Computer Science Journal of Moldova, vol.5, no.2(14), 1997

The approximate solution of singular integro-differential equations systems on smooth contours in spaces L_p

Iu. Cărăuş

Abstract

This article generalizes the results which were obtained in the paper [1], written together with my scientific-adviser, doctor-habilitat, professor Zolotarevschi V. Theoretical foundation of the collocation method and of mechanical quadrature method for singular integro-differential equations systems (SIDE) in the case when the equations are given on a closed contour satisfying some conditions of smoothness, without their reduction to the unit circle, is given below. Let Γ be a smooth Jordan border limiting the one-spanned area F^+, containing a point $t = 0$, $F^- = C \setminus \{F^+ \cup \Gamma\}$, C is a full complex plane. Let $z = \psi(w)-$ be a function, mapping comformally and single-valuedly the surface $\Gamma_0 = \{|w| > 1\}$ on F^- so that $\psi(\infty) = \infty, \psi^{(\prime)}(\infty) > 0$. We shall assume that the function $z = \psi(w)$ has its second derivative, satisfying on Γ_0 the Hölder condition with some parameter ν $(0 < \nu < 1)$; the class of such contours is denoted by $C(2;\nu)[2, p.23]$.

1 Statement of the problem and formulation of main theorems

In complex space $[L_p(\Gamma)]_m (1 < p < \infty)$ of vector-functions $g(t) = (g_1(t), \ldots, g_m(t)); g_j(t) \in L_p(\Gamma)$ $j = \overline{1,m}$ with the norm

$$||g|| = \sum_{k=1}^{m} ||g_k||_p; \quad ||g_k||_p = \left(\frac{1}{l} \int_{\Gamma} |g_k|^p |d\tau| \right)^{\frac{1}{p}}, \tag{1}$$

where l is the length of Γ, we will consider SIDE [3,p.312]

$$(Mx \equiv) \sum_{r=0}^{q} [\tilde{A}_r(t)x^{(r)}(t) + \tilde{B}_r(t)\frac{1}{\pi i}\int_{\Gamma} \frac{x^{(r)}(\tau)}{\tau - t}d\tau +$$
$$+\frac{1}{2\pi i}\int_{\Gamma} K_r(t,\tau)\cdot x^{(r)}(\tau)d\tau] = f(t), \quad t \in \Gamma, \tag{2}$$

where $\tilde{A}_r(t), \tilde{B}_r(t)$ and $K_r(t,\tau)$ $(r = \overline{0,q})$ are given $m \times m$ matrix-functions (MF); $f(t)$ is the given vector-function (VF), $x^{(0)}(t) = x(t)$ is the required VF; $x^{(r)}(t) = \dfrac{d^r x(t)}{dt^r}$ $(r = \overline{1,q})$; q is a natural number.

We search the solution of equation (2) in the class of vector-functions, satisfying the condition

$$\frac{1}{2\pi i}\int_{\Gamma} x(\tau)\tau^{-k-1}d\tau = 0, \quad k = \overline{0, q-1}. \tag{3}$$

Equation (2) with the help of operators $P = \frac{1}{2}(I+S)$, $Q = I - P$, where I is an identical operator, and S is a singular (with Cauchy nucleus) one, can be written as follows:

$$(Mx \equiv) \sum_{r=0}^{q} [A_r(t)(Px^{(r)})(t) + B_r(t)(Qx^{(r)})(t) +$$
$$+\frac{1}{2\pi i}\int_{\Gamma} K_r(t,\tau)x^{(r)}(\tau)d\tau] = f(t), \quad t \in \Gamma \tag{4}$$

where $A_r(t) = \tilde{A}_r(t) + \tilde{B}_r(t)$, $B_r(t) = \tilde{A}_r(t) - \tilde{B}_r(t)$, $\quad r = \overline{0,q}$.
We search the approximate solution of problem (2)-(3) in the form

$$x_n(t) = \sum_{k=0}^{n} \xi_k^{(n)} t^{k+q} + \sum_{k=-n}^{-1} \xi_k^{(n)} t^k, \quad t \in \Gamma, \tag{5}$$

where $\xi_k^{(n)} = \xi_k \quad (k = \overline{-n,n})$ are unknown m dimentional numerical vectors; we shall note that the VF $x_n(t)$, constructed by formula (5),obviously, satisfies conditions (3).

