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Strong stability and strong quasistability of vector trajectorial problem of lexicographic optimization *

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Abstract

Two types of stability of the lexicographic set for the multicriteria problem on a system of subsets of a finite set with the vector criterion of the most general kind are investigated. Lower bounds of stability radii have been found for the case where Chebyshev norm was defined in the space of vector criterion parameters.

1 Introduction

J.Hadamard [1] noted that a necessary attribute of a well-defined mathematical problem is stability of the problem. This implies that the solution depends continuously on the problem's parameters.

In the usual sense [2–7], the stability of an optimization problem is the property of upper and lower semicontinuity by Hauzdorf (or Berge) of the point-set mapping, which defines the choice function. If the set of admissible solutions is finite the property of upper semicontinuity can be replaced by an equivalent property of nonappearance of new optimal solutions under small perturbations of the problem's parameters (see, for example, [8–10] and also the survey [11]). A limit of such perturbations is called stability radius.

If we go over a single-criterion problem to a vector discrete optimization problem we get the notion of quasistability. The stability (quasistability) is a discrete analog of upper (lower) semicontinuity by

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Hauzdorf of Pareto-optimal mapping, i.e. is an existence of small perturbations of the problem's parameters such that the Pareto set can only narrow (extend). Results connected with such interpretation of the stability were described in detail in [6] (see also the survey [7]).

Papers [12–16] are devoted to study of different aspects of stability of a vector trajectorial (on a system of subsets of a finite set) problem of finding Pareto set in assumption that the vector criterion is an arbitrary combination of partial criteria of the kinds MINSUM, MINMAX and MINMIN, which are the most common in discrete optimization.

When the partial criteria are ordered with respect to their importance, a vector problem of lexicographic optimization arises. Different types of stability, in particular pseudostability and quasistability, of trajectorial problem of finding lexicographic set were considered in papers [17–20]. The pseudostability of the problem assumes that new lexicographic optimal trajectories do not appear under small perturbations of the problem's parameters. When we relax this demand we get the concept of the strong pseudostability, which was introduced first by V.K. Leontev for the single-criterion problem in [9]. This type of stability means that new lexicographic optimal trajectories can appear but, under any small perturbations, there exists a lexicographic optimal trajectory that keeps the lexicographic optimality.

The property of lower semicontinuity by Hauzdorf of our problem is equivalent to the property of preservation of all the lexicographic optima of the problem under small perturbations of its parameters. Following our terminology, we get a notion of quasistability, which has been investigated in [20]. When we relax this demand, we get the notion of the strong quasistability, which means that there exists a stable lexicographic optimal trajectory.

In this paper sufficient and also necessary conditions of strong pseudostability and strong quasistability of the vector trajectorial problem of lexicographic optimization with partial criteria of the kind Σ-MINMAX and Σ-MINMIN are obtained. These criteria include well-known in discrete optimization linear and bottleneck criteria. Lower bounds of radii of these two kinds of stability have been found for the case where Chebyshev norm was defined in the space of the vector criterion parameters.

2 Statement of the problem

Following [20], (E, T) is a system of subsets, where

$$E = \{e_1, e_2, \ldots, e_m\}$$

is a finite set of elements, $m > 1$, $T \subseteq 2^E \backslash \{\emptyset\}$, i.e. T is a family of non-empty subsets of the set E, which are called trajectories; $|T| > 1$.

On the set E, we define a vector weight function

$$a(e) = (a_1(e), a_2(e), \ldots, a_n(e)) \in \mathbf{R}^n, n \geq 1,$$

and, on the set T, a vector criterion

$$f(t) = (f_1(t), f_2(t), \ldots, f_n(t)).$$

The partial criteria of the vector criterion are functions of the following two kinds:

$$\Sigma\text{-MINMAX} \quad f_i(t) =$$

$$= \max\{\sum_{e \in q} a_i(e) : \ q \subseteq t, \ |q| = \min\{|t|, k_i\}\} \to \min_T, \quad (2.1)$$

