Computer Science Journal of Moldova, vol.**32**, no.1(94), 2024

Outer independent total double Italian domination number

Seyed Mahmoud Sheikholeslami, Lutz Volkmann

Abstract

If G is a graph with vertex set $V(G)$, then let $N[u]$ be the closed neighborhood of the vertex $u \in V(G)$. A total double Italian dominating function (TDIDF) on a graph G is a function $f : V(G) \to \{0, 1, 2, 3\}$ satisfying (i) $f(N[u]) \geq 3$ for every vertex $u \in V(G)$ with $f(u) \in \{0, 1\}$ and (ii) the subgraph induced by the vertices with a non-zero label has no isolated vertices. A TDIDF is an outer-independent total double Italian dominating function (OITDIDF) on G if the set of vertices labeled 0 induces an edgeless subgraph. The weight of an OITDIDF is the sum of its function values over all vertices, and the outer independent total double Italian domination number $\gamma^{oi}_{tdI}(G)$ is the minimum weight of an OITDIDF on G. In this paper, we establish various bounds on $\gamma^{oi}_{tdI}(G)$, and we determine this parameter for some special classes of graphs.

Keywords: (Total) double Italian domination. Outer independent (total) double Italian domination.

MSC 2010: 05C69.

1 Introduction

For notation and graph theory terminology, we in general follow Haynes, Hedetniemi and Slater [11]. The starting point of Roman and Italian domination in graphs, as well as all its variants, can be attributed to the mathematical formalization of a defensive model of the Roman Empire described by Stewart in [18]. The formal definition was given by Cockayne et al. [9] as follows. Given a graph $G = (V, E)$,

doi:10.56415/csjm.v32.02

with vertex set $V = V(G)$ and edge set $E = E(G)$, a *Roman dominating function* (RDF) on G is a function $f : V \to \{0, 1, 2\}$, that assigns labels to vertices of G, such that every vertex labeled with 0 must be adjacent to a vertex with a label 2. The sum of all vertex labels, $w(f) = f(V) = \sum_{v\in V} f(v)$, is called the weight of the RDF f and the minimum weight over all possible RDF's is the *Roman domination number*, $\gamma_R(G)$, of the graph G. For the sake of simplicity, an RDF with minimum weight is known as a $\gamma_R(G)$-function or a γ_R-function of G. Clearly, there is a one-on-one relation between Roman dominating functions and the set of subsets $\{V_0^f, V_1^f, V_2^f\}$ of $V(G)$, where $V_i^f = \{v \in V \mid f(v) = i\}$. That is why an RDF f is usually represented as $f = (V_0^f, V_1^f, V_2^f)$ or simply by (V_0, V_1, V_2), if there is no possibility of confusion. An *Italian dominating function* (IDF) on a graph G is defined in [5] as a function $f : V \to \{0, 1, 2\}$ satisfying $f(N(u)) \geq 2$ for each vertex u with $f(u) = 0$. An IDF $f = (V_0, V_1, V_2)$ is an *outer independent total Italian dominating function* (OITIDF) if V_0 is an independent set and $G[V_1 \cup V_2]$ is a subgraph without isolated vertices. The *outer independent total Italian domination number* $\gamma_{tI}^{oi}(G)$ equals the minimum weight of an OITIDF on G, and an OITIDF of G with weight $\gamma_{tI}^{oi}(G)$ is called a $\gamma_{tI}^{oi}(G)$-function. For more details on Roman and Italian domination and its variants, the reader can consult the following book chapters [6], [7] and the survey [8].

In this paper, we only consider simple graphs $G = (V, E)$ with vertex set $V = V(G)$ and edge set $E = E(G)$. The *size* of a graph is its number of edges and its *order* is the number of elements in V. The *open* (resp. *closed*) *neighborhood* $N(v)$ (resp. $N[v]$) of a vertex v is the set $\{u \in V(G) \mid uv \in E(G)\}$ (resp. $N[v] = N(v) \cup \{v\}$). The number of adjacent vertices with v is its *degree*, $\deg(v) = |N(v)|$. We denote by $\delta = \delta(G)$ (resp., $\Delta = \Delta(G)$) the *minimum* (resp., *maximum*) *degree* of a graph G. A *leaf* in a graph is a vertex whose degree is equal to 1 and its neighbor is a *support vertex*. Let $\alpha(G)$ and $\beta(G)$ be the *independence number* and the *covering number* of a graph, respectively. If G is a graph of order n without isolated vertices, then $\alpha(G) + \beta(G) = n$.

