Meadows in the Heptgrid and Possible Generalizations

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Abstract

In this paper, we summarize the results about flowers in the heptagrid, the tessellation $(7, 3)$ of the hyperbolic plane, results presented for MCU'2024 at Nice, France. In particular, we define meadows, more precisely n-meadows for any integer n at least 3. We recall that for each integer n at least three, there is an n-meadow on the heptagrid. We suggest that those results can be generalized to tilings $(p, 3)$ of the hyperbolic plane, where $p \geq 7$.

Keywords: hyperbolic geometry, tilings, flowers, meadows.

1 Introduction

The paper investigates particular tilings inside the tessellation $(7, 3)$ of the hyperbolic plane. Section 2 recalls the basics of hyperbolic geometry needed to understand the paper. In the same section, we define the heptagrid, the tessellation $(7, 3)$ of the hyperbolic plane as already mentioned. In Section 3, we define the flowers and the meadows announced in the abstract. In Section 5, we sketchily prove the results and consider the possible generalization of those constructions and of those related results to the tilings $(p, 3)$ of the hyperbolic plane. Section 6 temporarily concludes that piece of research.

2 The Tessellations $(p, 3)$, $p \geq 7$

Sub-section 2.1 defines the model of the hyperbolic plane we use in the paper, Poincaré's disc. Sub-section 2.2 defines the tessellations $(p, 3)$. Sub-section 2.3 defines the navigation tools which allow us to describe the structures presented in Section 3 and to present the ideas of the proofs of the Theorem given in Section 5.

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doi:10.56415/csjm.v32.21

2.1 Poincaré's Disc, a Model of Hyperbolic Geometry

Closing a long history of the impossible search to prove the fifth axiom of Euclid's *Elements*, namely, as stated by Playfair in the 17$^{\text{th}}$ century, in the Euclidean plane, the existence of a single straight line passing through a given point A and parallel to a given straight line ℓ which does not pass through A, hyperbolic geometry appeared by the end of the first third of 19$^{\text{th}}$ century,

Independently of each other, Lobachevsky and Bolyai discovered what was later called hyperbolic geometry in which through a point A out of a straight line ℓ there are two parallels to ℓ passing through A and infinitely many straight lines passing through A which do not cut ℓ. Poincaré defined a model of that geometry in the disc and in the half-plane. In this paper, we shall use Poincaré's disc to illustrate the notions we shall introduce and study. What we just said about the hyperbolic plane is illustrated by Figure 1. An easy presentation of that geometry can be found in [1].

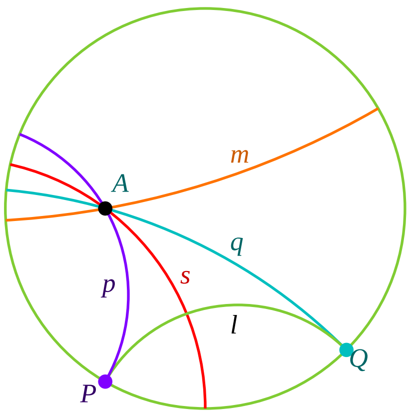

Figure 1. *Poincaré's disc. Straight lines are the traces in the unit disc U of the circles which are orthogonal to ∂U, the unit circle. The straight line s passes through the point A out of the straight line ℓ and cuts that line. The straight lines p and q also pass through A and are both parallel to ℓ. The straight line m is an example of a line passing through A and which does not cut ℓ.*

2.2 The Tessellation $(p, 3)$, $p \geq 7$

In a geometric plane, a tessellation is a tiling which is obtained from a regular convex polygon P by the reflection of P in its sides and, recursively of the images in their sides.

In the Euclidean plane, up to homogeneous dilatations, three tessellations are possible: those based upon the equilateral triangle, the square, and the regular hexagon.

