Computer Science Journal of Moldova, vol.**6**, no.1(16), 1998

Direct methods for solving singular integral equations with complex conjugation

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Abstract

The computational schemes of collocation and mechanical quadrature methods for solving singular integral equations (S.I.E.) containing the complex conjugate unknown function are proposed in this paper. A theoretical foundation of these two methods is obtained in Hölder functions space H_β $(0 < \beta < 1)$ and in L_p, $1 < p < \infty$ spaces. A rather common case of a non-standard contour of integration is investigated.

Let examine the S.I.E. of following form

$$(R\varphi \equiv) \qquad c_1(t)\varphi(t) + \frac{d_1(t)}{\pi i}\int\limits_\Gamma \frac{\varphi(\tau)}{\tau - t}d\tau + c_2(t)\bar{\varphi}(t) +$$

$$+\frac{d_2(t)}{\pi i}\int\limits_\Gamma \frac{\bar{\varphi}(\tau)}{\tau - t}d\tau + \frac{1}{2\pi i}\int\limits_\Gamma h_1(t,\tau)\varphi(\tau)d\tau$$

$$+\frac{1}{2\pi i}\int\limits_\Gamma h_2(t,\tau)\bar{\varphi}(\tau)d\tau = f(t), \quad t \in \Gamma \tag{1}$$

where $c_k(t)$, $d_k(t)$, $h_k(t,\tau)$ $(k = 1,2)$ and $f(t)$ are known functions of their arguments, $\varphi(t)$ is an unknown function. The singular integarls are understood in the sense of Cauchy's main meaning, $\Gamma-$ an arbitrary smooth simple closed contour, containing inside it the point $z = 0$.

The theory of this kind of equations is developed rather enough (for example [1;2]). In these works Neter conditions and the formula for calculation the index of equation (1) are established, the number of linear independent solutions in homogeneous equation (1) and the number of conditions of solving nonhomogeneous equations (1) are determined.

As is indicated in mentioned works, the exact solution of equation (1) can be found only in some rare particular cases. And even in these cases, the formulae for finding the solution contain multiple singular integrals, the calculation of which is connected with significant difficulties. In some cases these integrals (although they exist) can not be found at all.

That is why the necessity of developing approximative methods of solving S.I.E. with theoretical foundation in consequence arises.

The calculating schemes of collocation and mechanical quadrature methods for approximative solving of S.I.E. are proposed below. The theoretical foundation of these methods will be obtained in of scale Hölder spaces and Lebesgue spaces.

1 The reduction of S.I.E. (1) to an equivalent system of S.I.E.

Accomplishing the operation of conjugation in equation (1) and taking into consideration that

$$\bar{i} = -i, \overline{(\bar{\varphi}(t))} = \varphi(t) \quad \text{and} \quad \overline{\frac{1}{\pi i}\int\limits_{\Gamma} \frac{\varphi(\tau)}{\tau - t}d\tau} = -\frac{1}{\pi i}\int\limits_{\Gamma} \frac{\bar{\varphi}(\tau)}{\tau - t}d\tau$$

we obtain the equation

$$\bar{c}_1(t)\bar{\varphi}(t) - \frac{\bar{d}_1(t)}{\pi i}\int\limits_{\Gamma} \frac{\bar{\varphi}(\tau)}{\tau - t}d\tau + \bar{c}_2(t)\varphi(t) - \frac{\bar{d}_2(t)}{\pi i}\int\limits_{\Gamma} \frac{\bar{\varphi}(\tau)}{\tau - t}d\tau -$$

$$-\frac{1}{2\pi i}\int\limits_{\Gamma} h_1(t,\tau)\bar{\varphi}(\tau)d\tau - \frac{1}{2\pi i}\int\limits_{\Gamma} \bar{h}_2(t,\tau)\varphi(\tau)d\tau = \bar{f}(t), \quad t \in \Gamma \qquad (2)$$

