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Direct methods for solving singular integral equations with shifts in the unit circle

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Abstract

The computation schemes of spline-collocation methods for solving singular integral equations. A theoretical foundation of these two methods is obtained in space L_2.

In the present paper we give theoreticaly justification of the numerical schemes of spline-collocation method for solving the singular integral equations (SIE) of the following form

$$(A\varphi \equiv)\sum_{j=1}^{4}[a_j(t)(V_j\varphi)(t)+b_j(t)(SV_j\varphi)(t)]=f(t),$$
$$t\in\Gamma_0,\quad \Gamma_0=\{t,|t|=1\},\tag{1}$$

where $a_j(t),b_j(t)$ $(j=\overline{1,4})$ and known functions,$\varphi(t)$ is unknown function,

$$(V_1\varphi)(t)=\varphi(t),\quad (V_2\varphi)(t)=\varphi(\bar{t}),$$
$$(V_3\varphi)(t)=\varphi(-t),\quad (V_4\varphi)(t)=\varphi(-\bar{t});.$$

S is the operator of singular integration along the Γ_0.

At the same time as in [1] we consider the system of integral equations:

$$[D_1(t)I+D_2(t)\tilde{S}+D_3(t)K]\Phi(t)=F(t),\quad t\in\Gamma_0\tag{2}$$

where

$$D_1=\left\|\begin{array}{cccc} a_1 & a_2 & a_3 & a_4\\ a_2^{(1)} & a_1^{(1)} & a_4^{(1)} & a_3^{(1)}\\ a_3^{(2)} & a_4^{(2)} & a_1^{(2)} & a_2^{(2)}\\ a_4^{(3)} & a_3^{(3)} & a_2^{(3)} & a_1^{(3)}\end{array}\right\|$$

$$D_2 = \left\| \begin{matrix} b_1 & b_2 & b_3 & b_4 \\ -b_2^{(1)} & -b_1^{(1)} & -b_4^{(1)} & -b_3^{(1)} \\ b_3^{(2)} & b_4^{(2)} & b_1^{(2)} & b_2^{(2)} \\ -b_4^{(3)} & -b_3^{(3)} & b_2^{(3)} & -b_1^{(3)} \end{matrix} \right\|,$$

$$D_3 = \left\| \begin{matrix} 0 & 0 & 0 & 0 \\ b_2^{(1)} & b_1^{(1)} & b_4^{(1)} & b_3^{(1)} \\ 0 & 0 & 0 & 0 \\ b_4^{(3)} & b_3^{(3)} & b_2^{(3)} & b_1^{(3)} \end{matrix} \right\|, \quad K(\cdot) = \left(\frac{1}{\pi i} \int\limits_{\Gamma_0} \frac{(\cdot)}{\tau} d\tau \delta_{jk} \right)_{j,k=1}^{4}$$

$F(t) = (f(t), f^{(1)}(t), f^{(2)}(t), f^{(3)}(t))$, I is the identical operator and $\tilde{S}$ is the singular integration operator acting in the space of 4-dimensional vector-functions.

Here and further by the $g^{(1)}(t), g^{(2)}(t), g^{(3)}(t)$ we note the functions $g(\bar{t})$, $g(-t)$, $g(-\bar{t})$ correspondingly.

Theorem 1. *If the SIE (1) has the unique solution $\varphi(t)$ than the system of SIE (2) has the unique solution $\Phi(t)$ that is the vector-function of following form:*

$$\Phi(t) = (\varphi(t); \varphi^{(1)}(t); \varphi^{(2)}(t); \varphi^{(3)}(t)).$$

Inversely, if the system of SIE (2) is uniquely solvable and the vector-function $\Phi(t) = (\varphi_1(t),\ \varphi_2(t),\ \varphi_3(t),\ \varphi_4(t))$ is the solution of the system (2),then the SIE (1) is uniquely solvable too and the function

$$\varphi(t) = \frac{1}{4}[\varphi_1(t) + \varphi_2^{(1)}(t) + \varphi_3^{(2)}(t) + \varphi_4^{(3)}(t)]$$

is the solution of the SIE (1).

The justice of this theorem can be proved by direct verification.

