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The approximate solution of singular integro-differential equations on smooth contours in the spaces L_p.

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Theoretical foundation of the collocation method and mechanical quadrature method for singular integro-differential equations (SIDE) in the case when the equations are given on a closed contour satisfying some conditions of smoothness,without their reducing to the unit circle is given below.

Let Γ be a smooth Jordan border limiting the one-spanned area F^+, containing a point $t = 0$, $F^- = C \setminus \{F^+ \cup \Gamma\}$. Let $z = \psi(w)$ be a function, mapping comformally and single-valuedly $\Gamma_0 = \{|w| > 1\}$ on the surface Γ so that $\psi(\infty) = \infty, \psi^{(\prime)}(\infty) > 0$.

We shall assume that the function $z = \psi(w)$ has second derivative, satisfying on Γ_0 the Hölder condition with some parameter ν $(0 < \nu < 1)$; the class of such contours is denoted [I,p.23] by $C(2;\nu)$.

1 Statement of the problem and formulation of the main theorems

In complex space $L_p(\Gamma)(1 < p < \infty)$ of the functions $g(t) \in L_p(\Gamma)$ with the norm

$$\|g\| = \left(\frac{1}{l}\int\limits_{\Gamma} |g|^p|d\tau|\right)^{\frac{1}{p}}, \tag{1}$$

where l is the length of Γ, we will consider SIDE [2,p.312]

$$(Mx \equiv) \sum_{r=0}^{q} [\tilde{A}_r(t) x^{(r)}(t) + \tilde{B}_r(t) \frac{1}{\pi i} \int_{\Gamma} \frac{x^{(r)}(\tau)}{\tau - t} d\tau +$$

$$+\frac{1}{2\pi i} \int_{\Gamma} K_r(t,\tau) \cdot x^{(r)}(\tau) d\tau] = f(t), \quad t \in \Gamma, \tag{2}$$

where $\tilde{A}_r(t), \tilde{B}_r(t), f(t)$ and $K_r(t,\tau)$ $(r = \overline{0,q})$ are given functions ; $x^{(0)}(t) = x(t)$ is the required function; $x^{(r)}(t) = \dfrac{d^r x(t)}{dt^r}$ $(r = \overline{1,q}); q$ is a natural number.

We search the solution of equation (2) in the class of functions,satisfying the condition

$$\frac{1}{2\pi i} \int_{\Gamma} x(\tau) \tau^{-k-1} d\tau = 0, \quad k = \overline{0, q-1}. \tag{3}$$

Equation (2) with the help of operators $P = \frac{1}{2}(I + S)$, $Q = I - P$,where I is an identical operator, and S is a singular (with Causchy nucleus) one, can be written as follows:

$$(Mx \equiv) \sum_{r=0}^{q} [A_r(t)(Px^{(r)})(t) + B_r(t)(Qx^{(r)})(t) +$$

$$\frac{1}{2\pi i} \int_{\Gamma} K_r(t,\tau) x^{(r)}(\tau) d\tau] = f(t), \quad t \in \Gamma \tag{4}$$

where $A_r(t) = \tilde{A}_r(t) + \tilde{B}_r(t)$, $B_r(t) = \tilde{A}_r(t) - \tilde{B}_r(t)$, $\quad r = \overline{0,q}$.
We search the approximate solution of problem (2)-(3) in the form

$$x_n(t) = \sum_{k=0}^{n} \xi_k^{(n)} t^{k+q} + \sum_{k=-n}^{-1} \xi_k^{(n)} t^k, \quad t \in \Gamma, \tag{5}$$

where $\xi_k^{(n)} = \xi_k \quad (k = \overline{-n,n})$ are unknown; we shall note that the function $x_n(t)$, constructed by formula (5), obviously, satisfies conditions (3).

According to the collocation method , we determine the unknown $\xi_k \quad (k=\overline{-n,n})$ from a system of linear algebrical equations (SLAE):

$$\sum_{r=0}^{q}\{A_r(t_j)\sum_{k=0}^{n}\frac{(k+q)!}{(k+q-r)!}t^{k+q-r}\cdot\xi_k+$$

$$+B_r(t_j)\sum_{k=1}^{n}(-1)^r\frac{(k+r-1)!}{(k-1)!}t_j^{-k-r}\cdot\xi_{-k}+$$

$$+\frac{1}{2\pi i}\cdot\sum_{k=0}^{n}\frac{(k+q)!}{(k+q-r)!}\int\limits_{\Gamma}K_r(t_j,\tau)\tau^{k+q-r}d\tau\cdot\xi_k+$$

$$+\sum_{k=1}^{n}(-1)^r\frac{(k+r-1)!}{(k-1)!}\cdot\frac{1}{2\pi i}\int\limits_{\Gamma}K_r(t_j,\tau)\tau^{-k-r}d\tau\cdot\xi_{-k}\}=f(t_j),$$

$$j=\overline{0,2n}, \tag{6}$$

where $t_j(j=\overline{0,2n})$ is the set of different points on Γ.