$$\Sigma\text{-MINMIN} \quad f_i(t) =$$

$$= \min\{\sum_{e \in q} a_i(e) : \ q \subseteq t, \ |q| = \min\{|t|, k_i\}\} \to \min_T, \quad (2.2)$$

where $k_i, i \in N_n = \{1, 2, \ldots, n\}$, are given natural numbers such that

$$1 \leq k_i \leq p = \max\{|t| : \ t \in T\} \ \forall i \in N_n.$$

When $k_i = p, \ i \in N_n$, both criterion (2.1) and criterion (2.2) turn into the linear criterion

$$\text{MINSUM} \quad f_i(t) = \sum_{e \in t} a_i(e) \to \min_T . \quad (2.3)$$

When $k_i = 1,\ i \in N_n$, criterion (2.1) turns into the bottleneck criterion

$$\text{MINMAX} \quad f_i(t) = \max\{a_i(e) :\ e \in t\} \to \min_T$$

and criterion (2.2) turns into the criterion

$$\text{MINMIN} \quad f_i(t) = \min\{a_i(e) :\ e \in t\} \to \min_T .$$

Note that the problems with Σ-MINMAX and Σ-MINMIN criteria are related to necessities of optimal distribution [21].

By an n-criteria trajectorial problem, the problem of finding the lexicographic set is meant. The lexicographic set is a subset of the Pareto set and is defined as follows [22–26].

Suppose S_n is the set of all $n!$ permutations of the numbers $1, 2, \dots, n$. For any permutation $s = (s_1, s_2, \dots, s_n) \in S_n$, we introduce the binary relation of lexicographic order in criterion space $\mathbf{R}^n$:

$$\mathbf{x} \leq_s \mathbf{x}',$$

where $\mathbf{x} = (x_1, x_2, \dots, x_n)$ and $\mathbf{x}' = (x'_1, x'_2, \dots, x'_n)$, iff one of the following conditions holds:

1) $\mathbf{x} = \mathbf{x}'$;

2) $\exists j \in N_n\ \forall k \in N_{j-1}\ (x_{s_j} < x'_{s_j}\ \&\ x_{s_k} = x'_{s_k})$.

If $k = 1$, then the last equalities are absent ($N_o = \emptyset$).

The set $L^n = \bigcup_{s \in S_n} L^n(s)$, where $L^n(s) = \{t \in T : f(t) \leq_s f(t')\ \forall t' \in T\}$, is called a lexicographic set and its elements are called lexicographic optimal trajectories.

The vector weight function $a(e)$ can be represented as the matrix $A = \{a_{ij}\}_{n \times m}$, where $a_{ij} = a_i(e_j)$. Let I_1 and I_2 be the sets of those numbers from N_n, which enumerate criteria (2.1) and (2.2) respectively ($I_1 \bigcup I_2 = N_n$). If the numbers $k_1, k_2, \dots, k_n$ and the sets $E,\ T,\ I_1\ , I_2$ are fixed, then the individual n-criteria trajectorial problem of lexicographic optimization is uniquely determined by the matrix A. Therefore we denote the problem by $Z^n(A)$, the lexicographic set by $L_n(A)$, the vector criterion $f(t)$ by $f(t, A)$ and its partial criteria $f_i(t)$ by $f_i(t, A)$.

It is evident that the lexicographic optimization problem turns into the problem of finding the optimal solutions set if $n = 1$. All the combinatorial problems (in particular, all the problems on graphs) can be defined by the scheme of the single-criterion trajectorial problem.

As usual [12–20], we will perturb the matrix $A \in \mathbf{R}^{nm}$ by adding to A matrices from the set

$$\Re(\varepsilon) = \{B \in \mathbf{R}^{nm} : \|B\| < \varepsilon\},$$

where $\varepsilon > 0, \|.\|$ is the norm l_∞ (Chebyshev norm) in $\mathbf{R}^{nm}$, i.e.

$$\|B\| = \max\{|b_{ij}| : (i, j) \in N_n \times N_m\}, \; B = \{b_{ij}\}_{n \times m}.$$

Let $A, B \in \mathbf{R}^{nm}$. If we add a matrix B to the matrix A of the problem $Z^n(A)$, we get a perturbed problem $Z^n(A + B)$. The matrix B is called perturbing here.