We denote by P_n the *path graph* of order n, and by C_n the *cycle graph* of order n. The *corona* of a graph G denoted $G \circ K_1$, is the graph formed from a copy of G by adding for each $v \in V$, a new vertex v'

and the edge vv'.

A function $f : V \to \{0, 1, 2, 3\}$ is an *outer independent total double Roman dominating function* (OITDRDF) on a graph G if it meets the following requirements:

- Every vertex $v \in V$ with $f(v) = 0$ is adjacent to either a vertex w such that $f(w) = 3$ or to two vertices $w, w' \in V$ with $f(w) = f(w') = 2$.
- Every vertex $v \in V$ with $f(v) = 1$ is adjacent to a vertex $w \in V$ with $f(w) \geq 2$.
- The set of vertices with weight 0 induces an edgeless subgraph and the set of vertices with positive weight induces an isolated-free vertex subgraph.

The *outer independent total double Roman domination number* (*OITDRD-number for short*) $\gamma^{oi}_{tdR}(G)$ equals the minimum weight of an OITDRDF on G, and an OITDRDF of G with weight $\gamma^{oi}_{tdR}(G)$ is called a $\gamma^{oi}_{tdR}(G)$-function. The outer independent total double Roman domination was investigated by Teymourzadeh and Mojdeh [17]; Abdollahzadeh Ahangar, Chellali, Sheikholeslami, and Valenzuela-Tripodoro [2]; and Sheikholeslami and Volkmann [16].

In [12], Mojdeh and Volkmann defined a variant of double Roman domination, namely double Italian domination. A *double Italian dominating function* (DIDF) on a graph G is a function $f : V(G) \to \{0, 1, 2, 3\}$ satisfying $f(N[u]) \geq 3$ for every vertex $u \in V(G)$ with $f(u) \in \{0, 1\}$. According to Shao, Mojdeh, and Volkmann [15], a DIDF on a graph G with no isolated vertices is a *total double Italian dominating function* (TDIDF) if the subgraph induced by the vertices of the positive label has no isolated vertices. The *total double Italian domination number* $\gamma_{tdI}(G)$ is the minimum weight of a TDIDF on G. A TDIDF on G with weight $\gamma_{tdI}(G)$ is called a $\gamma_{tdI}(G)$-*function.* An *outer independent double Italian dominating function* (OIDIDF) of a graph G is a DIDF for which the vertices with weight 0 are independent. The *outer independent double Italian domination number* $\gamma_{oidI}(G)$ is the minimum weight of an OIDIDF on G (see [1], [3], [4], [19]). An OIDIDF on G with weight $\gamma_{oidI}(G)$ is called a $\gamma_{oidI}(G)$-*function.*

Our aim in this work is to continue the study of a new variation of Italian domination, namely the outer independent total double Italian domination. A TDIDF is an *outer independent total double Italian dominating function* (OITDIDF) on G if the set of vertices with weight 0 induces an edgeless subgraph. The *outer independent total double Italian domination number* (*OITDID-number for short*) $\gamma_{tdI}^{oi}(G)$ equals the minimum weight of an OITDIDF on G, and an OITDIDF of G with weight $\gamma_{tdI}^{oi}(G)$ is called a $\gamma_{tdI}^{oi}(G)$-function.

In this paper, we present basic properties and sharp bounds for the outer independent total double Italian domination number. In addition, we determine this parameter for special classes of graphs.

If G is a graph without isolated vertices, then the definitions lead to $\gamma_{tdI}(G) \leq \gamma_{tdI}^{oi}(G) \leq \gamma_{tdR}^{oi}(G)$.

We make use of the following results in this paper.