If the problem (2)-(3) is solved by the mechanical quadrature method, we also search the approximate solution in the form (5). However, we find the unknown ξ_k $(k=\overline{-n,n})$ as the solution of (SLAE) (6), in which the integrals are replaced by the quadrature formulae.

We shall aplly as quadrature formula the following one:

$$\frac{1}{2\pi i}\int\limits_{\Gamma}g(\tau)\tau^{l+k}d\tau\cong\frac{1}{2\pi i}\int\limits_{\Gamma}U_n(\tau^{l+1}\cdot g(\tau))\tau^{k-1}d\tau,$$

where $k=\overline{0,n}$, at $l=0,1,2,\dots$ and $k=\overline{-1,-n}$, for $l=-1,-2,\dots$; the operator of interpolation U_n is determined by the formula

$$(U_ng)(t)=\sum_{s=0}^{2n}g(t_s)\cdot l_s(t),$$

$$l_j(t)=\prod_{k=0,k\neq j}^{2n}\frac{t-t_k}{t_j-t_k}\left(\frac{t_j}{t}\right)^n\equiv\sum_{k=-n}^{n}\Lambda_k^{(j)}t^k,\quad t\in\Gamma,\quad j=\overline{0,2n}.$$

Thus, for the determination of the unknown ξ_k $(k=\overline{-n,n})$ by the mechanical quadrature method we get the following SIDE:

$$\sum_{r=0}^{q}\{A_r(t_j)\sum_{k=0}^{n}\frac{(k+q)!}{(k+q-r)!}t_j^{k+q-r}\cdot\xi_k+$$
$$+B_r(t_j)\sum_{k=1}^{n}(-1)^r\frac{(k+r-1)!}{(k-1)!}t_j^{-k-r}\cdot\xi_k+$$
$$+\sum_{k=0}^{n}\frac{(k+q)!}{(k+q-r)!}\sum_{s=0}^{2n}K_r(t_j,t_s)t_s^{1+k-r}\Lambda_{-k}^{(s)}\cdot\xi_k+$$
$$+\sum_{k=1}^{n}(-1)^r\frac{(k+r-1)!}{(k-1)!}\sum_{s=0}^{2n}K_r(t_j,t_s)t_s^{-k-r}\Lambda_{k}^{(s)}\cdot\xi_{-k}\}=f(t_j),$$
$$j=\overline{0,2n}. \qquad (7)$$

Let $\overset{\circ}{W}{}_p^{(q)}(\Gamma)=\{g;\ \ \exists\ \ g^{(r)},r=\overline{1,q},\ g^{(q)}\in L_P(\Gamma);\}$ and for $\forall g\in\overset{\circ}{W}{}_p^{(q)}$ the condition (3) is satisfied and the norm in $\overset{\circ}{W}{}_p^{(q)}$ is determined by the equality

$$\|g\|_{p,q}=\|g^{(q)}\|_{L_p}.$$

We shall denote by $L_{p,q}$ the image of space L_p with mapping $P+t^{-q}Q$ with the same norm as in L_p.

Lemma 1 . [3,p.44] *The differential operator* $D^q:\overset{\circ}{W}{}_p^{(q)}\to L_{p,q}$, $(D^qg)(t)=g^{(q)}(t)$ *is continuously reversable and its reverse operator is* $D^{-q}:L_{p,q}\to\overset{\circ}{W}{}_p^{(q)}$ *is determined by the equality*

$$(D^{-q}g)(t)=(N^+g)(t)+(N^-g)(t),$$
$$(N^+g)(t)=\frac{(-1)^q}{2\pi i(q-1)!}\int_\Gamma(Pg)(\tau)(\tau-t)^{q-1}\ln(1-\frac{t}{\tau})d\tau,$$
$$(N^-g)(t)=\frac{(-1)^{q-1}}{2\pi i(q-1)!}\int_\Gamma(Qg)(\tau)(\tau-t)^{q-1}\ln(1-\frac{\tau}{t})d\tau,$$