Following [20], we say that the problem $Z^n(A)$ is

– pseudostable if

$$\exists \; \varepsilon > 0 \; \forall B \in \Re(\varepsilon) \; L^n(A) \supseteq L^n(A + B);$$

– quasistable if

$$\exists \; \varepsilon > 0 \; \forall B \in \Re(\varepsilon) \; L^n(A) \subseteq L^n(A + B);$$

– stable if

$$\exists \; \varepsilon > 0 \; \forall B \in \Re(\varepsilon) \; L^n(A) = L^n(A + B).$$

Thus, the value

$$\rho_i^n(A) = \begin{cases} \sup \Omega_i(A) & \text{if } \Omega_i(A) \neq \emptyset, \\ 0 & \text{if } \Omega_i(A) = \emptyset, \end{cases}$$

where

$$\Omega_1(A) = \{\varepsilon > 0 : \; L^n(A) \supseteq L^n(A + B) \; \forall B \in \Re(\varepsilon)\},$$

$$\Omega_2(A) = \{\varepsilon > 0 : \ L^n(A) \subseteq L^n(A+B) \ \forall B \in \Re(\varepsilon)\},$$

$$\Omega_3(A) = \{\varepsilon > 0 : \ L^n(A) = L^n(A+B) \ \forall B \in \Re(\varepsilon)\},$$

is

– the pseudostability radius of the problem $Z^n(A), \ n \geq 1$, where $i = 1$,

– the quasistability radius of the problem $Z^n(A), \ n \geq 1$, where $i = 2$,

– the stability radius of the problem $Z^n(A), \ n \geq 1$, where $i = 3$.

Let I_{sum} be the set of those numbers from N_n that enumerate partial criteria MINSUM (2.3) of the vector criterion $f(t, A)$.

3 Strong pseudostability

As it was pointed out, the problem $Z^n(A)$ is said to be strongly pseudostable if

$$\exists \varepsilon > 0 \ \forall B \in \Re(\varepsilon) \ L^n(A) \cap L^n(A+B) \neq \emptyset.$$

Thus, the value

$$\rho_4^n(A) = \begin{cases} \sup \Omega_4(A) & \text{if } \Omega_4(A) \neq \emptyset, \\ 0 & \text{if } \Omega_4(A) = \emptyset, \end{cases}$$

where

$$\Omega_4(A) = \{\varepsilon > 0 : \forall B \in \Re(\varepsilon) \ L^n(A) \cap L^n(A+B) \neq \emptyset\},$$

is said to be the strong pseudostability radius of the problem $Z^n(A), \ n \geq 1$.

It is obvious that $\rho_4^n(A) \geq \rho_1^n(A) \ \forall n \geq 1, \ A \in \mathbf{R}^{nm}$.

In order to find a lower bound of the strong pseudostability radius, we will formulate some evident properties and prove a lemma.

Further, we will use the notion

$$\tau_i(t, t', A) = f_i(t, A) - f_i(t', A).$$

The next two properties follow directly from the definition of strong pseudostability radius. They are true if the vector criterion of the problem consists of any partial criteria (not only of (2.1) and (2.2) kinds).

Property 3.1. *Let $T = T_1 \cup T_2, T_1 \cap T_2 = \emptyset$. If there exists an index $i \in N_n$ such that for any trajectory $t \in T_1$ there is a trajectory $t' \in T_2$, for which*

$$\tau_i(t, t', A) > 0,$$

then

$$T_2 \cap L^n(A) \neq \emptyset.$$

Property 3.2. *Let $T = T_1 \cup T_2, T_1 \cap T_2 = \emptyset$. If for any index $i \in N_n$ there exists a trajectory $t \in T_1$ such that*

$$\tau_i(t, t', A) < 0 \ \forall t' \in T_2,$$

then

$$T_2 \cap L^n(A) = \emptyset.$$

Obviously, if $T = L^n(A)$, then the strong pseudostability radius $\rho_4^n(A)$ is equal to infinity. The problem $Z^n(A)$ is called nontrivial if

$$\bar{L}^n(A) = T \backslash L^n(A) \neq \emptyset.$$

The next two properties are true for such problems.

