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On graphs G of diameter two with $f(G) \leq |V(G)| + \Delta - \delta + 1$

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Abstract

It is known that for any graph G there exists a graph H whose median is isomorphic to G: $MedH \cong G$. For any graph G, let $f(G)$ denote the minimal number of vertices of a connected graph H satisfying $MedH \cong G$. It is known that if G of diameter two has n vertices and minimal (maximal) degree $\delta(\Delta)$ then $f(G) \geq n + \Delta - \delta$. We constructed a wide class of graphs G of diameter two for which $f(G) \leq n + \Delta - \delta + 1$.

1 Introduction

Let G be a simple connected graph with vertex set $V(G) = \{v_1, v_2, \dots, v_n\}$ and edge set $E(G)$, $m = |E(G)|$. For $v \in V(G)$, $deg_G v$ denotes the degree of v, $\delta = \delta(G) = \min\limits_i deg_G v_i$, $\Delta = \Delta(G) = \max\limits_i deg_G v_i$. If $u, v \in V(G)$, then $d_G(u,v)$ denotes the smallest number of edges in a path from u to v in G and is called the distance between u and v. For $v \in V(G)$ let $N_k(v) = \{u \in V(G) : d_G(u,v) = k\}$.

The distance of v in G is defined by $d_G(v) = \sum\limits_{u \in V(G)} d_G(v,u)$. A vertex of minimal distance is a median vertex of G, and the median $MedG$ of G is the subgraph of G, induced by its median velrtices.

Slater proved [1] that for any graph G there exists a graph H whose median is isomorphic to G: $MedH \cong G$. For any graph G, let $f(G)$ denotes the minimal number of vertices of a connected graph H satisfying $MedH \cong G$.

Miller showed [4] that $f(G) \leq 2|V(G)|$ if $\delta(G) \geq 1$ and Hendry showed [2] that $f(G) \leq 2|V(G)| - \delta(G) + 1$ for any G and gives a slightly better bound for certain G. Also Hendry [3] proved that

$$f(G) \geq |V(G)| + \max_{(u,v)\in E(G)} |deg_G u - deg_G v|$$

for graph G with diameter equal to two.

It is clear that

$$\max_{(u,v)\in E(G)} |deg_G u - deg_G v| \leq \Delta - \delta.$$

Consequently, $f(G) \geq |V(G)| + \Delta - \delta$.

In this paper we constructed a wide class of graphs G with diameter two for which

$$f(G) \leq |V(G)| + \Delta - \delta + 1.$$

In such a way, for these graphs G we obtain

$$|V(G)| + \Delta - \delta \leq f(G) \leq |V(G)| + \Delta - \delta + 1.$$

2 Construction of graph H

Let G be a graph with diameter two and $V(G) = \{v_1, v_2, \ldots, v_n\}$. Let H be the graph obtained from G as follows:

$$V(H) = V(G) \bigcup \{x_1, x_2, \ldots, x_{\Delta-\delta+1}\} \text{ and } \quad E(H) = E(G) \bigcup E^*.$$

E^* will consist of new edges, drawn between some of the vertices v_i and the new vertices x_j.

The algorithm that will construct the edges from E^* at every step connects a vertex v_i (or some of them) with exactly $\Delta + 1 - deg_G v_i$ new vertices x_j. While drawing edges (v_i, x_j) two priorities must be taken into account , the first priority being superior to the second one:

1. The vertex v_i, will be connected with such vertices x_j, that at the given moment satisfy the condition $d(v_i, x_j) > 2$.

2. The vertex v_i, will be connected with such vertices x_j, whose degree at the given moment is minimal.

The algorithm consists of the following steps.

Step 0. Vertex v_1 is connected with $\Delta + 1 - deg_G v_1$ arbitrary vertices x_j.

Step 1. It is taking one after another a vertex $v' \in N_1(v_1)$ and it is connected with $\Delta+1-deg_G v'$ vertices x_j (while drawing each edge the two priorities must be taken into account, as we mentioned above).

Ending vertices $v' \in N_1(v_1)$, we'll pass to the next step.