The hyperbolic plane contains infinitely many tessellations thanks to a theorem of Poincaré which states that there is a tessellation of the hyperbolic plane based on any triangle whose angles are $\frac{\pi}{p}$, $\frac{\pi}{q}$, and $\frac{\pi}{2}$ provided that the natural integers p and q satisfy the inequality $\dfrac{1}{p}+\dfrac{1}{q}<\dfrac{1}{2}$ which is the necessary condition for those angles to be angles of a triangle in the hyperbolic plane. As a corollary of the theorem, we get that there is a tessellation based on a regular convex polygon P with p sides and with the angle $\dfrac{2\pi}{3}$ as interior angle at each vertex of P, provided that p and 3 satisfy the above inequality which entails that $p \geq 7$. Such a tessellation is denoted by $(p, 3)$.

The tessellation $(7, 3)$ is called the **heptagrid**. A representation of the heptagrid in Poincaré's disc is given by the left-hand picture of Figure 2.

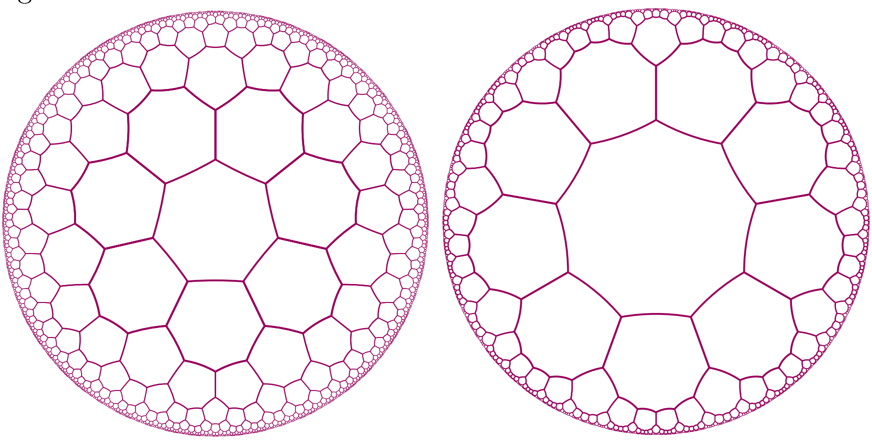

Figure 2. *To the left, the heptagrid. To the right, the tessellation* $(9, 3)$.

2.3 Navigation in $(p, 3)$, $p \geq 7$

We now present tools to navigate in a tessellation $(p, 3)$, where $p \geq 7$. To this aim, we introduce a way to split the tiling which is illustrated by Figure 3. The key point is that mid-points of sides of the tiles pairwise sharing a common vertex lie on the same straight line we call a mid-point line. Two such rays starting from the same mid-point define an acute angle α, and we call **sector** the set of tiles whose centre belongs to α. A sector itself can be split into a tile, $p-5$ images of a sector, and a third set of tiles we call a **strip**; see the right-hand side of Figure 3 which illustrates the case of the heptagrid.

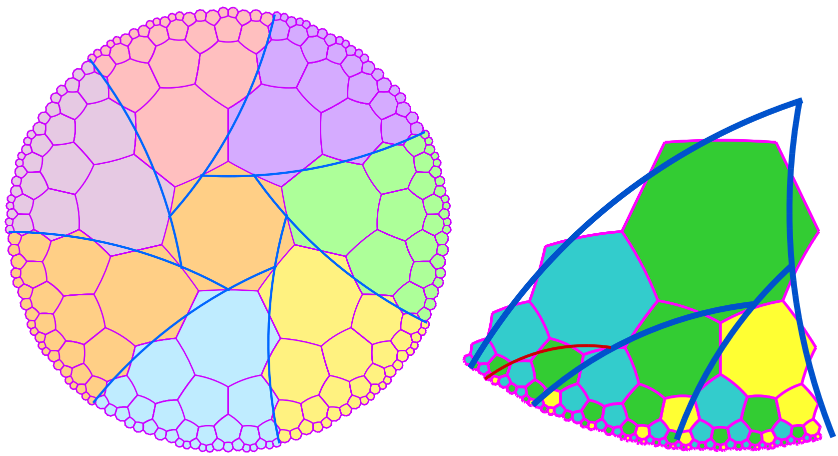

Figure 3. *Splittings in the heptagrid. To the left: seven sectors around a tile. To the right: in a sector, two sub-sectors and a strip. In a strip: one sub-sector and a sub-strip.*