Property 3.3. *Let $\varphi > 0$. If*

$$L^n(A) \cap L^n(A + B) \neq \emptyset \ \forall B \in \Re(\varphi),$$

then

$$\rho_4^n(A) \geq \varphi.$$

Property 3.4. *Let $\varphi \geq 0$. If, for any number $\varepsilon > \varphi$, there exists a perturbing matrix $B \in \Re(\varepsilon)$ such that*

$$L^n(A) \cap L^n(A + B) = \emptyset,$$

then

$$\rho_4^n(A) \le \varphi.$$

Let us introduce the following notation for any two different trajectories t, t':

$$\Delta_i(t,t') = \begin{cases} |t| + |t'| - 2|t \bigcap t'| & \text{if } i \in I_{sum}, \\ \min\{|t|,\ k_i\} + \min\{|t'|,\ k_i\} & \text{if } i \notin I_{sum}. \end{cases}$$

It is obvious that the inequality $\Delta_i(t,t') > 0$ holds for any index $i \in N_n$ and any trajectories $t \neq t'$.

Further, for any subset $t \subseteq E$, let $N(t)$ be the set of indexes $j \in N_m$ such that $e_j \in t$:

$$N(t) = \{j \in N_m :\ e_j \in t\}.$$

Lemma 3.1 [20]. *Let* $t, t' \in T,\ t \neq t',\ i \in N_n,\ \varepsilon > 0.$ *If*

$$\tau_i(t,t',A) \ge \varepsilon \Delta_i(t,t'),$$

then

$$\tau_i(t,t',A+B) > 0\ \forall B \in \Re(\varepsilon).$$

By definition, put

$$\varphi^n(A) = \max_{i \in N_n} \min_{t \in \bar{L}^n(A)} \max_{t' \in L^n(A)} \frac{\tau_i(t,t',A)}{\Delta_i(t,t')}.$$

It is obvious that $\varphi^n(A) \ge 0$.

Theorem 3.1. *Let* $A \in \mathbf{R}^{nm}$. *For any combination of partial criteria (2.1) and (2.2) of the nontrivial trajectorial problem* $Z^n(A),\ n \ge 1$, *we have*

$$\rho_4^n(A) \ge \varphi^n(A), \tag{3.1}$$

moreover

$$\rho_4^n(A) = \varphi^n(A) \tag{3.2}$$

if $I_{sum} = N_n$.

Proof. Inequality (3.1) is evident if $\varphi^n(A) = 0$.

Let $\varphi = \varphi^n(A) > 0$. Then $\Re(\varphi) \neq \emptyset$ and, by definition of the number φ, there exists an index $i \in N_n$ such that for any trajectory $t \in \bar{L}^n(A)$ there is a trajectory $t' \in L^n(A)$, for which

$$\tau_i(t, t', A) \geq \varphi \Delta_i(t, t').$$

Therefore, by lemma 3.1, we obtain

$$\tau_i(t, t', A + B) > 0 \ \forall B \in \Re(\varphi).$$

Hence, using property 3.1, we get

$$L^n(A) \cap L^n(A + B) \neq \emptyset \ \forall B \in \Re(\varphi).$$

Consequently, taking into account property 3.3, we have (3.1).

In order to prove equality (3.2) we will show that $\rho_4^n(A) \leq \varphi$ if $I_{sum} = N_n$.