According to the collocation method, we determine the unknown $\xi_k \quad (k = \overline{-n, n})$ from a system of linear algebrical equations (SLAE):

$$\sum_{r=0}^{q}\{A_r(t_j)\sum_{k=0}^{n}\frac{(k+q)!}{(k+q-r)!}t^{k+q-r}\xi_k+$$

$$+B_r(t_j)\sum_{k=1}^{n}(-1)^r\frac{(k+r-1)!}{(k-1)!}\cdot t_j^{-k-r}\cdot\xi_{-k}+$$

$$+\frac{1}{2\pi i}\cdot\sum_{k=0}^{n}\frac{(k+q)!}{(k+q-r)!}\int_{\Gamma}K_r(t_j,\tau)\tau^{k+q-r}d\tau\cdot\xi_k+$$

$$+\sum_{k=1}^{n}(-1)^r\frac{(k+r-1)!}{(k-1)!}\cdot\frac{1}{2\pi i}\int_{\Gamma}K_r(t_j,\tau)\tau^{-k-r}d\tau\cdot\xi_{-k}\}=$$

$$=f(t_j),\qquad j=\overline{0,2n}, \qquad (6)$$

where $t_j(j = \overline{0, 2n})$ is a set of different points on Γ.

If the problem (2)-(3) is solved by the mecanical quadrature method, then we also search the approximate solution in the form (5). However,we find the unknown ξ_k $(k = \overline{-n, n})$ as the solution of (SLAE) (6), in which the integrals are replaced by the quadrature formulae.

We shall apply as a quadrature formula the following one [4,p.70]:

$$\frac{1}{2\pi i}\int_{\Gamma}g(\tau)\tau^{l+k}d\tau\cong\frac{1}{2\pi i}\int_{\Gamma}U_n(\tau^{l+1}\cdot g(\tau))\tau^{k-1}d\tau,$$

where $k = \overline{0, n}$, at $l = 0, 1, 2, \ldots$ and $k = \overline{-1, -n}$, for $l = -1, -2, \ldots;$ the operator of interpolation U_n is determined by the formula [4,p.26]

$$(U_n g)(t) = \sum_{s=0}^{2n} g(t_s)\cdot l_s(t);$$

$$l_j(t) = \prod_{k=0,k\neq j}^{2n}\frac{t-t_k}{t_j-t_k}\left(\frac{t_j}{t}\right)^n \equiv \sum_{k=-n}^{n}\Lambda_k^{(j)}t^k, \quad t\in\Gamma, \quad j=\overline{0,2n}.$$

Thus, for the determination of the unknown ξ_k $(k=\overline{-n,n})$ by the mechanical quadrature method we get the following SIDE:

$$\sum_{r=0}^{q}\{A_r(t_j)\sum_{k=0}^{n}\frac{(k+q)!}{(k+q-r)!}t_j^{k+q-r}\xi_k+$$
$$+B_r(t_j)\sum_{k=1}^{n}(-1)^r\frac{(k+r-1)!}{(k-1)!}t_j^{-k-r}\cdot\xi_k+$$
$$+\sum_{k=0}^{n}\frac{(k+q)!}{(k+q-r)!}\sum_{s=0}^{2n}K_r(t_j,t_s)t_s^{1+k-r}\Lambda_{-k}^{(s)}\xi_k+$$
$$+\sum_{k=1}^{n}(-1)^r\frac{(k+r-1)!}{(k-1)!}\sum_{s=0}^{2n}K_r(t_j,t_s)t_s^{-k-r}\Lambda_{k}^{(s)}\xi_{-k}\}=$$
$$=f(t_j),\qquad j=\overline{0,2n}. \tag{7}$$

Let $[\overset{\circ}{W}{}_p^{(q)}]_m=\{g;\ \exists\ g^{(r)}\in C(\Gamma),r=\overline{1,q-1},\ g^{(q)}\in[L_p(\Gamma)]_m;\}$ and for $\forall g\in[\overset{\circ}{W}{}_p^{(q)}]_m$ the condition (3) is satisfied and norm in $\overset{\circ}{W}{}_p^{(q)}$ is determined by the equality

$$||g||_{p,q}=||g^{(q)}||_{[L_p]_m}.$$

We shall denote by $[L_{p,q}]_m$ the image of space $[L_p]_m$ with mapping $P+t^{-q}Q$ with the same norm as in $[L_p]_m$.

