** | 10 | 10 |
| (1, 3) | $+\infty$ | 2 | 2 | 2 | 2 |
| (1, 2) (1, 3) | 10 | 2 | 2 | 2 | 2 |
| $\max_{\mathcal{B}_I}$ | $+\infty$ | $+\infty$ | 10 | 10\10 | |
| $\max_{\mathcal{B}_I(S_{II})}$ | 10 | 2 | 10 | | 2\10 |

The games $J$, $J^+$ have the equilibrium situation. The game $J^+(*)$ does not have.

If we narrow the sets of admissible strategies so that from every vertex except the fourth at least one edge is going out, then we will have the payoff matrix:

| I | II: (2, 4) (3, 4) | $\min_{\mathcal{B}_{II}}$ | $\min_{\mathcal{B}_{II}(S_I)}$ |
|---|---|---|---|
| (1, 2) | **10** | 10 | 10 |
| (1, 3) | 2 | 2 | 2 |
| (1, 2) (1, 3) | 2 | 2 | 2 |
| $\max_{\mathcal{B}_I}$ | 10 | 10\10 | |
| $\max_{\mathcal{B}_I(S_{II})}$ | 10 | | 10\10 |

and the games $J$, $J^+$, $J^+(*)$ will have the same equilibrium situation.

Thus, for the same game $J$ we will have two pairs of games $J^+$, $J^+(*)$. For one pair the game $J^+(*)$ does not have equilibrium situation, for another all games $J$, $J^+$, $J^+(*)$ have the same equilibrium situation. △

It is known, that the problem of minimal cost flow may be formulated as the linear programming problem [7]. Let us renumber vertices and edges of $G$ so that vertices and edges of the first player be the first in the numbering. Let us define now the incidence matrix $A = [a_{ij}]$ of the graph $G$, where:

$$a_{ij} = \begin{cases} +1, & \text{if } e_j \text{ is output from vertex } i, \\ -1, & \text{if } e_j \text{ is input in vertex } i, \\ 0, & \text{otherwise,} \end{cases}$$

$i = \overline{1,\, n}$; $j = \overline{1,\, m}$. If we denote $f = \begin{bmatrix} f_I \\ f_{II} \end{bmatrix} = (f_1, \ldots, f_m)^T \in R^m$, $f_j$ – flow through edge $e_j$; $b, c \in R^m$, $b_j$ – capacity, $c_j$ – unit cost of edge $e_j$ flow;

$$d \in R^m, \qquad d_i = \begin{cases} -1, & \text{if } v_i = v_s, \\ +1, & \text{if } v_i = v_t, \\ 0, & \text{otherwise,} \end{cases}$$

then we obtain the following minimal cost flow problem in the net $G$:

$$c^T f \to \min, \tag{6}$$

$$\begin{cases} Af &=& d\varphi_0, \\ f &\le& b, \\ f &\ge& 0. \end{cases} \tag{7}$$

To the first $n$ restrictions, corresponding to the law of $\varphi_0$ flow safe, let us put in correspondence dual variables $\pi_i$, to the following $m$ restrictions of the flow capacity let us put in correspondence dual variables $\gamma_k$.

Then, the problem (6) – (7) has the following dual problem:

$$\varphi_0(\pi_s - \pi_k) + b^T\gamma \to \max, \tag{8}$$

$$\begin{cases} \pi_i - \pi_j + \gamma_k \le c_k, & \text{for } e_k = (v_i,\, v_j) \in E, \\ \gamma_k \le 0, & k = \overline{1,\, m}. \end{cases} \tag{9}$$

According to the main theorem of duality, (6) – (7) and (8) – (9) have optimal solutions if and only if the following system has a solution:

$$\begin{cases} c^T f &=& \varphi_0\pi_s - \varphi_0\pi_t + b^T\gamma, \\ Af &=& d\varphi_0, \\ f &\le& b, \\ f &\ge& 0, \\ \pi_i - \pi_j + \gamma_k &\le& c_k, \quad \text{for } e_k = (v_i,\, v_j) \in E, \\ \gamma_k &\le& 0, \qquad k = \overline{1, n}, \end{cases} \tag{10}$$

