Computer Science Journal of Moldova, vol.6, no.1(16), 1998

Algorithms for finding the minimum cycle mean in the weighted directed graph

D. Lozovanu C. Petic

Abstract

In this paper we study the problem of finding the minimum cycle mean in the weighted directed graph. The computational aspect of some algorithms for solving this problem is discussed and two algorithms for minimum cycle mean finding in the weighted directed graph are proposed.

1 Introduction

The problem of finding the minimum cycle mean in the weighted directed graphs appears as an auxiliary problem of finding optimal stationary strategies for optimal control problem [1]. In this paper we propose a new polynomial time algorithm for finding the minimum cycle mean in the weighted directed graph. There will be also considered and analyzed a more general problem of finding the optimal cycle mean.

2 Problem formulation

Let $G = (V, E)$ be a directed graph, where V, $|V| = n$, is the set of vertices and E, $|E| = m$, is the set of edges. On the edges set E there is defined a function $c : E \to R$ from E into the real numbers, associating each edge $e \in E$ a weight $c(e)$. The function c is called the function of cost and $c(e)$ is called the cost of the edge e.

The problem of finding minimum cycle mean in a digraph was formulated by Richard M. Karp as follows [2]. Given any sequence of edges

$\sigma = \{e_1, e_2, \ldots, e_p\}$ we define $W(\sigma)$, the weight of σ, as $\sum_{i=1}^{p} c(e_i)$ and define $M(\sigma)$, the mean weight of σ, as $\frac{W(\sigma)}{p}\cdot$ Then we assume that $\lambda^* = \min_c M(\sigma)$, where c ranges over all directed cycles in G. λ^* is called the minimum cycle mean.

Richard M. Karp has given a simple characterization of the minimum cycle mean in the directed weighted graph as well as an algorithm for computing it efficiently.

We consider that G is strongly connected. If G is not strongly connected, then we can find the minimum cycle mean by determining the minimum cycle mean for each strong component of G, and then taking the least of these. The strong components can be found in $O(n+m)$ computational steps.

3 Main results

The problem of finding minimum cycle mean in the weighted directed graph can be reduced to a linear programming problem [1,3], which represents the continue model of the minimum cycle mean problem in the directed graph. The mathematical formulation of the problem considered in this paper can be given as:

$$\sum_{e\in E} c(e)z(e) \rightarrow \min$$

$$\begin{cases} \sum\limits_{e\in V^+(v)} z(e) - \sum\limits_{e\in V^-(v)} z(e) = 0, \quad \forall v \in V \\ \\ \sum\limits_{e\in E} z(e) = 1 \\ \\ z(e) \geq 0 \end{cases} \tag{1}$$

where $V^+(v)$ is the set of edges, which have their extremities in v, $V^-(v)$ is the set of edges originated in v, and $z(e)$ is a variable associated to each edge $e \in E$.

Solution z of the system (1) determines in G some circulation with constant (equal to 1) sum of flux values by edges of the directed weighted graph G. It is easy to show, that any solution of the system (1) can be represented in the form of convex combination of the fluxes of some elementary directed cycles with constant (equal to 1) sum of flux values by edges of these cycles. Thus, associating each solution z of the system (1) the graph $G_z = (V_z, E_z)$ induced by the edges $e \in E$ with $z(e) > 0$, we obtain [4] that any of the extreme points of the polyhedral set Z of solutions of the system (1) will correspond to the graph G_z in G, which have the structure of elementary directed cycle.

Lemma 1 *If $z \in R^m$ is the solution of the system (1) and corresponds to an extreme point of the set Z, then the graph G_z reprezents an elementary cycle.*

Proof. First of all we shall prove that there exist a cycle in G_z. If G_z contains only edges for which $z(e) > 0$, then in any vertex v from the vertex set V_z get in an edge $z(e_i) > 0$ and get out an edge $z(e_j) > 0$. If G is a finite graph, then crossing the vertices from V_z we shall come to the first vertex from which we have started. Thus, there exist a cycle in G_z.

Further, we shall prove that G_z is an elementary cycle. Let assume that G_z is not an elementary cycle. Then, according to the demonstration made above in G_z there exist a cycle. We shall choose an elementary cycle from G_z. Let assume that the edges of this cycle are $\{e_1, e_2, \ldots, e_{n_2}\}$.