Theorem 1. *[10],[13] For a graph G with even order n and no isolated vertices, $\gamma(G) = n/2$ if and only if the components of G are the cycle C_4, or the corona $H \circ K_1$ for any connected graph H.*

Proposition 2. *[15] If C_n is a cycle of length $n \geq 3$, then $\gamma_{tdI}(C_n) = n$. If P_n is a path of order $n \geq 2$, then $\gamma_{tdI}(P_n) = n + 2$ when $n \equiv 1 \pmod 3$ and $\gamma_{tdI}(P_n) = n + 1$ otherwise.*

Proposition 3. *[16] If G is a graph of order n without isolated vertices, then $\gamma_{tdR}^{oi}(G) \leq 2n - \Delta(G)$.*

Proposition 4. *[2] For $n \geq 3$,*

$$(i)\ \gamma_{tdR}^{oi}(P_n) = \begin{cases} 6 & \text{if } n = 4, \\ \lceil \frac{6n}{5} \rceil & \text{otherwise.} \end{cases}$$

(ii) $\gamma_{tdR}^{oi}(C_n) = \lceil \frac{6n}{5} \rceil$.

The next lemma is easy to see, and therefore its proof is omitted.

Lemma 5. *Let G be a graph without isolated vertices. If v is a support vertex and u a leaf neighbor of v, then for any OITDIDF f of G, we have $f(u) + f(v) \geq 3$ and $f(v) \geq 1$.*

2 Special classes of graphs

In this section, we determine the outer independent total double Italian domination number for cycles, paths, and complete t-partite graphs.

Proposition 6. *If C_n is a cycle of length $n \geq 3$, then $\gamma^{oi}_{tdI}(C_n) = n$. If P_n is a path of order $n \geq 2$, then $\gamma^{oi}_{tdI}(P_n) = n + 2$ when $n \equiv 1 \pmod 3$ and $\gamma^{oi}_{tdI}(P_n) = n + 1$ otherwise.*

Proof. Define the function $f : V(C_n) \to \{0, 1, 2, 3\}$ by $f(x) = 1$ for each vertex $x \in V(C_n)$. Clearly, f is an OITDIDF on C_n of weight n, and thus $\gamma^{oi}_{tdI}(C_n) \leq n$. Using Proposition 2 we obtain

$$n = \gamma_{tdI}(C_n) \leq \gamma^{oi}_{tdI}(C_n) \leq n,$$

and thus $\gamma^{oi}_{tdI}(C_n) = n$.

Let now $P_n = v_1 v_2 \ldots v_n$. If $n = 3t + 1$ with an integer $t \geq 1$, then define f by $f(v_1) = f(v_n) = 2$ and $f(v_i) = 1$ for $2 \leq i \leq n - 2$. Then f is an OITDIDF on P_n of weight $n + 2$, and thus $\gamma^{oi}_{tdI}(P_n) \leq n + 2$. Using Proposition 2, we obtain

$$n + 2 = \gamma_{tdI}(P_n) \leq \gamma^{oi}_{tdI}(P_n) \leq n + 2,$$

and so $\gamma_{tdI}(P_n) = n + 2$ when $n \equiv 1 \pmod 3$.

Let next $n = 3t$ with an integer $t \geq 1$. Define f by $f(v_{3i}) = 0$ for $1 \leq i \leq t - 1$, $f(v_{3t}) = 1$, $f(v_{3i-1}) = 2$ for $1 \leq i \leq t$, and $f(v_{3i-2}) = 1$ for $1 \leq i \leq t$. Then f is an OITDIDF on P_n of weight $n + 1$, and thus $\gamma^{oi}_{tdI}(P_n) \leq n + 1$. It follows from Proposition 2 that

$$n + 1 = \gamma_{tdI}(P_n) \leq \gamma^{oi}_{tdI}(P_n) \leq n + 1$$

and so $\gamma_{tdI}(P_n) = n + 1$ in this case. Finally, let $n = 3t + 2$ with an integer $t \geq 0$. Define f by $f(v_{3i}) = 0$ for $1 \leq i \leq t$, $f(v_{3i-1}) = 2$ for $1 \leq i \leq t + 1$, and $f(v_{3i-2}) = 1$ for $1 \leq i \leq t + 1$. Then f is an OITDIDF on P_n of weight $n + 1$, and thus $\gamma^{oi}_{tdI}(P_n) \leq n + 1$. Again Proposition 2 leads to the desired result. □