Lemma 1. [5,p.44] *The differential operator* $D^q:[\overset{\circ}{W}{}_p^{(q)}]_m\to[L_{p,q}]_m$, $(D^qg)(t)=g^{(q)}(t)$ *is continuously reversible and its reverse operator* $D^{-q}:[L_{p,q}]_m\to[\overset{\circ}{W}{}_p^{(q)}]_m$ *is determined by the equality*

$$(D^{-q}g)(t)=(N^{+}g)(t)+(N^{-}g)(t),$$

$$(N^{+}g)(t)=\frac{(-1)^q}{2\pi i(q-1)!}\int_{\Gamma}(Pg)(\tau)(\tau-t)^{q-1}\ln(1-\frac{t}{\tau})d\tau,$$

$$(N^{-}g)(t) = \frac{(-1)^{q-1}}{2\pi i(q-1)!}\int\limits_{\Gamma}(Qg)(\tau)(\tau-t)^{q-1}\ln(1-\frac{\tau}{t})d\tau,$$

From lemma 1 it follows

Lemma 2 *The operator* $B : [\overset{\circ}{W}{}_p^{(q)}]_m \to [L_p]_m, B = (P + t^q Q)D^q$ *is reversible and*

$$B^{-1} = D^{-q}(P + t^{-q}Q)$$

The basic theorems in the given paper are the following :

Theorem 1 *Let the following conditions be satisfied:*

1) *the outline* $\Gamma \in C(2,\nu), \quad 0 < \nu < 1;$
2) *MF* $A_r(t)$ *and* $B_r(t)$ *belong to the space* $[H_\alpha(\Gamma)]_{m\times m}$ $0 < \alpha < 1, \quad r = \overline{0,q}$
3) *det* $A_q(t)\cdot$ *det* $B_q(t) \neq 0, \quad t \in \Gamma$
4) *the left partial indexes MF* $t^q B_q^{-1}(t)A_q(t)$ *are all equal to zero;*
5) *MF* $K_r(t,\tau) \quad (r = \overline{0,q}) \in H_\beta[(\Gamma\times\Gamma)]_{m\times m}, \ 0 < \beta \leq 1,$ *and VF* $f(t) \in [C(\Gamma)]_m;$
6) *the operator* $M : [\overset{\circ}{W}{}_p^{(q)}]_m \to [L_p(\Gamma)]_m$ *is linearly reversible;*
7) *the points* $t_j \ (j = \overline{0,2n})$ *form a system of Feier knots[6,p.36] on* Γ :

$$t_j = \psi\left[\exp\left(\frac{2\pi i}{2n+1}(j-n)\right)\right], \quad j = \overline{0,2n}, \quad i^2 = -1.$$

Then, beginning with $n \geq N_1$ *(*N_1 *depends on the coefficients of SIDE), SLAE (6) has the unique solution* ξ_k $(k = \overline{-n,n})$. *The approximate solutions* $x_n(t)$, *constructed by formula (5), converge when*

$n \to \infty$ in the norm of space $[\overset{\circ}{W}{}_p^{(q)}]_m$ to the exact solution $x(t)$ of the problem (2)-(3) and the following for estimation the convergence speed holds:

$$||x^{(q)} - x_n^{(q)}||_{p,q} = O\left(\frac{1}{n^\alpha}\right) + O(\omega(f;\frac{1}{n})) + O(\omega^t(h;\frac{1}{n})) \overset{def}{=} \delta_n \quad (8)$$

(MF $h(t,\tau)$ is a continuous MF relative to t and τ on Γ, defined below.)