Step $k (k \geq 2)$**.** It is taking one after another one vertex $v^{(k)} \in N_k(v_1)$ and is connected with $\Delta + 1 - deg_G v^{(k)}$ vertices x_j (with the help of $\Delta + 1 - deg_G v^{(k)}$ new edges).

Ending vertices $v^{(k)}$ from $N_k(v_1)$ we'll pass to step $k + 1$.

Remark. *The number of steps is $e(v_1) + 1$, where $e(v_1)$ is the eccentricity of v_1 in G.*

Theorem 1 *The graph H, constructed by the above algorithm, has the following properties:*

1° $deg_H v_i = \Delta + 1, \ i = \overline{1, n}$;

2° $d_H(v_i, x_j) \leq 2, \ i = \overline{1, n}, \ j = \overline{1, \Delta - \delta + 1}$;

3° $|deg_H x_i - deg_H x_j| \leq 1$, $i = \overline{1, n}, \ j = \overline{1, \Delta - \delta + 1}$.

Proof. We'll take one after another a vertex $v \in V(G)$ for checking up the properties $1^\circ, 1^\circ, 3^\circ$. We'll suppose that the vertices will be taken in the same order, in which they were taken in the algorithm above. It is necessary to mention that the priority 1 is ensured by the algorithm (by it's definition).

Let's consider for the first the vertex v_1. This vertex is directly connected (at the distance 1) or through the intermediary of its neighbours (their number being equal to $deg_G v_1$) at distance 2 with all vertices x_j by s edges,

$$s = \Delta + 1 - deg_G v_1 + \sum_{v' \in N_1(v_1)} (\Delta + 1 - deg_G v')\,. \qquad (1)$$

It easy to verify that $s > \Delta + 1 - \delta$. Indeed, (1) implies

$$s - (\Delta + 1 - \delta) = \delta - deg_G v_1 + \sum_{v' \in N_1(v_1)} (\Delta - deg_G v') + deg_G v_1 > 0,$$

that is $s \geq \Delta + 1 - \delta$. Because the edges are drawn according to priorities 1 and 2, this inequality ensures that the condition 2° will be fulfiled for the vertex v_1.

It is obvios that respecting of the priorities 1 and 2 we will ensure the fulfilment of the condition 1° and 3°.

Now, we'll consider the vertex $v' \in N_1(v_1)$ (as we agreed v' will be the first of the vertices taken at the first step) and for it we'll prove the condition 2°: $d(v', x_j) \leq 2$ for any j.

Lemma. *If*

$$N_1(v') \subseteq N_1(v_1) \bigcup \{v_1\}\ , \qquad (2)$$

then for v' the condition 2° is fulfilled.

Proof. Let's suppose that there exists one vertex x_j for which $d(v', x_j) > 2$. Then due to the first priority, the vertex v' isn't directly connected (at the distance 1) with any vertex x_i, with which v_1 is directly connected. Indeed, at the moment when we had to connect v' directly with $\Delta + 1 - deg_G v'$ vertices, it was

$$d(v', x_i) = d(v', v_1) + d(v_1, x_i) = 2$$

and, thus, in accordance to the priority 1, v' had to be connected with vertices, situated at a distance more than 2 (there were such vertices, for example, x_j). The vertex v' isn't directly connected with vertices x_i either with which the vertices $v \in N_1(v_1) \backslash N_1(v')$ are directly connected. Indeed, at the moment when we had to connect each of these vertices v with respective vertices x_i, the vertex v' had been already connected with them. The degrees of these vertices x_i, after the connection with v' is 1 and by priority 2 every vertex $v \in N_1(v_1)\backslash N_1(v')$ had to be connected with vertices x_i which had degree 0. Consequently, if there exists x_j than $d(v', x_j) > 2$, then we'll get $d(v, x_j) > 2$ for any $v \in N_1(v_1)\backslash N_1(v')$ and, therefore, $d(v_1, x_j) > 2$. This will contradict with $d(v_1, x_j) \leq 2$ for any x_j.

Thus, lemma is prooved.