(the first equality is the binding). For every situation $(S_I, S_{II}) \in \mathcal{B}_I \times \mathcal{B}_{II}$ we will get the analogous system. Let $\Phi(S_I, S_{II})$ denotes the set of all solutions of the corresponding system. Then, the cost function may be written down:

$$\overline{C}(S_I, S_{II}) = \begin{cases} c^T f, & \text{if } \Phi(S_I, S_{II}) \neq \emptyset \ \text{ where } (f, \pi, \gamma)^T \in \Phi(S_I, S_{II}), \\ -\infty, & \text{if } \Phi(S_I, S'_{II}) = \emptyset \ \text{ for all } S'_{II} \in \mathcal{B}_{II}, \\ +\infty, & \text{otherwise.} \end{cases}$$

Further on, using notions of linear programming, we will show that the problem of finding maximin and minimax situations in the flow game is equivalent to maximin and minimax linear problems. It is clear, that the set of feasible solutions of the problems (6) – (7) is an polyhedron in $R^m$, and the minimal solution is its vertex. Then, the aim of the first player to maximize the flow cost by rational choice of the net $G_S$ structure is equivalent to the aim to maximize the flow cost by optimal choice of the basic columns of the matrix $A$, corresponding to the choice edges of $E_I$. The aim of the second player to minimize the cost of the flow by rational choice of the net $G_S$ structure is equivalent to the aim to minimize flow cost by optimal choice of the basic columns of the matrix $A$. Therefore, for the first player it is advantageous to choice feasible solution, for which at least columns corresponding to edges from $E_I$ have non-negative dual estimations. For the second player it is advantageous to choice feasible solution, for which at least columns corresponding to edges from $E_{II}$ have non-positive dual estimations. Therefore a feasible solution is the saddle point being the equilibrium situation. Then, the problem of determining the equilibrium situation in the flow game is equivalent to the maximin linear problem:

$$\max_{f_I} \min_{f_{II}} c^T \begin{bmatrix} f_I \\ f_{II} \end{bmatrix}, \tag{11}$$

$$\begin{cases} A \begin{bmatrix} f_I \\ f_{II} \end{bmatrix} = d\varphi_0, \\ 0 \leq \begin{bmatrix} f_I \\ f_{II} \end{bmatrix} \leq b. \end{cases} \tag{12}$$

Using other notations, (11)–(12) may be written down as:

$$\varphi(f_I) = \min_{f_{II}}(c_I f_I + c_{II} f_{II}) = c_I f_I + \min_{f_{II}} c_{II} f_{II} \to \max . \qquad (13)$$

$$\begin{cases} A_{II} f_{II} & = & d\varphi_0 - A_I f_I, \\ 0 \leq & f_I & \leq b_I, \\ 0 \leq & f_{II} & \leq b_{II}. \end{cases} \qquad (14)$$

The function $\varphi(f_I)$ is determined as the solution of the linear parametric program with restrictions (14). It is known that solutions of such problems are piecewise linear convex functions [4]. Therefore, $\varphi(f_I)$ is piecewise-linear convex function.

Analogically, the function

$$\psi(f_{II}) = \max_{f_I}(c_I f_I + c_{II} f_{II}) = \max_{f_I}\ c_I f_I + c_{II} f_{II}$$

is piecewise-linear concave on (14).

**Theorem 4.** $\varphi(f_I)$ $(\psi(f_{II}))$ *is piecewise-linear convex (concave) function.*

Therefore, the problems of maximizing $\varphi(f_I)$ and minimizing $\psi(f_{II})$ are the problems of concave programming, which, as is well known, are $NP$ – hard even over an unit hypercube [8]. Consequently, taking into consideration that (13) – (14) may be represented as the problem of maximizing a piecewise-linear convex function over a hyperparalelepiped we get

**Theorem 5.** *Maximin (minimax) cost flow problem is NP – hard.*

## 4 Games of $p$ players on digraphs

Matrix games of two players can be generalized to the case of arbitrary number of players $p \leq n$. The vertex set $V$ is partitioned into the disjoint subsets of player positions:

$$V_1, V_2, \ldots, V_p \quad \left( \bigcup_{i=1}^{p} V_i = V,\ \ V_i \cap V_j = \emptyset, \ \ \text{for} \ \ i \neq j \right),$$

which evidently define the corresponding sets of player edges (possible passages):

$$E_i = \{(u,v) \in E | u \in V_i\},\ i = \overline{1,p}.$$

All players independently choose their strategies

$$(S_1, S_2, \ldots, S_p) \in 2^{E_1} \times \ldots \times 2^{E_p}.$$

After that, $\overline{M}(S_1, \ldots, S_p)$ (defined analogically to above) is determined. Every player determines his gain using his own cost function:

$$c_i(S_1,\ldots,S_p) = \begin{cases} c_i(\overline{M}(S_1,\ldots,S_p)), & \text{if } \overline{M}(S_1,\ldots,S_p) \neq \emptyset, \\ -\infty, & \text{if } \overline{M}(S'_1,\ldots,S'_p) = \emptyset,\ \forall\, S'_k \in 2^{E_k},\ k \neq i, \\ +\infty, & \text{otherwise,} \end{cases}$$

where $i = 1, \ldots, p$. Thus, the vector cost (payoff) function is defined

$$c:\ 2^{E_1} \times \ldots \times 2^{E_p} \to R^p,$$

which set the correspondence of every set of player strategies to player gains.