Theorem 2. *Let all conditions 1)-7) of theorem I be satisfied. Then, beginning with the numbers $n \geq N_2(\geq N_1)$ SLAE (7) has a unique solution ξ_k, $k = \overline{-n,n}$. The approximate solutions of (5) converge when $n \to \infty$ in the norm $[\overset{\circ}{W}{}_p^{(q)}]_m$ to exact solution $x(t)$ of the problem (2)-(3) and the following estimation for the convergence speed takes place:*

$$||x - x_n||_{p,q} = \delta_n + O(\omega^\tau(h;\frac{1}{n})) \quad (9)$$

We shall note that in the case of the standard contour (the segment of a real straightline or a unit circle) the similar theorems were obtained earlier in [7,8]. Before we proceed to the proof of the theorems 1 and 2, we shall bring some statements from [9], which will be necessary further.

As is proved in [9], VF $\frac{d^q(Px)(t)}{dt^q}$ and $\frac{d^q(Qx)(t)}{dt^q}$ can be represented by integrals of Cauchy type with the same density $v(t)$:

$$\left.\begin{aligned} \frac{d^q(Px)(t)}{dt^q} &= \frac{1}{2\pi i}\int_\Gamma \frac{v(\tau)}{\tau - t}d\tau, \quad t \in F^+ \\ \frac{d^q(Qx)(t)}{dt^q} &= \frac{t^{-q}}{2\pi i}\int_\Gamma \frac{v(\tau)}{\tau - t}d\tau, \quad t \in F^-. \end{aligned}\right\} \quad (10)$$

With the help of these representations the problem (2)-(3) can be reduced to an equivalent (in the sense of its solvability) singular integral equations system(SIE).

$$(Rv \equiv) C(t)v(t) + \frac{D(t)}{\pi i}\int\limits_{\Gamma}\frac{v(\tau)}{\tau - t}d\tau +$$

$$+\frac{1}{2\pi i}\int\limits_{\Gamma} h(t,\tau)v(\tau)d\tau = f(t), \quad t \in \Gamma, \tag{11}$$

where $C(t) = \frac{1}{2}[A_q(t) + t^{-q}B_q(t)]$, $\quad D(t) = \frac{1}{2}[A_q(t) - t^{-q}B_q(t)]$, and $h(t,\tau)$ from the condition 5) of the theorem I is a MF, belonging to the class $[C(\Gamma\times\Gamma)]_{m\times m}$ for both variables; the obvious form of this function is given in [9].

The system SIE (11) and problem (2)-(3) are equivalent in the sense that to each solution $v(t)$ of system (11) corresponds by the formulae

$$(Px)(t) = \frac{(-1)^q}{2\pi i(q-1)!}\int\limits_{\Gamma} v(\tau)[(\tau - t)^{q-1}\ln\left(1 - \frac{t}{\tau}\right) + \sum_{k=1}^{q-1}\alpha_k \tau^{q-k-1}t^k]d\tau$$

$$(Qx)(t) = \frac{(-1)^q}{2\pi i(q-1)!}\int\limits_{\Gamma} v(\tau)\tau^{-q}[(\tau - t)^{q-1}\ln\left(1 - \frac{\tau}{t}\right) + \sum_{k=1}^{q-2}\beta_k \tau^{q-k-1}t^k]d\tau \tag{12}$$

(α_k, $\quad k = \overline{1, q-1}$ and $k = \overline{1, q-2}$ are real numbers), the solution $x(t)$ of the problem (2)-(3) and, conversely; to each solution $x(t)$ of the problem (2)-(3) corresponds by formula

$$v(t) = \frac{d^q(Px)(t)}{dt^q} + t^q\frac{d^q(Qx)(t)}{dt^q},$$

the solution $v(t)$ of the system (11); and the linear-independent solutions of the problem (2)-(3) correspond to the linear-independent solutions of the system (11) (and conversely).

2. The proof of the theorem 1.

We shall show that for sufficiently large $n \quad (\geq N_1)$ the operator U_nMU_n is reversible as an operator, considered from the subspace $\overset{\circ}{X}_n = \left\{ t^q \sum\limits_{k=0}^{n} \alpha_k t^k + \sum\limits_{k=-n}^{-1} \alpha_k t^k \right\}$ in subspace $[R_n]_m$ of polynomials of type

$$\sum_{k=-n}^{n} r_k t^k, \quad t \in \Gamma$$

We consider that in these subspaces the norm is the same as in subspaces $[\overset{\circ}{W}{}_p^{(q)}]_m$ and $[L_p(\Gamma)]_m$ respectively.