From lemma 1 it follows:

Lemma 2 *The operator* $B : \overset{\circ}{W}{}_p^{(q)} \to L_p, B = (P + t^q Q) D^q$ *is reversable and*

$$B^{-1} = D^{-q}(P + t^{-q} Q)$$

The basic theorems in the given paper are the following
Theorem 1 *Let the following conditions be satisfied:*

- *1) the outline* $\Gamma \in C(2, \nu), \quad 0 < \nu < 1;$
- *2) the functions* $A_r(t)$ *and* $B_r(t)$ *belong to the space* $H_\alpha(\Gamma)$ $0 < \alpha < 1, \quad r = \overline{0, q}$
- *3)* $A_q(t) \cdot B_q(t) \neq 0, \quad t \in \Gamma$
- *4) the index of function* $t^q B_q^{-1}(t) A_q(t)$ *is equal to zero;*
- *5) the functions* $K_r(t, \tau) \quad (r = \overline{0, q}) \in H_\beta(\Gamma \times \Gamma), \quad 0 < \beta \leq 1,$ *and the function* $f(t) \in C(\Gamma)$;
- *6) the operator* $M : \overset{\circ}{W}{}_p^{(q)} \to L_p$ *is linearly reversable;*
- *7) the points* t_j $(j = \overline{0, 2n})$ *form a system of Feier knots [4,p.36] on* Γ :

$$t_j = \psi \left[\exp \left(\frac{2\pi i}{2n+1} (j - n) \right) \right], \quad j = \overline{0, 2n}, \quad i^2 = -1.$$

Then, begining with $n \geq N_1$, *(*N_1 *depends on the coefficients of SIDE), SLAE (6) has the unique solution* ξ_k $(k = \overline{-n, n})$. *The approximate solutions* $x_n(t)$, *constructed by the formula (5), converge when* $n \to \infty$ *in the norm of space* $\overset{\circ}{W}{}_p^{(q)}$ *to the exact solution* $x(t)$ *of the problem (2)-(3) and the following estimation the convergence speed holds:*

$$\|x^{(q)} - x_n^{(q)}\|_{p,q} = O\left(\frac{1}{n^\alpha}\right) + O(\omega(f; \frac{1}{n})) + O(\omega^\tau(h; \frac{1}{n})) \overset{def}{=} \delta_n \quad (8)$$

($h(t, \tau)$ *is a continuos function relative to* t *and* τ *on* Γ *defined below.)*

Theorem 2. *Let all conditions 1)-7) of the theorem 1 be satisfied. Then, begining with the numbers* $n \geq N_2 (\geq N_1)$ *SLAE (7) has a unique solution* $\xi_k,\ \ k = \overline{-n, n}$. *The approximate solution of (5) converge when* $n \to \infty$ *in the norm of* $\overset{\circ}{W}{}_p^{(q)}$ *to the exact solution* $x(t)$ *of the problem (2)-(3) and the following estimation for the convergence speed takes place:*

$$\|x - x_n\|_{p,q} = \delta_n + O(\omega^{\tau}(h; \frac{1}{n})) \tag{9}$$

We shall note that in the case of the standard contour (the segment of a real straightline or an unit circle) the similar theorems were obtained earlier in [5,6]. Before we proceed to the proof of the theorems 1 and 2, we shall bring some statements from [7], which will be necessary further.

As is proved in [7], functions $\dfrac{d^q(Px)(t)}{dt^q}$ and $\dfrac{d^q(Qx)(t)}{dt^q}$ can be represented by integrals of Cauchy type with the same density $v(t)$:

$$\left.\begin{aligned} \frac{d^q(Px)(t)}{dt^q} &= \frac{1}{2\pi i}\int\limits_{\Gamma} \frac{v(\tau)}{\tau - t} d\tau, \quad t \in F^+ \\ \frac{d^q(Qx)(t)}{dt^q} &= \frac{t^{-q}}{2\pi i}\int\limits_{\Gamma} \frac{v(\tau)}{\tau - t} d\tau, \quad t \in F^-. \end{aligned}\right\} \tag{10}$$

With the help of these representations the problem (2)-(3) can be reduced to an equivalent (in the sense of solving) singular integral equation (SIE).

$$(Rv \equiv) C(t)v(t) + \frac{D(t)}{\pi i}\int\limits_{\Gamma} \frac{v(\tau)}{\tau - t} d\tau + \frac{1}{2\pi i}\int\limits_{\Gamma} h(t,\tau)v(\tau)d\tau = f(t),$$

$$t \in \Gamma, \tag{11}$$

where $C(t) = \frac{1}{2}[A_q(t) + t^{-q}B_q(t)],\quad D(t) = \frac{1}{2}[A_q(t) - t^{-q}B_q(t)]$, and $h(t,\tau)$ according to the condition 5) of the theorem I is a function belonging to the class $C(\Gamma \times \Gamma)$; the obvious form of this function is given in [7].