Let $z = (z(e_1), z(e_2), \ldots, z(e_{n_1}), 0, \ldots, 0)$ be the solution of the system (1), associated to G_z. Then we can reprezent it in the following form: $z = (z(e_1), z(e_2), \ldots, z(e_{n_2}), z(e_{n_2+1}), \ldots, z(e_{n_1}), 0, \ldots, 0)$, where the first n_2 values of z are the values of the solution on the edges of the elementary cycle which we have chosen. Let us denote $\theta = \min\limits_{e_i \in C_0} z(e_i) > 0$, where C_0 is the elementary chosen cycle. Further, we shall consider the following two solutions:

$$z^1 = \frac{1}{1 - n_2\theta}(z(e_1) - \theta, \ldots, z(e_{n_2}) - \theta, z(e_{n_2+1}), \ldots, z(e_{n_1}), 0, \ldots, 0)$$

and

$$z^2 = \frac{1}{n_2\theta}(\underbrace{\theta, \theta, \dots, \theta}_{n_2}, 0, 0, \dots, 0), \qquad z^1, z^2 \in R^m.$$

We'll show that z^1 and z^2 are feasible solutions of the problem (1). It is easy to see that, considering the sum of first n_1 values of the solution z^1 we obtain: $\sum_{i=1}^{n_1} z(e_i) - n_2\theta = 1 - n_2\theta$, because $\sum_{i=1}^{n_1} z(e_i) = 1$. Applying the same procedure we obtain, that z^2 is also a feasible solution of the problem (1).

Solution z, associated to G_z, can be reprezented as the linear combination of two feasible solutions z^1 and z^2:

$$z = (1 - n_2\theta)z^1 + n_2\theta z^2, \quad \text{where } n_2\theta < 1.$$

In that case z is not an extreme point of the convex polyhedron Z, which is the set of all solutions of the problem (1). So then, G_z is an elementary cycle.

Theorem 1 *If z^* is the optimal solution of the problem (1), then the cycle, generated by the set E_{z^*}, is the minimum cycle mean in G, and vice versa if G_{z^*} represents the minimum cycle mean in the graph G, then Z^* is the optimal solution of the problem (1).*

Proof. Let assume that z^* is the optimal solution of the problem (1). Since z^* is the solution of the system (1), and corresponds to an extreme point of the polyhedral set Z, then the graph G_{z^*}, associated to z^*, represents an elementary cycle. Since z^* is the optimal solution of the problem (1), then G_{z^*} is the minimum elementary cycle of G, which is the minimum cycle mean in the weighted directed graph.

And vice versa, if G_{z^*} is the minimum cycle mean in G, then this graph is generated by the solution z^*, which corresponds to an extreme point of the set Z, and it represents the minimum solution of the problem (1).

The theorem is proved.

Theorem 2 *Let $G = (V, E)$ be a directed weighted graph. Then, there exist the function $\varepsilon : V \to R$ and the number p, such that*

$$\varepsilon(v) - \varepsilon(u) + p \leq c(e), \quad \forall e = (v, u)$$

and

$$\min_{u \in V^+(v)} \{\varepsilon(u) - \varepsilon(v) - p + c(e)\} = 0, \quad \forall e = (v, u),\ \forall v \in V,$$

where $p = \lambda^*$ is the length of optimal cycle.

Proof. For proving this theorem, we shall consider the dual problem of the problem (1), which can be written as:

$$\begin{cases} \max p \\ \varepsilon(v) - \varepsilon(u) + p \leq c(e), \quad \forall e = (v, u) \end{cases} \tag{2}$$

According to the theorem of duality there exist z^*, and ε, p are the optimal solutions of the problem (2) then and only then, when

$$z^*[\varepsilon(u) - \varepsilon(v) - p + c(v, u)] = 0, \quad \forall (v, u) \in E.$$

From this it follows, that for edges $e = (v, u)$ of minimum cycle mean

$$\min_{u \in V^+(v)} \{\varepsilon(u) - \varepsilon(v) - p + c(v, u)\} = 0,$$

where $(v, u) \in C^*$, C^* is the minimum cycle mean. Further, the numbers $\varepsilon(v)$ for $v \in V$, which do not belong to the optimal cycle will be found in the following way: $\varepsilon(v)$ will represent the least length of the path from the vertex v to the optimal cycle. Then it is clear, that the relation

$$\min_{u \in V^+(v)} \{\varepsilon(u) - \varepsilon(v) - p + c(v, u)\} = 0, \quad \forall v \in V$$

will take place. The theorem is proved.