Similarly to the formulae (10) we shall represent the VF $\dfrac{d^q(Px_n)(t)}{dt^q}$ and $\dfrac{d^q(Qx_n)(t)}{dt^q}$ by integrals of Cauchy type with the same density $v_n(t)$:

$$\left.\begin{aligned} \frac{d^q(Px_n)(t)}{dt^q} &= \frac{1}{2\pi i} \int\limits_{\Gamma} \frac{v_n(\tau)}{\tau - t} d\tau, \quad t \in F^+ \\ \frac{d^q(Qx_n)(t)}{dt^q} &= \frac{t^{-q}}{2\pi i} \int\limits_{\Gamma} \frac{v_n(\tau)}{\tau - t} d\tau, \quad t \in F^-. \end{aligned}\right\} \tag{13}$$

It is easy to prove that

$$v_n(t) = \sum_{k=0}^{n} \frac{(k+q)!}{k!} t^k \xi_k + (-1)^q \sum_{k=1}^{n} \frac{(k+q-1)!}{(k-1)!} t^{-k} \xi_{-k}$$

and therefore $v_n(t) \in [R_n]_m, \quad t \in \Gamma$.

With the help of representation (13) the equation $U_nMU_nx_n = U_nf$, in the same way as the problem (2)–(3), is reduced to an equivalent in the same sense as the above equation

$$U_nRU_nx_n = U_nf, \tag{14}$$

considered as an equation in the subspace $[R_n]_m$.

The last equation, obviously, represents an equation of the collocation method for the system (SIE) (11). From (13) and $v_n \in [R_n]_m$ we conclude that if $v_n(t)$ is the solution of the equation (14), then VF $y_n(t)$, defined by the equality

$$\begin{aligned}(Py_n)(t) &= \frac{(-1)^q}{2\pi i(q-1)!}\int_\Gamma v_n(\tau)[(\tau-t)^{q-1}\ln(1-\frac{t}{\tau})+\\ &+\sum_{k=1}^{q-1}\alpha_k\tau^{q-k-1}t^k]d\tau;\\ (Qy_n)(t) &= \frac{(-1)^q}{2\pi i(q-1)!}\int_\Gamma v_n(\tau)\tau^{-q}[(\tau-t)^{q-1}\ln(1-\frac{\tau}{t})+\\ &+\sum_{k=1}^{q-1}\beta_k\tau^{q-k-1}t^k]d\tau; \qquad (15)\end{aligned}$$

is the solution of the equation $U_nMU_nx_n = U_nf$ and conversely. As it is mentioned above, VF $y_n(t)$ is defined with help of $v_n(t)$ by formulae (15) in the unique manner.

It follows that if the equation (14) has the unique solution $v_n(t)$ in subspace $[R_n]_m$, then the equality $y_n(t) = x_n(t)$ should be satisfied and $x_n(t)$ is determined in a unique way.

The (14) is an equation of the collocation method for (11). We shall show that for this equation all conditions of the theorem 8.3 are satisfied from [4,p.76], which gives the theoretical foundation of the collocation method for systems SIE in spaces $L_p(\Gamma)$. From the conditions 3), 4), 6), lemma 1, lemma2, it follows the reversibility of operator $R : [L_p(\Gamma)]_m \to [L_p(\Gamma)]_m$; this together with the conditions 1), 2), 5), 7) coincide with theorem 8.3 from [4]. Therefore, beginning with the numbers $n \geq N_1$, the equation (14) has the unique solution $v_n(t) \in [R_n]_m$. Hence, the equation $U_nMU_nx_n = U_nf$, and the SLAE (6) are solvable in a unique way for such n.

Besides, according to the theorem 8.3. from [4,p.76] the following estimation is true:

$$||v - v_n|| = O\left(\frac{1}{n^\alpha}\right) + O(\omega(f;\frac{1}{n})) + O(\omega^t(h;\frac{1}{n})). \tag{16}$$

From the relations (15) and $y_n(t) = x_n(t)$ it follows that

$$||x^{(q)} - x_n^{(q)}|| \leq c||v - v_n||,$$

From the previous inequality and with the help of (16) we obtain (8). The theorem I is proved.