By definition of the number φ, for any index $i \in N_n$ there exists a trajectory $t \in \bar{L}^n(A)$ such that

$$\tau_i(t, t', A) \leq \varphi = \Delta_i(t, t') \ \forall t' \in L^n(A). \tag{3.3}$$

Hence, if for every number $\varepsilon > \varphi$ we take the perturbing matrix $B \in \Re(\varepsilon)$ with the elements

$$b_{ij} = \begin{cases} -b & \text{if } i \in N_n, \ j \in N(t), \\ b & \text{if } i \in N^n, \ j \notin N(t), \end{cases}$$

where $\varphi < b < \varepsilon$, and use (3.3) we get

$$\tau_i(t, t', A + B) = \tau_i(t, t', A) - b\Delta_i(t, t') <$$

$$< \tau_i(t, t', A) - \varphi \Delta_i(t, t') \leq 0 \quad \forall t' \in L^n(A).$$

Thus, by property 3.2 we obtain

$$L^n(A) \cap L^n(A + B) = \emptyset.$$

Consequently, taking into account property 3.4, we have

$$\rho_4^n(A) \leq \varphi.$$

This inequality combined with inequality (3.1) proves the theorem.

In [18] the next formula for the pseudostability radius of the nontrivial linear ($I_{sum} = N_n$) problem $Z^n(A)$ was obtained:

$$\rho_1^n(A) = \min_{i\in N_n} \min_{t\in \bar{L}^n(A)} \max_{t'\in T\setminus\{t\}} \frac{\tau_i(t,t',A)}{\Delta_i(t,t')}. \qquad (3.4)$$

The next well-known result follows from this formula and theorem 3.1.

Corollary 3.1 [9]. *For any linear single-criterion trajectorial problem $Z^1(A)$ the next equality is true*

$$\rho_4^1(A) = \rho_1^1(A).$$

We say that the lexicographic set $L^n(A)$ of the nontrivial problem $Z^n(A)$ is strong if

$$\exists i \in N_n \ \forall t \in \bar{L}^n(A) \ \exists t' \in L^n(A) \ (\tau_i(t,t',A+B) > 0).$$

Corollary 3.2. *In order that the nontrivial trajectorial problem $Z^n(A), n \geq 1$, with any combination of partial criteria (2.1) and (2.2) be strongly pseudostable it is sufficient, and also necessary in the case $I_{sum} = N_n$, for lexicographic set to be strong.*

This corollary implies the following well-known result [9]: *any linear problem $Z^1(A)$ is strongly stable.*

The next example shows that the sufficient condition of corollary 3.2 is not a necessary condition and the radius of strong pseudostability can be greater than $\varphi^n(A)$ if $I_{sum} \neq N_n$.

Example 3.1.

Suppose $n = 2, m = 4, A = \begin{pmatrix} -1 & 0 & -2 & 1 \\ -1 & 1 & 0 & 0 \end{pmatrix}$,

$$T = \{t_1, t_2, t_3\}, \ t_1 = \{e_1, e_2\}, \ t_2 = \{e_2, e_3\}, \ t_3 = \{e_3, e_4\},$$

$$f_1(t, A) = \max\{a_{1j} : \ j \in N(t)\} \to \min_T,$$

$$f_2(t, A) = \sum_{j \in N(t)} a_{2j} \to \min_{T}.$$

Then $L^2(A) = \{t_1\}$.

As for any matrix $B \in \Re(1/2)$ the expressions

$$\tau_1(t_1, t_2, A + B) = 0,\ \tau_2(t_1, t_2, A + B) < 0,\ \tau_1(t_1, t_3, A + B) < 0$$

are true, t_1 is a lexicographic optimum of a perturbed problem $Z^2(A+B)\ \forall B \in \Re(1/2)$. Consequently, the problem $Z^2(A)$ is strongly pseudostable and $\rho_4^2(A) > 0$.

On the other hand, we have

$$\tau_1(t_1, t_2, A) = 0, \tau_2(t_1, t_3, A) = 0.$$

Hence, the set $L^2(A)$ is not strong and $\varphi^2(A) = 0$, i.e. $\rho_4^2(A) > \varphi^2(A)$.

4 Strong quasistability

As it was pointed out, the problem $Z^n(A)$ is said to be strongly quasistable if

$$\exists \varepsilon > 0 \quad \exists t \in L^n(A) \quad \forall B \in \Re(\varepsilon) \quad (t \in L^n(A + B)).$$

Thus, the value

$$\rho_5^n(A) = \begin{cases} \sup \Omega_5(A) & \text{if } \Omega_5(A) \neq \emptyset, \\ 0 & \text{if } \Omega_5(A) = \emptyset, \end{cases},$$

where

$$\Omega_5(A) = \{\varepsilon > 0 :\ \exists t \in L^n(A)\ \forall B \in \Re(\varepsilon)\ (t \in L^n(A + B))\},$$

is said to be the strong quasistability radius of the problem $Z^n(A),\ n \geq 1$.