Proposition 4 implies $\gamma^{oi}_{tdR}(C_n) = \lceil \frac{6n}{5} \rceil$ and $\gamma^{oi}_{tdR}(P_n) = \lceil \frac{6n}{5} \rceil$ for $n \geq 5$. Therefore, Proposition 6 shows that the difference $\gamma^{oi}_{tdR}(G) - \gamma^{oi}_{tdI}(G)$ can be arbitrarily large.

Proposition 7. *If $K_{p.q}$ is the complete bipartite graph with $3 \leq p \leq q$, then $\gamma_{tdI}^{oi}(K_{p,q}) = p+2$.*

Proof. Let X, Y be a bipartition of $K_{p,q}$ with $|X| = p$ and $|Y| = q$, and let $f = (V_0, V_1, V_2, V_3)$ be an OITDIDF on $K_{p,q}$. If $|V_0| = 0$, then $\gamma_{tdI}^{oi}(K_{p,q}) \geq p+q > p+2$. So let now $|V_0| \geq 1$, and assume, without loss of generality, that $V_0 \subseteq Y$. This implies $f(X) \geq p$ and $f(Y) \geq 1$. If $f(X) \geq p+1$, then $\gamma_{tdI}^{oi}(K_{p,q}) \geq f(X) + f(Y) \geq p+2$. In the remaining case that $f(X) = p$, we deduce that $f(x) = 1$ for each $x \in X$, and therefore $f(Y) \geq 2$. Also in this case we obtain $\gamma_{tdI}^{oi}(K_{p,q}) \geq p+2$.

Conversely, let $w \in Y$. Define the function g by $g(x) = 1$ for $x \in X$, $g(w) = 2$ and $g(y) = 0$ for $y \in Y \setminus \{w\}$. Then g is an OITDIDF on $K_{p,q}$ of weight $p+2$. Hence $\gamma_{tdI}^{oi}(K_{p,q}) \leq p+2$ and so $\gamma_{tdI}^{oi}(K_{p,q}) = p+2$. □

Note the following completion to Proposition 7.

Proposition 8. *If $q \geq 2$, then $\gamma_{tdI}^{oi}(K_{1,q}) = 4$. If $q \geq 3$, then $\gamma_{tdI}^{oi}(K_{2,q}) = 5$.*

Proposition 9. *Let $G = K_{n_1,n_2,\ldots,n_t}$ be the complete t-partite graph with $n_1 \leq n_2 \leq \cdots \leq n_t$, $n = n_1 + n_2 + \cdots + n_t$ and $t \geq 3$. If $n_1 = 1$ and $t = 3$, then $\gamma_{tdI}^{oi}(G) = n+1-n_t$ and $\gamma_{tdI}^{oi}(G) = n - n_t$ otherwise.*

Proof. Let $X_1, X_2, \ldots, X_t$ be the partite sets of G with $|X_i| = n_i$ for $1 \leq i \leq t$. If $f = (V_0, V_1, V_2, V_3)$ is an OITDIDF on G, then $|V_0| \leq n_t$, and therefore $\gamma_{tdI}^{oi}(G) \geq n - n_t$. If $n_1 \geq 2$ or $t \geq 4$, then the function g with $g(x) = 0$ for $x \in X_t$ and $g(x) = 1$ for $x \in V(G) \setminus X_t$, is an OITDIDF on G of weight $n - n_t$. Hence $\gamma_{tdI}^{oi}(G) \leq n - n_t$ and so $\gamma_{tdI}^{oi}(G) = n - n_t$ in this case.