Analogically with the case of matrix games, the games of $p$ players with connected strategies can be defined, requiring of course the ordering of player leads.

The solution of the game of $p$ players is the set of strategies, optimal by Pareto, Nash, etc. If the characteristics of some players have the same tendency of increasing or decreasing, then we get the coalition games. For $p = 2$ and $c_2(S_1, S_2) = -c(S_1, S_2)$, we obtain a zero-sum matrix game.

## 5 Dynamic games on digraphs

Further on, we will denote mentioned games as $I$ (matrix game) and $IP$ ($p$ players game). $I$ and $IP$ are static one step games. Players choose their strategies at the same step or time moment. At the same moment (on the same step) the feasible graph $G^*_S$ is constructed, the cost (costs) of the game is determined and the gain or loss is distributed to players.

The dynamic (multi-steps) game is defined as the development of the static (one step) games, in general of different types, but, we will suppose for simplicity that types of games $I$ and $IP$ are fixed. Let us denote such games $DI$ and $DIP$ correspondingly. We will denote the static game of the step $t$ $DI(t)$ and $DIP(t)$ correspondingly, their costs being $c(DI(t))$ and $c(DIP(t))$. If the game is considered on the finite discrete time interval $\{1, 2, \ldots, \tau\}$, then it is obvious that the player strategies are sequences of the form: $(S_i(1), S_i(2), \ldots, S_i(\tau))$, where $S_i(t),\ t = 1, 2, \ldots \tau$ is the strategy of the $i$th player on the step (time moment) $t$.

The cost of the dynamic game is determined on the base of the static games of the steps $1, 2, \ldots, \tau$. In the simplest case, the cost function may be defined as following:

$$1.\ c(DI) = \sum_{t=1}^{\tau} c(DI(t)); \quad 2.\ \overline{c}(DI) = \frac{1}{\tau} \sum_{t=1}^{\tau} c(DI(t)),$$

for matrix games, or

$$3.\ c_i(DIP) = \sum_{t=1}^{\tau} c_i(DIP(t)); \quad 4.\ \overline{c}_i(DIP) = \frac{1}{\tau} \sum_{t=1}^{\tau} c_i(DIP(t)),$$

$i = \overline{1, p}$ for games of $p$ players.

If we will suppose that the type of static games is fixed, the result of the t-step game do not depends of the results of the games of previous steps and the dynamic game is ending on the step $\tau$, then it is obvious, that for solvable static games $I$ with optimal strategies $(S_I^*, S_{II}^*)$, the corresponding dynamic games have optimal strategies $(\underbrace{S_I^*, \ldots, S_I^*}_{\tau})$ and $(\underbrace{S_{II}^*, \ldots, S_{II}^*}_{\tau})$ with the cost $c(DI) = \tau c(S_I^*, S_{II}^*)$ or $\overline{c}(DI) = c(S_I^*, S_{II}^*)$. Therefore such dynamic games are the same as the static games for matrix and $p$ player games.

The dynamic game become more interesting if we will define another ending criterion. For example, the game may end when, on the some step $\theta$ is constructed some feasible graph of a determined structure in

$G$. Time interval, on which the dynamic game is considered, may be unlimited.