It is obvious that $\rho_5^n(A) \geq \rho_2^n(A)\ \forall n \geq 1,\ A \in \mathbf{R}^{nm}$.

In order to find a lower bound of the strong quasistability radius, we will formulate some evident properties.

Property 4.1. *A trajectory t is a lexicographic optimal trajectory of the problem $Z^n(A)$ if there exists an index $i \in N_n$ such that*

$$\tau_i(t, t', A) < 0 \ \forall t' \in T \backslash \{t\}.$$

Property 4.2. *A trajectory t is not a lexicographic optimal trajectory of the problem $Z^n(A)$ if for any index $i \in N_n$ there exists a trajectory $t' \neq t$ such that*

$$\tau_i(t, t', A) > 0.$$

The next properties follow directly from the definition of the strong quasistability radius.

Property 4.3. *Let $\psi > 0$. If*

$$\exists t \in L^n(A) \quad \forall B \in \Re(\psi) \quad (t \in L^n(A+B)),$$

then

$$\rho_5^n(A) \geq \psi.$$

Property 4.4. *Let $\psi \geq 0$. If for any number $\varepsilon > \psi$ and any trajectory $t \in L^n(A)$ there exists a perturbing matrix $B \in \Re(\varepsilon)$ such that*

$$t \in \bar{L}^n(A+B),$$

then

$$\rho_5^n(A) \leq \psi.$$

By definition, put

$$\psi^n(A) = \max_{t \in L^n(A)} \max_{i \in N_n} \min_{t' \in T \backslash \{t\}} \gamma_i(t, t', A),$$

where

$$\gamma_i(t, t', A) = -\frac{\tau_i(t, t', A)}{\Delta_i(t, t')}. \tag{4.1}$$

It is obvious that $\psi^n(A) \geq 0$.

Theorem 4.1. *Let $A \in \mathbf{R}^{nm}$. For any combination of partial criteria (2.1) and (2.2) of the trajectorial problem $Z^n(A), n \geq 1$, we have*

$$\rho_5^n(A) \geq \psi^n(A), \tag{4.2}$$

moreover

$$\rho_5^n(A) = \psi^n(A) \tag{4.3}$$

if $I_{sum} = N_n$.

Proof. First we prove inequality (4.2). If $\psi^n(A) = 0$, then (4.2) is evident. Let $\psi = \psi^n(A) > 0$. Then $\Re(\psi) \neq \emptyset$ and by definition of the number ψ there exist a trajectory $t \in L^n(A)$ and an index $k \in N_n$ such that for any trajectory $t' \neq t$ we have

$$\psi \leq \gamma_k(t, t', A),$$

i.e.

$$\tau_k(t', t, A) \geq \psi \Delta_k(t, t').$$

Therefore, by lemma 3.1, we obtain

$$\tau_k(t, t', A + B) < 0 \ \forall B \in \Re(\psi) \ \forall t' \in T \backslash \{t\}.$$

Hence, using property 4.1, we get

$$t \in L^n(A + B) \ \forall B \in \Re(\psi).$$

Consequently, taking into account property 4.3, we have (4.2).

Now we prove that

$$\rho_5^n(A) \leq \psi \text{ if } I_{sum} = N_n.$$

By definition of the number ψ for any trajectory $t \in L^n(A)$ and for any index $i \in N_n$ there exists a trajectory $t' \neq t$ such that

$$\gamma_i(t, t', A) \leq \psi.$$

Hence, getting $\psi < \alpha < \varepsilon$ and taking the perturbing matrix $B \in \Re(\varepsilon)$ with elements

$$b_{ij} = \begin{cases} \alpha, & \text{where } i \in N_n, \ j \in N(t), \\ -\alpha, & \text{where } i \in N_n, \ j \notin N(t), \end{cases}$$