The strip itself, as illustrated by the figure, can also be split into a tile, $p-6$ images of a sector, and an image of a strip. The tile which is the closest to the vertex where the rays defining the sector meet is called the **head** of the sector. Similarly, we define the **head** of a strip. From those splittings which can recursively be repeated, we define rules showing the connection between heads of sectors and strips:

$$W \to BW^{p-5} \qquad B \to BW^{p-6}, \tag{1}$$

where W is attached to a head of a sector and B is attached to the head of a strip. From those considerations, it can be proved that the tiles of a sector are in bijection with the nodes of (1). The proof of the bijection can be illustrated by Figure 4 for the case of the heptagrid.

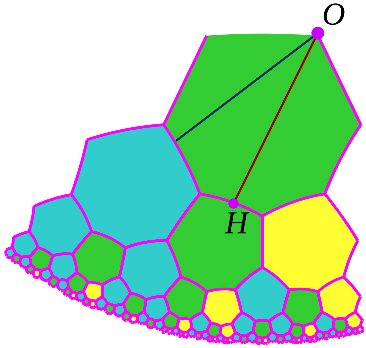

Figure 4. *Proof of the bijection between the tiles of a sector and the nodes of the tree in the case of the heptagrid. The surjection is plain. The injection comes from the fact that from one level of the tree to the next one, the distance of the tiles from the vertex O is growing.*

From the rules (1), it is easy to prove, see [2], that on the level n of the tree defined by those rules, there are m_{2n+1} nodes exactly where m_q is the q^{rmth} term of the sequence whose first two initial terms m_0 and m_1 are both 1 and the general term can be written as follows: $m_{q+2} = (p-4)m_{q+1} - m_q$ with q being non negative. The sequence is called a **pseudo-metallic sequence** and the tree above defined is called a **pseudo-metallic tree**. It is called **Fibonacci tree** in the case when $p = 7$ as far as the pseudo-metallic sequence is a subsequence of the Fibonacci sequence when $p = 7$. The nodes of a pseudo-metallic tree can be numbered starting from 1 attached to the root of the tree and then going from one level to the next one and, on each level, from the leftmost node to the rightmost one. We obtain coordinates in the heptagrid by attaching to a tile the number of the node to which it corresponds in the pseudo-metallic tree. We can write those numbers in the numbering systems obtained from the pseudo-metallic sequence. Such writings are called **codes** of a tile. To get the main property of the codes, we rewrite the rules (1) as:

$$W_\ell \to BW^{p-4}W_\ell W_r,\; W_r \to BW^{p-5}W_\ell W_r W_r,\; B \to BW^{p-5}W_\ell W_r. \quad (1b)$$

Note that in $(1b)$, two kinds of W-nodes are introduced. The reason is the following one: if ν is a code of a B- or a W_ℓ-tile, $\nu 0$ is the code of its last son. If ν is a code a W_r-tile, $\nu 0$ is the code of its penultimate son. These properties are proved in the quoted works, see Figure 5.