The totality of equations (1) and (2) can be written as the following system of S.I.E. with respect to the unknown functions $\varphi(t)$ and $\bar{\varphi}(t)$:

$$RX(t) \equiv C(t)X(t) + \frac{D(t)}{\pi i}\int\limits_{\Gamma} \frac{X(\tau)}{\tau - t} d\tau +$$
$$+\frac{1}{2\pi i}\int\limits_{\Gamma} r(t,\tau)X(\tau)d\tau = F(t), \quad t \in \Gamma \tag{3}$$

where $C(t)$, $D(t)$ and $r(t,\tau)$ are second order matrices of function and $F(t)$ and $X(t)$ are second order vectors of functions

$$C(t) = \begin{bmatrix} c_1(t) & c_2(t) \\ \bar{c}_2(t) & \bar{c}_1(t) \end{bmatrix} \quad D(t) = \begin{bmatrix} d_1(t) & -d_2(t) \\ \bar{d}_1(t) & -\bar{d}_2(t) \end{bmatrix}$$

$$r(t,\tau) = \begin{bmatrix} h_1(t,\tau) & h_2(t,\tau) \\ -\bar{h}_2(t,\tau) & -\bar{h}_1(t,\tau) \end{bmatrix} \quad F(t) = \begin{bmatrix} f(t) \\ \bar{f}(t) \end{bmatrix}$$

$$X(t) = \begin{bmatrix} \varphi(t) \\ \bar{\varphi}(t) \end{bmatrix}$$

It should be mentioned that according to [1;2] S.I.E. (1) and the system of S.I.E. are equivalent in the sence, that they are solvable or not at the same time; if $\varphi(t)$ is a solution for S.I.E. (1) then vector $(\varphi(t), \bar{\varphi}(t))$ is solution of system of S.I.E. (2). Vice versa, if vector $X(t) = (\varphi_1(t), \varphi_2(t))$ is a solution of the system of S.I.E. (2), then the function $\varphi(t) = \varphi_1(t)$ is the solution of S.I.E. (1).

According to [1], the system of S.I.E. (3) is a Neter one if

$$det[C(t) \pm D(t)] \neq 0, \quad t \in \Gamma.$$

It is easy to check that this condition is fulfilled, if

$$\Delta(t) = [c_1(t) + d_1(t)][\bar{c}_1(t) - \bar{d}_1(t)] -$$
$$-[c_2(t) - d_2(t)][\bar{c}_2(t) + \bar{d}_2(t)] \neq 0 \tag{4}$$

Further we shall consider that condition (4) is fulfilled and the functions $c_j(t)$, $d_j(t)$, $j = 1,2$, and $f(t)$ belong to $H_\alpha^{(r)}$, $r = 0,1,2,\dots$, $\alpha \in (0,1)$, and the functions $h_k(t,\tau)$, $k = 1,2$, belong to $C^{(r)}(\Gamma)$ by the variable τ uniformly with respect to t and belong to $H_\alpha^{(r)}$, $r \geq 0$, $\alpha \in (0,1)$ by the variable t uniformly with respect to τ.

Then, according to [1;2], if left-hand side particular indexes of the matrix of function

$$[C(t) + D(t)]^{-1}[C(t) - D(t)]$$

are all nonnegative, than the system S.I.E. (3) is solvable and its solution belongs to space $H_\alpha^{(r)}$, $r \geq 0$, $\alpha \in (0,1)$.

Future, approximate solutions of equation (1) will be obtained by an approximative solution of the system of S.I.E. (3).

2 The calculation scheme of collocation method

The approximative solution of system S.I.E. (3) will be found as a two-dimensional polynom

$$\Phi_n(t) = \sum_{k=-n}^{n} X_k^{(n)} t^k, \quad t \in \Gamma,$$

$$X_k^{(n)} = X_k = \begin{pmatrix} x_{1,k} \\ x_{2,k} \end{pmatrix}, \quad k = \overline{-n,n} \tag{5}$$

The unknown coefficients X_k, $k = \overline{-n,n}$ of which are determined from the system of linear algebraic equations (SLAE):

$$\sum_{k=-n}^{n} \{sign(k)[A_{jk} - B_{jk}] + D_{jk}\} X_k = F_j, \quad j = \overline{0,2n}. \tag{6}$$

Here $sign(k) = 1$, for $k \geq 0$, $sign(k) = -1$, for $k < 0$;

$$A_{jk} = [C(t_j) + D(t_j)]t_j^k, \quad B_{jk} = [C(t_j) - D(t_j)]t_j^k,$$

$$D_{jk} = \frac{1}{2\pi i} \int_{\Gamma} r(t_j, \tau)\tau^k d\tau, \quad F_j = F(t_j), k = -n, n, \quad j = 0, 2n$$

where $t_j (j = 0, 2n)$ are different nodes, situated on Γ.