Now we will consider a vertex $v' \in N_1(v_1)$. If $N_1(v') \subseteq N_1(v_1) \bigcup \{v_1\}$, then, by Lemma, $d(v', x_j) \leq 2$ for all vertices x_j. Suppose, that the condition 2° is not satisfied. Obviously, v' shall be connected from v_1 (at distance 2) with $\Delta + 1 - deg_G v_1$ from the vertices x_j and directly with $\Delta + 1 - deg_G v'$ vertices x_j. Two cases can occur.

Case 1. Through $\Delta + 1 - deg_G v'$ edges, which is to be shared, the vertex v' is directly connected only with such vertices x_j, with which also v_1 is directly connected (with all or only with a part of them). This means that

$$\Delta + 1 - deg_G v' \leq \Delta + 1 - deg_G v_1, \quad \text{that is} \quad \deg_G v' \geq deg_G v_1 .$$

The difference $\Delta + 1 - deg_G v_1 - (\Delta + 1 - deg_G v') = deg_G v' - deg_G v_1$ represents the number of vertices x_j, directly connected with v_1, but not connected directly with v'. For each of these $\Delta + 1 - deg_G v_1$ vertices $d(v', x_j) \leq 2$.

Let us see if v' is connected with the aid of its neighbour vertices (different with v_1; their number is $deg_G v' - 1$) with the other vertices x_j (their number is $\Delta + 1 - \delta - (\Delta + 1 - deg_G v_1)$, that is $deg_G v_1 - \delta$). From every of $deg_G v' - 1$ vertices adjacents to v' the edges were drawn towards the vertices x_j after the edges from v' had been drawn. Because $deg_G v' \geq deg_G v_1$ and taking into consideration the priority 1, the number of all these vertices is sufficiently to connect all the adjacents to v' vertices with any from $deg_G v_1 - \delta$ vertices.

Case 2. $\Delta + 1 - deg_G v' \geq \Delta + 1 - deg_G v_1$, that is $deg_G v' \leq deg_G v_1$. Two cases are possible here: $2a$) and $2b$).

$2a$). The vertex v' is connected directly with all those $\Delta + 1 - deg_G v_1$ vertices x_j, with which is connected v_1 and, besides this, v' is connected with still

$$s = \Delta + 1 - deg_G v' - (\Delta + 1 - deg_G v_1) = deg_G v_1 - deg_G v'$$

vertices x_j.

The difference $\Delta + 1 - \delta - (\Delta + 1 - deg_G v') = deg_G v' - \delta$ represents the number of the vertices x_j with which v' is not connected neither directly and nor through v_1. These vertices are connected directly with the vertices from $N_1(v') \backslash \{v_1\}$. Because

$$|N_1(v') \backslash \{v_1\}| = deg_G v' - 1 \quad \text{and} \quad \delta \geq 1,$$

their number is sufficient to be able to draw from them by one edge to each of the $deg_G v' - \delta$ vertices x_j, not directly connected neither with v_1 nor with v' (these connections are ensured by priorities 1 and 2). The others $\delta - 1$ vertices from $N_1(v') \backslash \{v_1\}$ are connected with vertices x_j, holding account of those two priorities.

So it is clear, that v' at the respective step could be connected directly or with the aid of its neighbours with the all vertices x_j, so as for each x_j we will have $d(v', x_j) \leq 2$ and also $|deg x_i - deg x_j| \leq 1$.

$2b$). The vertex v' is not directly connected with all the $\Delta + 1 - deg_G v_1$ vertices x_j, with which v_1 is directly connected, or even neither with one.

Obviously, there exists at most

$$\Delta + 1 - \delta - (\Delta + 1 - deg_G v') + 1 = deg_G v' - \delta + 1$$

vertices x_j not directly connected neither with v' nor with v_1. By reasons similar to those from the case $2a$) it can be shown, that with the aid of its neighbours from $N_1(v') \backslash \{v_1\}$ the vertex v' is connected with the all these vertices x_j.

Thus, also in case 2 the condition $d(v', x_j) \leq 2$ is satisfied for any x_j.

It is clear that the condition 3° will be ensured by priority 2.

Thus, Theorem 1 is proved.