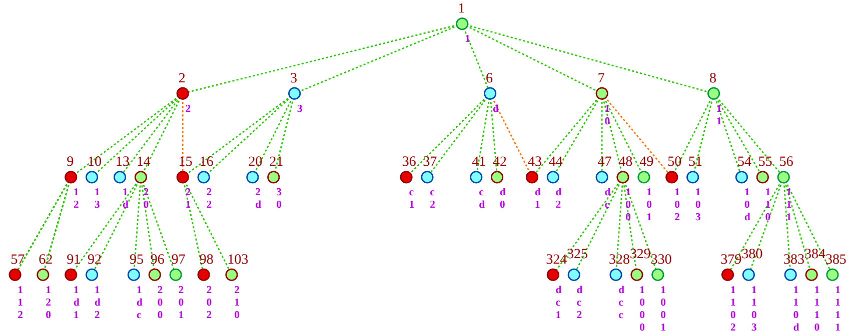

Figure 5. *The pseudo-metallic tree down to level* 3. *In that figure,* $p=9$ *and* $d=p-3$.

3 Flowers and Meadows in the Heptagrid

Let us now turn to the definition of the notions considered in this paper.

By definition, a **flower** of the heptagrid is a set of tiles T_i, $i \in [0..7]$ such that the T_i's with $i > 0$ are the neighbours of T_0. We say that tiles U and V of the heptagrid are neighbours of one another if both tiles share a common side. The tile T_0 is called the **centre** of the flower and its neighbours are also called its **petals**.

A **path** from tile A to tile B is defined as a finite sequence of tiles $\{T_i\}_{i\in[0..n]}$ such that $T_0 = A$, $T_n = B$ and T_{i+1} is a neighbour of T_i for $0 \leq i < n$. We say that n is the length of the path. The **distance** from tile A to tile B, denoted by $\text{dist}(A, B)$, is defined as the shortest length among those of the paths from A to B. A **shortest path** from A to B, is a path whose length is $\text{dist}(A, B)$. We easily get that the distance defined in that way satisfies the triangular inequality so that it is a distance in the topological meaning. A **circle** around A of radius n is the set of tiles whose distance to A is precisely n. A **ball** around A of radius n is the set of tiles whose distance to A is at most n. Accordingly, a flower is a ball around its centre of radius 1.

By definition, a **meadow** is a tiling of the heptagrid in which there are infinitely many flowers. Two flowers F_0 and F_1 are called **neighbours** if and only if there is no centre of a third flower on the shortest path from the centre of F_0 to that of F_1. An n-**meadow** is a meadow in which all neighbouring flowers have the same distance n between their centres. From now on, we define a **meadow** to be an n-meadow for some integer n with $n \geq 3$.

4 Results

The goal of the present Section is to prove the following result:

Theorem **1.** *For any integer n with $n \geq 3$, there is an n-meadow in the heptagrid.*

Note that 3 is the minimal distance between two flowers. It was proved in [2] that for any n, n-balls tile the heptagrid. From that result, we get that the theorem is true for odd values of n. Figure 6 illustrates the case when $n = 3$, *i.e.*, when flowers are contiguous and do not overlap, covering the whole heptagrid.

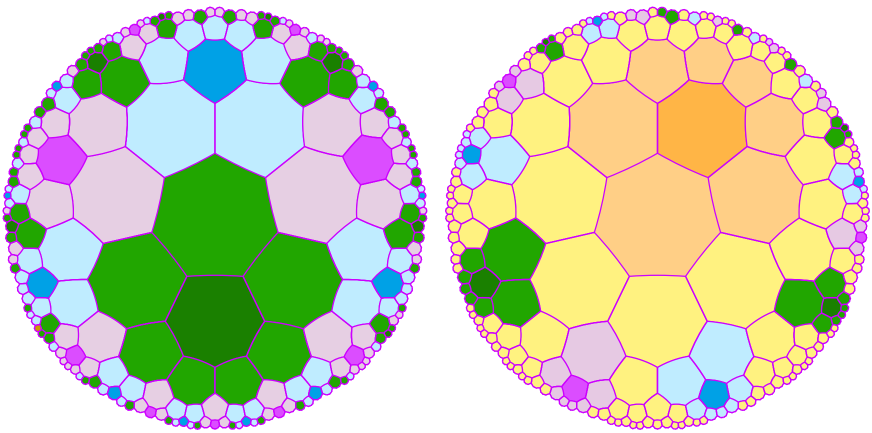

Figure 6. *To the left, flowers in a 3-meadow in which each flower is surrounded by 14 ones. To the right, an image of a 4-meadow. Each flower is surrounded by 21 flowers.*