The proof of theorem 2

It is easy to verify that SLAE (7) is equivalent to the operational equation

$$U_n\left\{\sum_{r=0}^{q}[A_r(t)(Px_n^{(r)})(t) + B_r(t)(Qx_n^{(r)})(t)+\right.$$
$$+\frac{1}{2\pi i}\int\limits_\Gamma \frac{1}{\tau}U_n^{(\tau)}[\tau^{q+1-r}K(t,\tau)](Px_n^{(r)})(\tau)d\tau+$$
$$\left.+\frac{1}{2\pi i}\int\limits_\Gamma \frac{1}{\tau}U_n^{(\tau)}[\tau^{-r-1}K(t,\tau)](Qx_n^{(r)})(\tau)d\tau]\right\} = U_n f, \tag{17}$$

which after the application of integrate representation (13) is equivalent (in the same sense as it was mentioned above) to the equation

$$U_n\left\{C(t)v_n(t) + D(t)(Sv_n)(t) +\right.$$
$$\left.+\frac{1}{2\pi i}\int\limits_\Gamma \frac{1}{\tau}U_n^{(\tau)}[\tau h(t,\tau)]\cdot v_n(\tau)d\tau\right\} = U_n f, \tag{18}$$

where the VF $C(t), D(t)$ and $h(t,\tau)$ are determined above.
The equation (18) represents an equation of the mechanical quadrature method for system SIE (11). It is easy to verify like in the proof of theorem 1, that from the conditions of theorem 2 the fulfillment

of conditions of theorem 8.4 from [4,p.77] follows, arranging the use of mechanical quadrature method to system SIE (11). Therefore, by virtue of cited theorem 8.4., beginning with the numbers $n \geq N_2(\geq N_1)$ the equation (18) has the unique solution $v_n(t) \in [R_n]_m$, and the following estimation is true:

$$||v - v_n|| = O\left(\frac{1}{n^\alpha}\right) + O(\omega(f; \frac{1}{n})) + O(\omega^\tau(h; \frac{1}{n})) + O(\omega^t(h; \frac{1}{n})). \quad (19)$$

Then for such n the equation (17) has the unique solution $x_n(t) \in \overset{\circ}{X}_n$, which is connected with $v_n(t)$ by the formulae (15). Besides, as the exact solutions $x(t)$ and $v(t)$ of the problem (2)-(3) and system (11), respectively, are connected by the formulae (12), then we shall obtain (9), taking into account (19). The theorem 2 is proved.

References

[1] Zolotarevschi V., Cărăuş Iu. *The approximate solution of singular integro-differential equations on smooth contours in the spaces L_p*. Kishinev, Computer Science Journal of Moldova, Vol.5, No.1, 1997, p.44–55.

[2] Vekua I.N. *Generalized analytical functions.* M.: Science 1988. (Russian)

[3] Vekua N.P. *The systems of singular integral equations.* M.: Science, 1970. (Russian)

[4] Zolotarevschi V.A. *Finite methods of solving the singular integral equations on the closed contours of integration.* Kishinev, Shtiintsa, 1991. (Russian)

[5] Saks R.S. *The regional problems for elliptical systems of differential equations.* Novosibirsk: University of Novosibirsk, 1975. (Russian)

[6] Smirnov V.I., Lebedev N.A. *The constructive theory of the functions of the complex variables.* M.-L.: Science, 1964. (Russian)

[7] B. Gabdulalhaev. *Direct methods of solving some operational equations.* News of higher schools. Mathematics. No.4, 1972, p.32–43. (Russian)

[8] B. Gabdulalhaev. *The polynomial approximations of singular integral and integro-differential equations by Dziadik.* News of higher schools. Mathematics. No.6, 1978, p.51–62. (Russian)

[9] Krikunov Iu.V. *The scientific notes of the Kazan university.* 1966, V116, Book.4, Vol.4, p.3–29. (Russian)

Iu. Cărăuş,
Institute of Mathematics
Academy of Sciences of Moldova,
5, Academiei str., Kishinev,
MD-2028, Moldova

Received April 30, 1997