4 Algorithms

Algorithm 1. (Find the minimum cycle mean of the weighted directed graph G)

Input: A weighted directed graph $G = (V, E)$ and the function of cost c, defined on the edge set E.

Output: The minimum cycle mean of the graph G.

Step 1. Form the linear programming problem (1).
Step 2. Determine the optimal solution z^* of the problem (1).
Step 3. Determine edges $e \in E$ for which $z^*(e) > 0$. These edges generate the minmum cycle mean in G.

Algorithm 2. (Find the minimum cycle mean of G by determining the cycle of negative weight)

Input: A weighted directed graph $G = (V, E)$ and the function of cost c, defined on the edge set E.

Output: The minimum cycle mean of G.

Step 1. Let $S^1 = 0$, $S^2 = \max\limits_{e \in E}\{c(e)\}$.

Step 2. If $S^2 - S^1 < \dfrac{1}{2^n \max c(e)}$, then $c(e) \to c(e) - S^1$, $\forall e \in E$ and determine the cycle of negative weight, which will be the minimum cycle mean of the graph G. If not, then go to **Step 3**.

Step 3. Let $h = \dfrac{S^2 - S^1}{2}$ and $p = S^1 + h$. Make the substitution: $c(e) \to c(e) - p$.
Step 4. If there exist a cycle of negative weight, then $S^1 = S^1 + p$ and go to **Step 2**. If not, then $S^2 = S^2 - p$ and go to **Step 2**.

The algorithm for determining the cycle of negative weight is given in [5].

5 Generalization

The problem formulated above can be generalized in the following way. It is given a graph G on the edge set of which there are defined two functions $c : E \to R$ and $d : E \to R$. Where c is the function of edges costs and d is a function, which associate every edge $e \in E$ some other number $d(e) \geq 0$. It is necessary to find such a cycle Φ, for which the aim function

$$z(\Phi) = \frac{\sum_{e\in E} c(e)}{\sum_{e\in E} d(e)}$$

is minimum. The problem of finding the cycle Φ in the graph G with double weights, for which the relation $z^*(\Phi)$ is minimum, can be solved applying the algorithm of determining the cycle of negative weight in the graph G.

This problem can be reduced to a linear programming problem in the following way:

$$\sum_{e\in E} c(e)z(e) \to \min$$

$$\begin{cases} \sum\limits_{e\in V^+(v)} z(e) - \sum\limits_{e\in V^-(v)} z(e) = 0, \quad \forall v \in V \\ \\ \sum\limits_{e\in E} d(e)z(e) = 1 \\ \\ z(e) \geq 0 \end{cases} \tag{3}$$

For this problem lemma 1 takes place. That means, that if $z \in R^m$ is the solution of the system (3) and corresponds to an extreme point of the set Z, then the graph $G_z = (V_z, E_z)$, associated to the solution z, represents an elementary cycle. And also, for problem (3) it is true, that if z^* is the optimal solution of this problem, then G_{z^*} represents the minimum cycle mean of the graph G.

References

[1] I.V.Romanovski. Optimization of stationary control of discrete deterministic process. Cybernetics 2, 1967.

[2] R.M.Karp. A characterization of the minimum cycle mean in a digraph. Discr. Math. 1978, Vol.23, No.3, p.309–311.

[3] D.D.Lozovanu. Extreme-combinatorial problems and algorithms of their solving, Chişinău, Ştiinţa, 1991.

[4] I.V.Romanovski. Algorithms of solving extreme problems. M.: Nauka, 1977. (Russian)

[5] N.Cristofides. Graph theory. An agorithmic approach, Academic Press, New York, London, San Francisco, 1975.

D.Lozovanu, C.Petic,
Institute of mathematics,
Academy of Sciences of Moldova,
5 Academiei str., Kishinev,
MD-2028, Moldova

Received December 19, 1997