From the definition, it follows that:

LEMMA **1.** *Let π be a shortest path from a tile A to a tile B. If C and D are two tiles of π, the sub path of π from C to D is a shortest path from C to D.*

The proof of Theorem 1 makes use of the following result also proved in [2]:

LEMMA **2.** *In the Fibonacci tree, a shortest path from its root to a node νs given by following the part of the branch which starts from the root and which passes through ν, such a branch being unique in the tree.*

LEMMA **3.** *Assume that in a sector identified with the Fibonacci tree attached to it, the sides of each tile are numbered in such a way that side* 1 *is shared by the father and, for the root, side 3 is shared by its B-son. Then the B-, W-sons of a B-node ν share the 4, 5 sides of ν, respectively; the B- and W-sons of a W-node μ share the 3 and 4, 5 sides of μ, respectively.*

The proof goes as follows: we already know that when n is odd, the theorem is a corollary of the proposition proved in [2] that for any n, it is possible to tile the heptagrid with balls of radius n only. Indeed, the distance of the centers of two adjacent balls of radius k is $2k+1$. It follows from the definition of the distance.

LEMMA **4.** *Let σ be the shortest path from A to B. From the definition of the shortest path of length ℓ from a tile U to another tile V, we get that for any number k with $k \in [0..\ell]$, there is a tile W such that $dist(U, W) = k$ and there is also a tile X such that $dist(X, V) = k$.*

The proof of the lemmas 1, 3 and 4 can be found in the proceedings of MCU'2024; when they will appear, look at:

https://link.springer.com/conference/mcu.

From those lemmas, Theorem 1 can easily be proved, see the mentioned proceedings.

The proof also provides us with an algorithm to construct an n-meadow for a given even n, $n \geq 4$. A 4-meadow is illustrated by Figure 7. Algorithm 1 gathers the centres of flowers which lie on a circle C around a tile P of radius n as a sequence $\mathcal{G}$ of tiles so that two consecutive terms are at distance n from each other, the last term and the first one are also at distance n from each other. Algorithm 2 constructs an n-meadow as a sequence $\mathcal{F}$ of the centers of the flowers defining that meadow. We define T_0 a tile chosen at random in the heptagrid, and we decide that T_0 is the center of a flower. We set $P := T_0$, and we define C as the circle around P of radius n.

Algorithm **1. collect**

$\mathcal{G} := \{U_i\}_{i\in[1..k]}$;
with $U_i \in C$, *dist*$(U_i, U_{i+1}) = n$, $i \in [1..k{-}1]$; *dist*$(U_1, U_k) = n$;

Algorithm **2. meadow**

begin loop $\mathcal{F} := \{T_0\}$; $P := T_0$; $\mathcal{G} :=$ **collect**; $\mathcal{F} = \mathcal{F} \bigcup \{\mathcal{G} \backslash \mathcal{F}\}$;
number $\mathcal{F}$; **if** $P = \mathcal{F}_i$ **then** $P := \mathcal{F}_{i+1}$; *define* C;
end loop;