Corollary. *For any vertex $v_i \in V(G)$ we have $deg_H v_i = \Delta + 1$ and the number of the new edges (of the form (v_i, x_j)) is*

$$\sum_{i=1}^{n}(\Delta + 1 - deg_G v_i) = n(\Delta + 1) - 2m.$$

Since $|deg_H x_i - deg_H x_j| \leq 1$ for any x_i and x_j, it follows

$$\max_i deg_H x_i = \left\lceil \frac{n(\Delta+1) - 2m}{\Delta - \delta + 1} \right\rceil . \qquad (3)$$

$\Big($ Recall that if $a > 0$, then

$$\lceil a \rceil = \left\{ \begin{array}{cl} a, & \text{if } a \text{ is integer;} \\ [a] + 1, & \text{if } a \text{ is not integer} \end{array} \right.\Big).$$

The result below is implied by Corollary.

Theorem 2 *$MedH \cong G$ if and only if*

$$\left\lceil \frac{n(\Delta+1) - 2m}{\Delta - \delta + 1} \right\rceil \leq \Delta. \qquad (4)$$

3 The graphs G of diameter two with $f(G) \leq n + \Delta - \delta + 1$

Let δ and n are two integer numbers satisfying

$$\delta \geq 2 \ , \frac{n - (\delta+1)}{\delta} \geq \delta - 1 \ , \frac{n - (\delta+1)}{\delta} = k + \frac{q}{\delta} \ ,$$

k is an integer, $k \geq \delta - 1$, q is an integer, $0 \leq q \leq \delta - 1$. We will construct the graph G^1 with

$$V(G^1) = \{v_1, v_2, \ldots, v_n\}, E(G^1) = E_1 \bigcup E_2 \bigcup \ldots \bigcup E_{\delta+1},$$

$$E_1 = \{(v_1, v_j) : j = \overline{2, \delta + 1}\}, E_i = \{(v_i, v_j) :$$

$$j = \overline{\delta + (i-2)k + i, \delta + (i-2)k + i + k}\}, \quad \text{for } i = \overline{2, q+1};$$

$$E_i = \{(v_i, v_j) : j = \overline{\delta + 2 + q + (i-2)k, \delta + 2 + q + (i-2)k + k + 1}\}$$

for $i = \overline{q+2, \delta+1}$.

It is clear that $deg_{G^1} v_1 = \delta, deg_{G^1} v_i = k+1$ for $i = \overline{2, q+1}$, and $deg_{G^1} v_i = k$ for $i = \overline{q+2, \delta+1}$.

On each set of vertices $N_1(v_i) \backslash \{v_1\}, i = \overline{2, q+1}$, we will construct one complete graph $K_{k+1}^{(i)}$, and on each set $N_1(v_i) \backslash \{v_1\}, i = \overline{q+2, \delta+1}$ - one complete graph $K_k^{(i)}$. The graph so obtained from G^1 is denoted by G^2.

It is clear, that $V(G^2) = \{v_1, v_2, \ldots, v_n\}$ and $E(G^2) = E(G^1) \bigcup E^\star$, where $E^\star$ represents the set of edges of complete constructed subgraphs. The diameter of graph G^2 is equal to 4.

Now we will construct the graph G. For this we will add the new edges of type (v_i, v_j) for v_i and v_j from different complete subgraphs. Exactly, every vertex v_i from $K_{k+1}^{(1)}$ will be connected exactly with one vertex v_j from each subgraph $K_{k+1}^{(2)}, \ldots, K_k^{(\delta-q)}$. Every vertex from $K_{k+1}^{(2)}$ will be connected with one vertex from every subgraph $K_{k+1}^{(3)}, \ldots, K_k^{(\delta-q)}$. Similarly, we will proceed with the vertices from $K_{k+1}^{(3)}$ and so on. We'll have to take into account a single condition: the vertex v_i will be connected with such vertices, which at this moment have the minimal degree. The graph which is obtained is G and it's diameter is equal to 2. Evidently, $\min\limits_i deg_G v_i = deg_G v_1 = \delta$.