The theorem 1 makes it possible to reduce the theoretical foundation of the numerical schemes for approximate solution of SIE (1) to the theoretical foundation of the corresponding schemes for system of SIE (2), that is the easier problem.

1 The deduction of the numerical scheme for SIE (1)

Let $\left\{\frac{j}{n}\right\}$ be the decomposition of the real axis $\mathbf{R}$ by the equidistant points $t_j = \frac{j}{n}$, $j = 0; \pm 1; \ldots, n-$ natural number.

Let us note by S_n^m the space of all 1-periodical smooth splines of an add degree $m = 2r - 1$ with defect 1 corresponding to this decomposition ([2]).

Thus, every function from S_n^m with its derivatives up to the $m-1$ order inclusive, is continuous and 1-periodical on $\mathbf{R}$, but its contraction on the interval $(\frac{j}{n}, \frac{j+1}{n})$ is polynomial of degree less or equal to m.

Let

$$\{\mathcal{X}_k(t)\}_{k=0}^{n-1} = \left\{\mathcal{X}_k(e^{2\pi \cdot i \cdot t})\right\}_{k=0}^{n-1}, \quad t \in [0, 1)$$

be an interpolation basis from S_n^m, satisfying the conditions

$$\mathcal{X}_k(t_j) = \delta_{jk}, \quad j, k = \overline{0, n-1}$$

where δ_{jk} is Kronecher delta.

We see an approximative solution of SIE (1) in the form of the following spline

$$\varphi_n(t) = \frac{1}{4}\sum_{s=1}^{4} \varphi_{sn}(t) \equiv \frac{1}{4}\sum_{s=1}^{4}\sum_{k=0}^{n-1} \alpha_{sk}^{(n)} \mathcal{X}_k(t), \quad t \in \Gamma_0 \tag{3}$$

with unknown complex numbers $\alpha_{sk}^{(n)} = \alpha_{sk}$, $s = \overline{1, 4}$, $k = \overline{0, n-1}$.

We specify the unknown splines $\varphi_{sn}(t)$, $s = \overline{1, 4}$ from the following equation of the spline collocation method

$$P_n(D_1 I + D_2 \tilde{S} + D_3 K)\Phi_n = 4P_n F \tag{4}$$

where P_n is interpolation projection operator on S_n^m,
$\Phi_n(t) = (\varphi_{1n}(t); \varphi_{2n}^{(1)}(t); \varphi_{3n}^{(2)}(t); \varphi_{4n}^{(3)}(t))$; – is unknown 4-dimensional vector -spline.

2 Deduction of the numerical scheme for system of SIE

Let us consider an arbitrary system of SIE

$$(MZ \equiv) C(t)Z(t) + D(t)(SZ)(t) + (RZ)(t) = Y(t), \quad t \in \Gamma_0 \qquad (5)$$

where $C(t) = (c_{ls}(t))_{l,s=1}^{q}$, $D(t) = (d_{ls}(t))_{l,s=1}^{q}$ are matrix functions of dimension $q \times q$, q is natural number, R is the integral operator, kernel $h(t,\tau)$ of which is $q \times q$ matrix-function determined on the torus $\Gamma_0 \times \Gamma_0$, $Y(t)$ is vector -function of dimension q, $Z(t)$ is unknown $q-$ dimensional vector-function.

We seek an approximate solution of system (5) as a $q-$ dimensional vector-function $s_n(t) \in [S_n^m]_q$

$$s_n(t) = \sum_{k=0}^{n-1} \gamma_k^{(n)} \mathcal{X}_k(e^{2\pi i\theta}), \quad \theta \in [0;1)$$

where $\gamma_k^{(n)} = \gamma_k = (\gamma_{k1}, \gamma_{k,2}, \ldots, \gamma_{k,q})$ $k = \overline{0, n-1}$ are unknown $q-$ dimensional complex numbers. We specify these numbers from the following conditions. We substitute the vector-function $z(t)$ in the left side of the (5) by $s_n(t)$ and then insist that the obtained expression coincides with the right part $Y(t)$ in the choosen points of decomposition. After calculation we obtain the following system of linear algebric equations S.L.A.E. for unknowns γ_k, $k = \overline{0, n-1}$:

$$C(t_j)\gamma_j + D(t_j) \sum_{k=0}^{n-1} \begin{pmatrix} (S\mathcal{X}_k)(t_j) & \ldots & 0 \\ \ldots & \ldots & \ldots \\ 0 & \ldots & (S\mathcal{X}_k)(t_j) \end{pmatrix} \gamma_k +$$

$$+ \sum_{k=0}^{n-1} (R\mathcal{X}_k)(t_j)\gamma_k = Y(t_j), \quad j = \overline{0, n-1} \qquad (6)$$