Let now $t = 3$ and $n_1 = 1$ with $X_1 = \{w\}$. If $|V_0| = 0$, then $\gamma_{tdI}^{oi}(G) \geq n \geq n+1-n_t$. So let now $|V_0| \geq 1$, and assume, without loss of generality, that $V_0 \subseteq X_3 = X_t$. This implies $f(X_1) \geq n_1$ and $f(X_2) \geq n_2$. If $f(X_3) \geq 1$, then we deduce that $\gamma_{tdI}^{oi}(G) \geq f(X_1) + f(X_2) + f(X_3) \geq n_1 + n_2 + 1 = n+1-n_3 = n+1-n_t$. Let now $f(X_3) = 0$. If $f(X_2) \geq n_2+1$, then $\gamma_{tdI}^{oi}(G) \geq f(X_1)+f(X_2)+f(X_3) \geq n_1 + n_2 + 1 = n+1-n_t$. In the remaining case that $f(X_2) = n_2$, we deduce that $f(x) = 1$ for each $x \in X_2$, and therefore $f(w) \geq 2$. Also in this case we obtain $\gamma^{oi}_{tdI}(G) \geq n+1-n_t$. Conversely, define the function g by $g(x) = 1$ for $x \in X_2$, $g(w) = 2$ and $g(x) = 0$ for $x \in X_3$. Then g is an OITDIDF on G of weight $n+1-n_t$. Hence $\gamma^{oi}_{tdI}(G) \leq n+1-n_t$ and so $\gamma^{oi}_{tdI}(G) = n+1-n_t$. □

3 Bounds on $\gamma^{oi}_{tdI}(G)$

In this section, we establish upper and lower bounds for the outer independent total double Italian domination number of graphs. We start with a simple result.

Proposition 10. *Let G be a graph of order n. If $\delta(G) \geq 2$, then $\gamma^{oi}_{tdI}(G) \leq n$.*

Proof. If $\delta(G) \geq 2$, then define the function $f : V(G) \to \{0, 1, 2, 3\}$ by $f(x) = 1$ for each vertex $x \in V(G)$. Clearly, f is an OITDIDF on G of weight n, and thus $\gamma^{oi}_{tdI}(G) \leq n$. □

For connected graphs G with minimum degree at least two and $\Delta(G) = n(G) - 1$, we can improve the bound of Proposition 10 slightly.

Proposition 11. *Let G be a connected graph of order $n \geq 4$ with $\delta(G) \geq 2$ and $\Delta(G) = n - 1$. Then $\gamma^{oi}_{tdI}(G) \leq n - 1$. This bound is sharp for complete graphs.*

Proof. Let $v \in V(G)$ be a vertex of degree $n - 1$. If G is complete, then $\gamma^{oi}_{tdI}(G) = n - 1$ by Proposition 9. If G is not complete, then two neighbors, say w and z of v are independent. Since $n \geq 4$ and $\delta(G) \geq 2$, we observe that $(\{w, z\}, V(G) \setminus \{v, w, z\}, \{v\}, \emptyset)$ is an OITDIDF on G of weight $n - 1$. Thus $\gamma^{oi}_{tdR}(G) \leq n - 1$, and the proof is complete. □

Proposition 12. *If G is a graph of order n without isolated vertices, then $\gamma^{oi}_{tdI}(G) \leq \gamma(G) + n \leq n + \lfloor \frac{n}{2} \rfloor$.*

Proof. Given a minimum dominating set D of G, the function f defined by $f(x) = 2$ if $x \in D$ and $f(x) = 1$ otherwise, is an OITDIDF on G, implying that $\gamma^{oi}_{tdI}(G) \leq |D| + n = \gamma(G) + n$. Using Ore's result $\gamma(G) \leq \lfloor \frac{n}{2} \rfloor$ for graphs without isolated vertices, we obtain $\gamma^{oi}_{tdI}(G) \leq n + \lfloor \frac{n}{2} \rfloor$. □

The next result is an immediate consequence of Theorem 1 and Propositions 6 and 12.

Corollary 13. *If G is a graph of order n without isolated vertices, then $\gamma_{tdI}^{oi}(G) \le \frac{3}{2}n$ with equality if and only if the components of G are the corona $H \circ K_1$ for any connected graph H.*

Next we focus on graphs with minimum degree at least three. A set of vertices $P \subseteq V(G)$ is a 2-*packing* of G if the distance in G between any pair of distinct vertices from P is larger than two. The maximum cardinality of a 2-packing of G is the *packing number* of G and is denoted by $\rho(G)$.