Further on we will calculate the number of edges of the graph G. The maximum degree of vertices v_i from $K_k^{(s)}$, $s = \overline{1, \delta - q}$, is equal with $k + \delta - 1 + \left\lceil \frac{q}{k} \right\rceil$. From all k vertices of each subgraph $K_k^{(s)}$, exactly $k + q - k \left\lceil \frac{q}{k} \right\rceil$ vertices have the degree $k + \delta - 1 + \left\lceil \frac{q}{k} \right\rceil$, but $k \left\lceil \frac{q}{k} \right\rceil - q$ vertices have the degree $k + \delta - 2 + \left\lceil \frac{q}{k} \right\rceil$. Therefore, these $\delta - q$ subgraphs contain $(\delta - q)(k + q - k \left\lceil \frac{q}{k} \right\rceil)$ vertices with the degree $\delta + k - 1 + \left\lceil \frac{q}{k} \right\rceil$

and $(\delta - q)(k\left\lceil\frac{q}{k}\right\rceil - q)$ vertices with the degree of $\delta + k - 2 + \left\lceil\frac{q}{k}\right\rceil$. In $K_{k+1}^{(s)}$, $s = \overline{1,q}$, the degree of each vertex is $\delta + k - 1$. Because $\sum_{i=1}^{n} deg_G v_i = 2m, m = |E(G)|$, we have

$$2m = \delta + q(k+2) + (\delta - q)(k+1) + q(k+1)(\delta + k) +$$

$$(\delta - q)\left(k + q - k\left\lceil\frac{q}{k}\right\rceil\right)\left(\delta + k - 1 + \left\lceil\frac{q}{k}\right\rceil\right) + (\delta - q)\left(k\left\lceil\frac{q}{k}\right\rceil - q\right)\left(\delta + k - 2 - \left\lceil\frac{q}{k}\right\rceil\right)$$

. From here it is easy to obtain :

$$2m = 2\delta + q + (k+\delta)(2q + k\delta) - q^2 \tag{5}$$

or

$$2m = k\delta^2 + \delta(2 + 2q + k^2) + q(2k + 1 - q) \tag{6}$$

(k, q - integers, $k \geq 1, q \geq 0$).

The sufficient conditions for

$$\left\lceil\frac{n(\Delta+1) - 2m}{\Delta - \delta + 1}\right\rceil \leq \Delta\ .$$

Case 1. $q = 0$. Then $\Delta = \delta + k - 1$ and $\Delta - \delta + 1 = k$. The inequality $\left\lceil\frac{n(\Delta+1) - 2m}{\Delta - \delta + 1}\right\rceil \leq \Delta$ may be written as $\left\lceil\frac{n(\delta + k) - 2m}{k}\right\rceil \leq \delta + k - 1$. This is equivalent to

$$\frac{n(\delta + k) - 2m}{k} \leq \delta + k - 1 \tag{7}$$

Indeed, the sufficient conditions for the last inequality (7) will be sufficient for that one too. This results out of that if $a = \frac{n(\delta + k) - 2m}{k}$ is an integer, then the inequality $a \leq \delta + k - 1$ is equivalent to $\lceil a \rceil \leq \delta + k - 1$. If however a isn't an integer, then $a < \delta + k - 1$ and therefore $\lceil a \rceil = [a] + 1 \leq \delta + k - 1$.

Because $n = \delta k + \delta + 1$, on account of (6), for $q = 0$ we obtain $(\delta k + \delta + 1)(\delta + k) - (k\delta^2 + 2\delta + k^2\delta) \le k\delta + k^2 - k$, or

$$k^2 - 2k - (\delta^2 - \delta) \ge 0 \quad . \tag{8}$$

This condition is sufficient for (7). The roots of the trinomial are $k_{1,2} = 1 \pm \sqrt{1 + \delta^2 - \delta}$. Thus we have obtained