The equation (11) and problem (2)-(3) are equivalent in the sense that to each solution $v(t)$ of equation (11) corresponds with the formulae

$$(Px)(t) = \frac{(-1)^q}{2\pi i(q-1)!}\int\limits_{\Gamma} v(\tau)[(\tau-t)^{q-1}\ln\left(1-\frac{t}{\tau}\right)+$$

$$+\sum_{k=1}^{q-1}\alpha_k\tau^{q-k-1}t^k]d\tau$$

$$(Qx)(t) = \frac{(-1)^q}{2\pi i(q-1)!}\int\limits_{\Gamma} v(\tau)\tau^{-q}[(\tau-t)^{q-1}\ln\left(1-\frac{\tau}{t}\right)+$$

$$+\sum_{k=1}^{q-2}\beta_k\tau^{q-k-1}t^k]d\tau \qquad (12)$$

(α_k, $k=\overline{1,q-1}$ and $k=\overline{1,q-2}$ are real numbers), the solution $x(t)$ of the problem (2)-(3) and , conversely ; to each solution $x(t)$ of the problem (2)-(3) corresponds with formula

$$v(t) = \frac{d^q(Px)(t)}{dt^q} + t^q\frac{d^q(Qx)(t)}{dt^q},$$

the solution $v(t)$ of the equation (11), and the linear-independent solutions of the problem (2)-(3) correspond to the linear-independent solutions of equation (11) (and conversely) .

2. The proof of the theorem 1. We shall show that for sufficiently large n $(\geq N_1)$ the operator U_nMU_n is reversable as the operator, considered from the subspace $\dot{X}_n = \left\{t^q\sum\limits_{k=0}^{n}\alpha_kt^k + \sum\limits_{k=-n}^{-1}\alpha_kt^k\right\}$ in the subspace R_n of polynomials of type

$$\sum_{k=-n}^{n} r_kt^k, \quad t\in\Gamma.$$

We consider that in these subspaces the norm is the same as in subspaces $\overset{\circ}{W}{}_p^{(q)}$ and $L_p(\Gamma)$, respectively.

Similarly to the formulae (10) we shall represent the functions $\dfrac{d^q(Px_n)(t)}{dt^q}$ and $\dfrac{d^q(Qx_n)(t)}{dt^q}$ by integrals of Cauchy type with the same density $v_n(t)$:

$$\left.\begin{aligned} \frac{d^q(Px_n)(t)}{dt^q} &= \frac{1}{2\pi i}\int\limits_{\Gamma}\frac{v_n(\tau)}{\tau-t}d\tau, \quad t\in F^+ \\ \frac{d^q(Qx_n)(t)}{dt^q} &= \frac{t^{-q}}{2\pi i}\int\limits_{\Gamma}\frac{v_n(\tau)}{\tau-t}d\tau, \quad t\in F^-. \end{aligned}\right\} \tag{13}$$

It is easy to prove that

$$v_n(t)=\sum_{k=0}^{n}\frac{(k+q)!}{k!}t^k\xi_k+(-1)^q\sum_{k=1}^{n}\frac{(k+q-1)!}{(k-1)!}t^{-k}\xi_{-k}$$

and therefore $v_n(t)\in R_n, \quad t\in\Gamma$.

With the help of representation (13) the equation $U_nMU_nx_n=U_nf$ is reduced to the equivalent equation in the same way as the problem (2)-(3):

$$U_nRU_nx_n=U_nf, \tag{14}$$

considered as an equation in the subspace R_n. The last equation, obviously, represents an equation of the collocation method for SIE (11). From (13) and $v_n\in R_n$ we conclude that if $v_n(t)$ is the solution of the equation (14), then the function $y_n(t)$, defined by the equality

$$(Py_n)(t)=\frac{(-1)^q}{2\pi i(q-1)!}\int\limits_{\Gamma}v_n(\tau)[(\tau-t)^{q-1}\ln(1-\frac{t}{\tau})+$$

$$+\sum_{k=1}^{q-1}\alpha_k\tau^{q-k-1}t^k]d\tau;$$

$$(Qy_n)(t)=\frac{(-1)^q}{2\pi i(q-1)!}\int\limits_{\Gamma}v_n(\tau)\tau^{-q}[(\tau-t)^{q-1}\ln(1-\frac{\tau}{t})+$$

$$+\sum_{k=1}^{q-1}\beta_k\tau^{q-k-1}t^k]d\tau; \tag{15}$$

is the solution of the equation $U_nMU_nx_n = U_nf$ and conversely. As it is mentioned above, the function $y_n(t)$ is defined with the help of $v_n(t)$ by fomulae (15) in the unique manner.