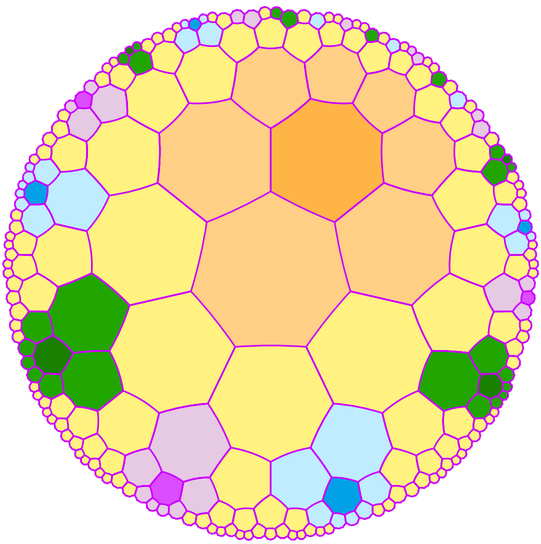

Figure 7. *Image of a* 4*-meadow in the heptagrid. Note the* 21 *flowers around the orange one.*

It can be noticed that Algorithm 2 makes use of Algorithm 1. That latter algorithm uses the variables P and C, modifying them for the next execution of the loop. Note that the loop is infinite which makes it certain that P eventually runs other all $\mathcal{F}$ so that the meadow will eventually fill up the heptagrid.

5 Generalization

During the discussion after my talk at MCU'2024, I was asked whether the result about meadows in the heptagrid could be generalised to other tilings of the hyperbolic plane. As it seemed to me that it was possible, I said yes. Here, I mention that it can be proved in the tessellations $(p, 3)$. As far as the vertex angle in a regular convex heptagon is $\dfrac{2\pi}{3}$, most of the arguments held in the heptagrid can be extended to the tessellations $(p, 3)$. Figure 8 illustrates flowers and a 3-meadow in the tessellation, where $p = 9$.

To see how arguments can be extended, we remain with the case when n is even, *i.e.*, when $n = 2p{+}1$, with $p \geq 2$. In the case of the heptagrid, the idea of the proof was to consider a Fibonacci tree $\mathcal{F}$ rooted at the centre C of one flower. We fix a tile T_0 on the border of a circle around C of radius n. From T_0, we climb along the branch leading form C to T_0 up to the level $p{-}1$ of $\mathcal{F}$, reaching a tile ν. From ν, we take a branch of $\mathcal{F}$ reaching the border at T_1 which is the centre of a flower whose distance to C is n and whose distance to T_0 is also n. The branch taken from ν is the rightmost branch of the subtree of $\mathcal{F}$ rooted at ν. It is easy to show that the path obtained by joining the path from T_0 to $T\nu$ to that from ν to T_1 is a shortest path. In that way, starting from T_1 and repeating the construction, we arrive to the $\nu{+}1$ on the level $p{-}1$ of $\mathcal{F}$.

The same proof holds for the tessellation $(p, 3)$. Here the sides of a tile are numbered from 1 up to p. Lemma 3 must be changed by 3 up to $p{-}2$ for a W-node and by 4 up to $p{-}2$ for a B-one. The following property can be proved easily:

LEMMA **5.** *Let $T_{i i\in[0..k-1]}$ be a path π of length k from T_0 to T_{k-1}. Let $\mathcal{F}$ be the pseudo-metallic tree rooted at T_0 such that the subpath from T_0 to T_1 is the side 1 of T_0. Assume that, for all $i \in [1..k-1]$, T_i is seen from T_{i-1} from its j-side with $j \in [a..k-2]$, where $a = 4$ unless T_{i-1}*

is a W-node, in which case $a = 3$ is possible. Then, π is the shortest path.

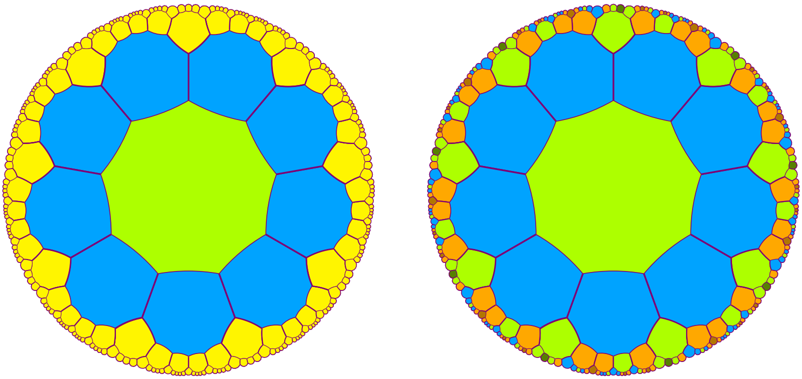

Figure 8. *To the left, a flower in the tessellation $(p, 3)$, where $p = 9$. To the right, a 3-meadow in that tessellation. Note that there are 36 flowers around each flower.*