by virtue of linearity of the function $\tau_i(t, t', A)$ we have

$$\tau_i(t, t', A + B) = \tau_i(t, t', A) + \alpha\Delta_i(t, t') >$$

$$> \tau_i(t, t', A) + \gamma_i(t, t', A)\Delta_i(t, t') = 0$$

$$\forall i \in N_n.$$

Thus by property 4.2 the trajectory t is not a lexicographic optimum of the perturbed problem $Z^n(A+B)$. Consequently, taking into account property 4.4, we obtain

$$\rho_5^n(A) \leq \psi.$$

Combining this with (4.2) we get theorem 4.1.

In [20] the statement of theorem 4.1 was formulated and proved in terms of the stability kernel radius of the problem $Z^n(A)$.

Let us introduce the set of all strict lexicographic optimal trajectories of the trajectorial problem $Z^n(A)$. By definition, put

$$S^n(A) = \{t \in L^n(A) : \ \exists i \in N_n \ \forall t' \in T, \ t' \neq t \ (\tau_i(t, t', A) < 0)\}.$$

The next corollary follows from theorem 4.1.

Corollary 4.1. *In order that the trajectorial problem $Z^n(A), n \geq 1$, be strongly quasistable it is sufficient, and also necessary in the case $I_{sum} = N_n$, to have*

$$S^n(A) \neq \emptyset.$$

In [18] the next formula for the quasistability radius of the linear ($I_{sum} = N_n$) problem $Z^n(A)$ was obtained:

$$\rho_2^n(A) = \min_{t \in L^n(A)} \ \max_{i \in N_n} \ \min_{t' \in T \setminus \{t\}} \gamma_i(t, t', A),$$

where $\gamma_i(t, t', A)$ is calculated according to formula (4.1).

By the above, formula (3.4) and corollary 4.1 we get the next well-known result.

Corollary 4.2. [9] *Linear single-criterion problem $Z^1(A)$ is strongly quasistable iff it has a single optimal trajectory. In this case*

$$\rho_1^1(A) = \rho_2^1(A) = \rho_3^1(A) = \rho_4^1(A) = \rho_5^1(A).$$

The next example shows that the condition $S^n(A) \neq \emptyset$ is not a necessary condition of the strong quasistability and the strong quasistability radius can be greater than the value $\psi^n(A)$ if $I_{sum} \neq N_n$.

Example 4.1.

Suppose $n = 2, m = 4, A = \begin{pmatrix} 0 & 2 & 0 & 3 \\ 0 & 0 & 2 & 0 \end{pmatrix}$,

$$T = \{t_1, t_2, t_3\}, \quad t_1 = \{e_1, e_2\}, \quad t_2 = \{e_2, e_3\}, \quad t_3 = \{e_2, e_4\},$$

$$f_1(t, A) = \max\{a_{1j} : \ j \in N(t)\} \rightarrow \min_T,$$

$$f_2(t, A) = \sum_{j \in N(t)} a_{2j} \rightarrow \min_T.$$

Then $L^2(A) = \{t_1\}$.

As for any matrix $B \in \Re(1/2)$ the expressions

$$\tau_1(t_1, t_2, A + B) = 0, \ \tau_2(t_1, t_2, A + B) < 0, \ \tau_1(t_1, t_3, A + B) < 0$$

are true, $t_1 \in L^2(A + B) \ \forall B \in \Re(1/2)$. Consequently, the problem $Z^2(A)$ is strongly quasistable, i.e. $\rho_5^2(A) > 0$.

On the other hand, we have $\tau_1(t_1, t_2, A) = 0$ and $\tau_2(t_1, t_2, A) = 0$. Hence

$$S^n(A) = \emptyset, \ \psi^2(A) = 0,$$

i.e.

$$\rho_5^2(A) > \psi^2(A).$$

Remark 4.1. *By virtue of equivalence of all the norms in a finite-dimentsional space (see [27]) corollaries 3.2, 4.1 and the first part of corollary 4.2 are valid for any norm in the space $\mathbf{R}^{nm}$ of perturbing matrices.*

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