According to [3]

$$(S\mathcal{X}_k)(t_j) = \begin{cases} 1, & \text{when} \quad j = 0 \\ \sigma_m(t_j) = \dfrac{\sigma_m^-(t_j)}{\sigma_m^+(t_j)}, & j = \overline{1, n-1} \end{cases}$$

where

$$\sigma_m^{\pm}(s) = \sum_{k=0}^{\infty}(k+s)^{-m-1} \pm \sum_{k=1}^{\infty}(k-s)^{-m-1}$$

and

$$(R\mathcal{X}_k)(t_j) = \left(\frac{1}{2\pi i}\int\limits_{\Gamma_0} h_{ls}(t_j,\tau)\mathcal{X}_k(\tau)d\tau\right)_{l,s=1}^{q}$$

It is easy to check that the matrix of S.L.A.E. (6) represent system of $n \cdot q$ unknown γ_{ks}, $k = \overline{0, n-1}$; $s = \overline{1, q}$ with $n \cdot q$ equations. Matrix extended system contain the sum of the diagonal block and circulant block matrices [4], the circulant matrices defined with the help of unitary matrix of form U :

$$U = \sqrt{\frac{1}{n}}\left(e^{2\pi ij\cdot\frac{k}{n}}\right)_{j,k=0}^{n-1}$$

This circumstance has the special importance for creation the computer programs for the method.

3 Theoretical foundation of the algorithms

We consider the system of SIE (5) in the space $[L_2(\Gamma_0)]_q$ with the norm in $L_2(\Gamma_0)$

$$||\cdot|| = \left(\frac{1}{2\pi}\int\limits_{\Gamma_0} |\cdot|^2\right)^{\frac{1}{2}}$$

We assume the coefficients, the kernel and the right part of the equation are continuous functions of their arguments. Using the results of the work [2,5] determined the correctly of the following theorem.

Theorem 2. *Let the following conditions be fulfilled*

1) Operator M is invertible in $L_2(\Gamma_0)$,

2) $Det[C(t) \pm D(t)] \neq 0$, $t \in \Gamma_0$,

3) *The left and the right special index of matrix function* $[C(t) \pm D(t)]$ *all equal to zero.*

When the beginning with numbers $n \geq n_0$, *S.L.A.E. (6) has unique solution* $\gamma_k^{(n)}$, $k = \overline{0, n-1}$, *with every right part* $Y(t)$ *from* $[L_2(\Gamma_0)]_q$. *The approximative solution* $Z_n(t) = \sum \gamma_k^{(n)} \mathcal{X}_k(t)$. *converge with* $n \to \infty$ *with the norm* $[L_2(\Gamma_0)]_q$ *to the exact solution* $Z(t)$ *system S.I.E. (5).*

Applying theorem 2 to the system (2), we will have:

Theorem 3. *Let the following conditions be fulfilled.*

1) *Operator* A *is invertible in* $L_2(\Gamma_0)$,

2) $det[D_1(t) \pm D_2(t)] \neq 0$, $t \in \Gamma_0$,

3) *The left and right special index of the matrix function* $[D_1(t) \pm D_2(t)]$ *all equal to zero.*

When the beginning with numbers $n \geq n_1$, *equation (4) has unique solution, with every right part* $F(t)$ *from* $[L_2(\Gamma_0)]_4$.

The approximate solution $\varphi_n(t) = \sum\limits_{k=0}^{n-1} x_{1,k} \cdot \mathcal{X}_k(t)$ *converge with* $n \to \infty$ *with the norm* $L_2(\Gamma_0)$ *to the exactly solution* $\varphi(t)$ *S.I.E (1).*

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