It follows that if the equation (14) has the unique solution $v_n(t)$ in subspace R_n, then the equality $y_n(t) = x_n(t)$ should be satisfied and $x_n(t)$ is the solution determined in the unique way.

The (14) is an equation of the collocation method for SIE (11). We shall show that for this equation all conditions of the theorem 8.3 are satisfied from [8,p.76], which gives the theoretical foundation of the collocation method for SIE L_p. From the conditions 3),4),6),
lemma 1,lemma 2,it follows the reversability of operator $R : L_p \to L_p$; this together with the conditions 1), 2), 5), 7) coincide with theorem 8.3 from [8]. Therefore, begining with the numbers $n \geq N_1$, the equation (14) has the unique solution $v_n(t) \in R_n$. Hence, the equation $U_nMU_nx_n = U_nf$, and the SLAE (6) are solvable in the unique way for such n.

Besides, according to the theorem 8.3. from [8,p.76] the following estimation is true:

$$\|v - v_n\| = O\left(\frac{1}{n^\alpha}\right) + O(\omega(f; \frac{1}{n})) + O(\omega^t(h; \frac{1}{n})). \qquad (16)$$

From the relations (15) and $y_n(t) = x_n(t)$ it follows that

$$\|x^{(q)} - x_n^{(q)}\| \leq c\|v - v_n\|,$$

From the previous inequality and with help of (16) we obtain (8). The theorem 1 is proved.

The proof of theorem 2 It is easy to verify that SLAE (7) is equivalent to the operational equation

$$U_n\left\{\sum_{r=0}^{q}[A_r(t)(Px_n^{(r)})(t) + B_r(t)(Qx_n^{(r)})(t)+\right.$$

$$\frac{1}{2\pi i}\int_{\Gamma}\frac{1}{\tau}U_n^{(\tau)}[\tau^{q+1-r}K(t,\tau)](Px_n^{(r)})(\tau)d\tau +$$

$$+\frac{1}{2\pi i}\int\limits_{\Gamma}\frac{1}{\tau}U_n^{(\tau)}[\tau^{-r-1}K(t,\tau)](Qx_n^{(r)})(\tau)d\tau]\Bigg\}=U_nf, \quad (17)$$

which after the application of the integrate representation (13) is equivalent (in the same sense as was mentioned above) to the equation

$$U_n\{C(t)v_n(t)+D(t)(Sv_n)(t)+$$

$$+\frac{1}{2\pi i}\int\limits_{\Gamma}\frac{1}{\tau}U_n^{(\tau)}[\tau h(t,\tau)]\cdot v_n(\tau)d\tau\}=U_nf, \quad (18)$$

where the functions $C(t), D(t)$ and $h(t,\tau)$ are determined above.

The equation (18) represents an equation of the mechanical quadrature method for SIE (11). It is easy to verify as in the proof of theorem 1, that from the conditions of theorem 2 follows the execution of conditions of theorem 8.4 from [8,p.77],arranging the use of mechanical quadrature method to SIE (11) . Therefore , by virtue of cited theorem 8.4., begining with the numbers $n\geq N_2(\geq N_1)$ the equation (18) has the unique solution $v_n(t)\in R_n$, and the following estimation (taking into account the inclusion) $v(t)\in C(\Gamma)$ is true:

$$\|v-v_n\|=O\left(\frac{1}{n^{\alpha}}\right)+O(\omega(f;\frac{1}{n}))+O(\omega^{\tau}(h;\frac{1}{n}))+O(\omega^{t}(h;\frac{1}{n})). \quad (19)$$

Then for such n the equation (17) has the unique solution $x_n(t)\in\dot{X}_n$, which is connected with $v_n(t)$ by the formulae (15). Besides,as the exact solutions $x(t)$ and $v(t)$ of the problem (2)-(3) and equation (11) respectively are connected by the formulae (12), then we shall obtain (9),taking into account (19). The theorem 2 is proved.

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