Theorem 3 *If $\delta \ge 2$, $k = \dfrac{n - (\delta + 1)}{\delta}$ is an integer and $k \ge \delta - 1$, then $\left\lceil \dfrac{n(\Delta + 1) - 2m}{\Delta - \delta + 1} \right\rceil \le \Delta$ for any*

$$k \ge \left\lceil 1 + \sqrt{1 + \delta^2 - \delta} \right\rceil \tag{9}$$

Remark. *Because $1 + \sqrt{\delta^2 + 1 - \delta} = 1 + \sqrt{(\delta - 1)^2 + q} = 1 + \delta - 1 + \beta = \delta + \beta, \beta \in (0, 1)$, the restriction (9) means $k \ge \delta + 1$.*

Corollary. *For any $\delta \ge 2$ there exists an infinitude of graphs G of diameter two, for which $f(G) \le n + \Delta - \delta + 1$.*

Case 2. $q > 0 (q = \overline{1, \delta - 1})$. We have seen that $\Delta = \max\limits_i deg_G v_i = \delta + k - 1 - \left\lceil \dfrac{q}{k} \right\rceil$ and , because $k \ge \delta - 1$, it follows $\left\lceil \dfrac{q}{k} \right\rceil = 1$. Hence, in this case

$$\Delta = \delta + k, \Delta - \delta + 1 = k + 1, \frac{n - (\delta + 1)}{\delta} = k + \frac{q}{\delta},$$

therefore

$$n = \delta k + \delta + q + 1, 2m = k\delta^2 + \delta(2 + 2q + k^2) + q(2k + 1 - q)$$

and the condition $\dfrac{n(\Delta + 1) - 2m}{\Delta - \delta + 1} \le \Delta$ may be written as follows

$$k^2 + (q - \delta)k + (\delta + q\delta - q^2 - \delta^2 - 1) \ge 0. \tag{10}$$

The roots of the trinomial are

$$k_{1,2} = \frac{\delta - q \pm \sqrt{5\delta^2 + 5q^2 + 4 - 2\delta(3q + 2)}}{2}$$

and they exist, because

$$5\delta^2+5q^2+4-2\delta(3q+2)=(\sqrt{5}\delta-\sqrt{5}q)^2+4+4\delta(q-1)\geq 4.$$

It is clear, that $k_1<0, k_2>0$. Thus we have shown

Theorem 4 *For any graph G of diameter two, constructed by procedure described in section 3, with parameters $n, \delta(\delta\geq 2)$, $\dfrac{n-(\delta+1)}{\delta}=k+\dfrac{q}{\delta}$, $k\geq\delta-1$, $1\leq q\leq\delta-1$, the constraint (4) is satisfied for any*

$$k\geq\left\lceil\frac{\delta-q+\sqrt{5\delta^{+}5q^2+4-2\delta(3q+2)}}{2}\right\rceil. \tag{11}$$

Corollary. *For any $\delta\geq 2, 1\leq q\leq\delta-1$, there exists an infinitude of graphs G of diameter two, which satisfies the constraint (11). For these graphs we have $f(G)\leq n+\Delta-\delta+1$.*

Example. Let $\delta=10$. The inequality (11), which is sufficient for (4), implies:

1) if $q=1$, then $k\geq 15$; 6) if $q=6$, then $k\geq 11$;

2) if $q=2$, then $k\geq 14$; 7) if $q=7$, then $k\geq 10$;

3) if $q=3$, then $k\geq 13$; 8) if $q=8$, then $k\geq 10$;

4) if $q=4$, then $k\geq 12$; 9) if $q=9$, then $k\geq 10$.

5) if $q=5$, then $k\geq 11$.

Remark. *If $q=\delta-1$, then $k\geq\delta$.*

Indeed, for $q=\delta-1$ the discriminant $D=2(\delta-1)+\epsilon$, $\epsilon\in(0,1)$ and

$$k\geq\left\lceil\frac{1+2(\delta-1)+\epsilon}{2}\right\rceil=\lceil\delta-1+\epsilon'\rceil=\delta\ \Big(\epsilon'\in(0,1)\Big)$$

.

We have seen that if $k\geq\delta+1$, then $f(G)\leq n+\Delta-\delta+1$. In case $q>0$ not every $k\geq\delta+1$ makes to be (4). However, there exists $k\geq\delta$, which satisfies (4). For example, we can take the graph H with $q=\delta-1$.

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