Next, we will associate with every step $t$ a state (condition) of the the dynamic game as the set of vertices and the set of edges which is changing from the step to step, and which determines the set of possible strategies of players on every step. If we will set the initial state $W(0) \subseteq V$ and $E(0) = E$ at the initial moment, the set of all possible strategies of the $i$th player at the moment $t$ may be defined as:

$$E_i(t) = \{(u, v) \in E | u \in V_i \cap W(t-1)\},\ t = \overline{1, \tau};\ i = \overline{1, p},$$

where $W(t-1)$ is the state of the game in the moment $t-1$, $V_i$ – the set of positions of the $i$th player. Every subset $S_i(t) \subseteq E_i(t)$ is called the strategy of player $i$ at the step $t$. The state of the dynamic game at the step $t$ is determined after examination of the corresponding static game at the step $t-1$ by the formula:

$$W(t) = \{v \in V | \, \exists (u, v) \in \bigcup_{i=1}^{p} S_i(t)\},\ \ t = \overline{1, \tau}.$$

As the player strategies at the step $t$ depend both on their positions and on the game state at the moment $t$, we have at consecutive moments $t$ and $t+1$ to solve static games of the same type, but, in general, with different sets of strategies of the same players. Then, as the player strategies are the edges which determine the application of $W(t-1)$ in $W(t)$, then players, in antagonistic interests, endeavour to increase (to decrease) the set of their own advantageous positions of the state $W(t)$, and to decrease (to increase) the set of advantageous (non-advantageous), positions of adversaries on the same state $W(t)$. Therefore, $G$ is the graph of all possible passages (strategies), which are limited by $2^m$. $V$ defines the set of all possible states, which are limited by $2^n$.

If the dynamical game is examined on unlimited time interval $\{1, 2, \ldots\}$, then, it follows from the limited number of states that some states of the game and corresponding strategies of players will be repeated in some sequences. It is obvious, that the cost functions $c(DI)$

and $c(DIP)$, having the form of the number sequences, will be unlimited on $\tau \to +\infty$. Therefore, the definition of such game must be completed with special ending criterion: or the value cost function is larger then determined limit, or at some moment the graph of determined structure is constructed, etc. In the case of the cost function $\overline{c}(DI)$ (or $\overline{c}(DIP)$) we will have:

$$\overline{c}(DI) = \lim_{\tau \to +\infty} \Big( \frac{1}{\tau} \sum_{t=0}^{\tau} c(DI(t) \Big),$$

for which there exists (as is mentioned above) repeated sequences of game states with limited value of the cost function, such that $\overline{c}(DI)$ is equal to fixed number. In this case, the problem to find cycle of the states may be considered.

Next, lengths of edges can be functions depending on $t$ and the cost of a dynamic game is calculated using static game costs only at some steps.

As mentioned, it is clear that the contents and type of dynamic games are depending on: static game; initial state and restriction of cardinality of the game states; cost function; edge length function; time interval on which the game is examined; ending criterion, etc. In investigation of dynamic games $DI$ it is useful sometimes to use the property that every dynamic game $DI$ can be represented as the matrix game.

## 6 Concluding remarks

It is necessary to mention that strategies of players may be defined also as subsets of vertices, or the pair of subsets of vertices and edges. The investigation of such games, the determination of solvable games and the elaboration of corresponding algorithms are problems for future work.

## References

[1] C. Berge, Theory of graphs and its applications, John Wiley & Sons, Inc., New York, 1962

[2] R. Boliac, D. Lozovanu, Finding of minimax paths tree in weighted digraph, Buletinul A.Ş. a R.M., 1996, Nr.3(66), pp.74–82 (Russian)

[3] N. Christofides, Graph theory: an algoritmic approach, Academic Press, London, 1975

[4] E. Golshtein, D. Yudin, New directions in linear programming, Moscow, Sovetskoe radio, 1966 (Russian)

[5] V. Gurvitch, A. Karzanov, L. Khatchiyan, Cyclic games: finding of minimax mean cycles in digraphs, J. Comp. Mathem. and Math. Phys., 1988, Nr.9(28), pp.1407–1417 (Russian)

[6] D. Lozovanu, A strongly polynomial time algorithm for finding minimax paths in network and solving cyclic games, Cybernetics and System Analysis, 1993, Nr.5, pp.145–151 (Russian)

[7] Chr. Papadimitriou, K. Steiglitz, Combinatorial Optimization: Algorithms and Complexity, Prentice-Hall Inc., Englewood Cliffs, New Jersey, 1982

[8] V. Ungureanu, Minimizing a concave quadratic function over a hypercube, Buletinul A.Ş. a R.M.: Mathematics Series, 1997, Nr.2(24), pp.69–76 (Românian)

[9] V. Ungureanu, A. Leconiuc, Maximin trees in digraphs, Conferinţa corpului didactico-ştiinţific al U.S.M., Chişinău, 30.09–5.10.1998, p.51

V. Ungureanu,
Institute of Mathematics
Moldovian Academy of Sciences
5, Academiei str., Kishinev
MD-2028, Moldova
Phone: 3732-73-35-83

Received December 10, 1998