Accordingly, we may state the following assertion:

THEOREM **2.** *In each tessellation $(p, 3)$ with $p \geq 7$ and for any n with $n \geq 4$, there is an n-meadow.*

6 Further Remarks

A lot of questions arise. In [2], a stronger result is proved about balls in the heptagrid:

THEOREM **3.** *Let $\mathcal{I}$ be a non empty set of positive integers. Then there is a tiling $\mathcal{T}$ of the heptagrid consisting of balls B_n with $n \in \mathcal{I}$ such that, for any $n \in \mathcal{I}$, infinitely many balls of $\mathcal{T}$ have a radius n with $n \in \mathcal{I}$.*

The idea of the proof is that in a ball $\mathcal{B}$ of radius n around A, there are many tiles T at the distance n from A such that there is a side in each tile supported by a line ℓ so that $\mathcal{B}$ is contained in the half-plane defined by ℓ which also contains T. Many of those tiles contain two contiguous such sides so that we can conclude that a ball of radius n is contained in the angle defined by a vertex V of a tile T and two rays issued from V containing the sides of T meeting at V. It can be proved that in a tiling by balls, there is an angle $\frac{5\pi}{21}$ at each extremal vertex of the contact between two balls, that angle being mainly outside both balls, so that there is room for any ball in that angle which is the angle between the rays defining a sector. Note that the remark provides us with an algorithm to tile the heptagrid with such balls.

Consider a tiling of the heptagrid by balls whose radiuses belong to a set $\mathcal{I}$ of integers not smaller than 3 and such that, for any $n \in \mathcal{I}$, the tiling contains infinitely many balls of radius n. Denote such a tiling by $\mathcal{T}(\mathcal{I})$. The distances between the centres of those balls are of the form $n+m+1$, where $n, m \in \mathcal{I}$. Let $\mathbb{C}(\mathcal{T}, \mathcal{I})$ denote the set of the centres of the balls in $\mathcal{T}(\mathcal{I})$. We call **wild meadow directed by $\mathcal{T}(\mathcal{I})$** a set of flowers such that the set of their centres is $\mathbb{C}(\mathcal{T}, \mathcal{I})$.

Clearly, from Theorem 3, we can conclude:

Corollary **1.** *For any tiling by balls $\mathcal{T}(\mathcal{I})$, where $\mathcal{I}$ is a set of integers not smaller than* 3*, there is a wild meadow in the heptagrid directed by $\mathcal{T}(\mathcal{I})$.*

Again in [2], it is shown that three colours are enough to colour the flowers of a 3-meadow such that two neighbouring flowers do not share the same colour. That result does not extend to n-meadows for any $n \geq 3$ as far as it does not hold for a 4-meadow in the heptagrid. Indeed, in a 4-meadow of the heptagrid, around a flower F, there are 21 closest flowers so that, with three colours, at least two flowers around F that would be at distance 4 from each other would share the same colour.

Another corollary of the existence of the 4-meadow is that considering the balls of radius 2 around the centre of each flower, those balls share a common tile if considering three flowers pairwise at distance 4.

We remark that the coloration problem is more complex in the tessellations $(p, 3)$.

And so, there is at least some work ahead.

Acknowledgements

I thank Rudi Freund and Sergey Verlan for inviting me to write the present paper.

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Received October 05, 2024
Revised October 23, 2024
Accepted October 23, 2024

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