Than the approximative solution $\varphi_n(t)$ of equation (1) will be obtained by the formula:

$$\varphi_n(t) = \sum_{k=-n}^{n} x_{1,k} t^k, t \in \Gamma, \tag{7}$$

where $x_{1,k}$ is the solution of SLAE (6).

3 The calculation scheme of mechanical quadratures method

The approximative solution of the system (3) is searched also in the form (5), but in this case the unknowns $X_k, k = \overline{-n, n}$ are obtained from the following SLAE:

$$\sum_{k=-n}^{n} \{sign(k)[A_{jk} - B_{jk}] + C_{jk}\} X_k = F_j, j = 0, 2n. \tag{8}$$

Here the numbers C_{jk} are obtained from numbers D_{jk} replacing the integral by a quadrature formula. Taking into consideration the specific character of numbers D_{jk}, we shall use as a quadrature formula the following one:

$$\frac{1}{2\pi i} \int_{\Gamma} g(\tau)\tau^k d\tau \cong \frac{1}{2\pi i} \int_{\Gamma} U_n(\tau \cdot g(\tau))\tau^{k-1} d\tau \tag{9}$$

where U_n is Lagrange interpolation operator, constructed on points $t_j, j = 0, 2n :$

$$(U_n g)(t) = \sum_{j=0}^{2n} g(t_j) l_j(t), \quad t \in \Gamma,$$

$$l_j(t) = \left(\frac{t_j}{t}\right)^n \prod_{k=0,k\neq j}^{2n} \frac{t - t_k}{t_j - t_k} \equiv \sum_{r=-n}^{n} \Lambda_r^{(j)} t^r, \quad t \in \Gamma,\ j = \overline{0, 2n}.$$

In this case, formula (9) turns into

$$\frac{1}{2\pi i} \int_\Gamma g(\tau)\tau^k d\tau \cong \sum_{s=0}^{2n} \Lambda_{-k}^{(s)} g(\tau^s)$$

and in consequence, numbers C_{jk} are calculated by formula

$$C_{jk} = \sum_{s=0}^{2n} \Lambda_{-k}^{(s)} r(t_j, t_s), \quad k = \overline{-n, n}; \quad j = \overline{0, 2n}.$$

4 The theoretical foundation of calculation schemes

Theorems that give a theoretical foundation of calculation schemes developed in point 2,3 will be formulated and proved below. The convergence of the methods will be established in scale of Hölder spaces and Lebesgue spaces $\mathbf{L_p}(\mathbf{1} < \mathbf{p} < \infty)$.

Theorem 1. *Let Γ - be an arbitrary simple closed smooth contour, containing inside the point $z = 0$. The functions $c_k(t)$, $d_k t)$, $k = 1, 2$ belong to $H_\alpha^{(r)}$, $r \geq 0$, $\alpha \in (0; \beta)$, $\beta < 1$, $\Delta(t) \neq 0$, $t \in \Gamma$, and all left - hand side particular indexes of matrix of function $[C(t)+D(t)]^{-1}[C(t)-D(t)]$ are equal to zero; functions $h_1(t, r), h_2(t, \tau) \in H_\alpha^{(r)}$ by the variable t uniformly with respect to τ and $h_1(t, r), h_2(t, \tau) \in C^{(r)}$ by the variable τ uniformly with respect to t.*

Besides, let $dimKerR = 0$. If the points $t_j (j = \overline{0, 2n})$ form a system of Fejer nodes on Γ [3], then, beginning with numbers $n \geq n_1$, for which

$$\frac{\gamma_1 + \gamma_2 \ln n}{n^{\alpha-\beta+r}} \leq q_1 < 1,$$

the SLAE [1] *(6) (collocation method) has a unique solution* $X_k = \begin{pmatrix} x_{1,k} \\ x_{2,k} \end{pmatrix}$, $(k = -n, n)$ *for any right-hand side* $f(t) \in H_\alpha^{(r)}$. *The approximative solutions* $\varphi_n(t)$, *obtained by the formula (7), converge as* $n \to \infty$ *by the* H_β *space norm to the exact solution* $\varphi(t)$ *of S.I.E. (1).*

The rate of convergence is estimated by the value

$$\|\varphi - \varphi_n\|_\beta \leq \frac{\gamma_3 + \gamma_4 \ln n}{n^{\alpha-\beta+r}}$$

Theorem 2. *Let all conditions of theorem 1 be fulfilled,* $h_1(t,\tau)$, $h_2(t,\tau) \in H_\alpha^{(r)}$ *being on both variables. Then, beginning with numbers* $n \geq n_2 (\geq n_1)$ *for which*

$$\frac{\gamma_5 + \gamma_6 \ln n + \gamma_7 \ln^2 n}{n^{\alpha-\beta+r}} \leq q_2 < 1,$$

the SLAE (8) of mechanical quadrature method has a unique solution $X_k = \begin{pmatrix} x_{1,k} \\ x_{2,k} \end{pmatrix}$, $(k = -n, n)$, *for any right part* $f(t) \in H_\alpha^{(r)}$. *The approximative solutions* $\varphi_n(t)$ *(7), converge as* $n \to \infty$ *by the* H_β *space norm, to the exact solution* $\varphi(t)$ *of (1).*

For the rate of convergence use the formula

$$\|\varphi - \varphi_n\|_\beta \leq \frac{\gamma_8 + \gamma_9 \ln n + \gamma_{10} \ln^2 n}{n^{\alpha-\beta+r}}$$

Theorem 3. *Let* Γ *- be a simple closed smooth contour from group* C_μ^2, $\mu \in (0;1)$ *[5;p. 23]. The functions* $c_k(t)$, $d_k(t)$, $k = 1,2$ *belong to* $H_\alpha^{(r)}, r \geq 0, \alpha \in (0;1), \Delta(t) \neq 0, t \in \Gamma$, *and all left-hand side particular indexes of matrix of functions* $[C(t) + D(t)]^{-1}[C(t) - D(t)]$ *are equal to zero. The functions* $h_k(t,\tau)$, $k = 1,2$ *belong to* $C^{(r)}$ *on both variables.*

[1] $\gamma_k, k = 1,2,$ - the constants, that don't depend on n.

Points $t_j (j = \overline{0, 2n})$ form the system of Fejer's nodes on Γ.
Than, beginning with numbers n, for which

$$\eta_{n,1} = \frac{\gamma_{11}}{n^{\alpha-\beta+r}} + \gamma_{12}\frac{1}{n^r}\omega^t(h; 1/n) \leq q_3 < 1, \tag{10}$$

the SLAE (6) of collocation method has a unique solution $X_k = \begin{pmatrix} x_{1,k} \\ x_{2,k} \end{pmatrix}$, $(k = \overline{-n, n})$ *for any right part $f(t)$ from $C^{(r)}$.*

The approximative solution $\varphi_n(t) = \sum\limits_{k=-n}^{n} x_{1,k}t^k$ *converges as $n \to \infty$ by the $L_p (1 < p < \infty)$ space norm to the exact solution $\varphi(t)$ of (1) with the rate*

$$\|\varphi - \varphi_n\|_{L_p} \leq \frac{\gamma_{13}}{n^{\alpha-\beta+r}} + \gamma_{14}\frac{1}{n^r}\omega^t(h; 1/n) = \eta_{n,2} \tag{11}$$

Theorem 4. *Let the conditions of theorem 3 be fulfilled. Than, all the statements of this theory, concerning th SLAE (8) of mechanical quadratures method (8) with replacement of value (10) by*

$$\eta_{n,1} + \gamma_{15}\frac{1}{n^r}\omega^\tau(h; 1/n) \leq q_4 < 1$$

and of value (11) by

$$\|\varphi - \varphi_n\|_\beta \leq \eta_{n,2} + \gamma_{16}\frac{1}{n^r}\omega^\tau(h; 1/n)$$

are justified.

Denote, that in the work [4] the theoretical foundation of collocation and of mechanical quadratures methods are obtained for the S.I.E. system for more restricted conditions on regulary nucleus $h_k(t, r), k = 1, 2$ and on condition that the points of discretisation $t_j, j = \overline{0, 2n}$, are the root's shape of $2n + 1$ orders from 1. The group of admitted nucleuses $h_k(t, \tau), k = 1, 2$ and the set of points on which the methods are based, are enlarged in this work.

The proofs of theorems 1-4 are obtained by schemes similary to [4].

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Received December 17, 1997

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