Theorem 14. *If G is a graph of order n with $\delta(G) \ge 3$, then $\gamma_{tdR}^{oi}(G) \le n - \rho(G)$. Moreover, this bound is sharp.*

Proof. Suppose that $A = \{v_1, v_2, \ldots, v_\rho\}$ is a 2-packing of G. Define the function f by $f(v_i) = 0$ for $1 \le i \le \rho$ and $f(x) = 1$ for $x \in V(G)\setminus A$. Since $\delta(G) \ge 3$, f is an OITDIDF on G, and thus $\gamma_{tdR}^{oi}(G) \le n - \rho(G)$.

Now let $H_1, H_2, \ldots, H_t$ be isomorphic to the complete graph K_p with $p \ge 4$, and let $a_i, b_i \in V(H_i)$ for $1 \le i \le t$. Define H as $H_1 \cup H_2 \cup \ldots \cup H_t$ together with the edges $b_i a_{i+1}$ for $1 \le i \le t-1$. If $g = (V_0, V_1, V_2, V_3)$ is an OITDIDF on H, then we observe that $|V_0 \cap V(H_i)| \le 1$ for $1 \le i \le t$, and therefore $|V_0| \le t$. It follows that

$$\omega(g) = |V_1| + 2|V_2| + 3|V_3| \ge |V_1| + |V_2| + |V_3| = n(H) - |V_0| \ge n(H) - t.$$

Since $\rho(H) = t$, the bound above implies $\gamma_{tdR}^{oi}(H) \le n(H) - \rho(H)$. Hence we obtain $\gamma_{tdR}^{oi}(H) = n(H) - t = n(H) - \rho(H)$. □

Using Proposition 3, we get the next result.

Theorem 15. *Let G be a connected graph of order $n \ge 2$. Then $\gamma_{tdI}^{oi}(G) \le 2n - \Delta(G)$ with equality if and only if $G = F \circ K_1$, where F is a connected graph with maximum degree $\Delta(F) = n(F) - 1$.*

Proof. Proposition 3 implies that $\gamma_{tdI}^{oi}(G) \le 2n - \Delta(G)$.

If $G = F \circ K_1$, where F is a connected graph with maximum degree $\Delta(F) = n(F) - 1$, then $\Delta(G) = n(F)$ and $\gamma_{tdI}^{oi}(G) = 3n(F) = n(G) + n(F) = 2n(G) - n(F) = 2n(G) - \Delta(G)$.

Conversely assume that $\gamma_{tdI}^{oi}(G) = 2n - \Delta(G)$. Proposition 10 yields $\delta(G) = 1$. If $\Delta(G) = 2$, then G is a path and applying Proposition 6, we obtain $G \in \{P_2, P_4\}$ and so G satisfied the condition. Assume that $\Delta(G) \geq 3$. Let v be a vertex of maximum degree $\Delta(G)$ and let $N(v) = \{u_1, u_2, \ldots, u_t\}$. Assume that $X = V(G) - N[v]$. If $X = \emptyset$, then suppose, without loss of generality, that $\deg(u_1) = 1$ and define the function f on G with $f(v) = 3$, $f(u_1) = f(u_2) = 0$ and $f(x) = 1$ otherwise. Clearly, f is an OITDIDF of G of weight $n < 2n - \Delta(G)$, which is a contradiction. Thus $X \neq \emptyset$. Let $X = \{w_1, w_2, \ldots, w_s\}$. If there is an edge $w_i w_j \in E(G)$, then the function f defined on G by $f(w_i) = 1$, $f(u_1) = \cdots = f(u_t) = 1$ and $f(x) = 2$ otherwise, is an OITDIDF of G of weight $2n - \Delta(G) - 1 < 2n - \Delta(G)$, which is a contradiction. Thus X is an independent set. If some w_i has two neighbors in $N(v)$, then the function f defined on G by $f(w_i) = 1$, $f(u_1) = \cdots = f(u_t) = 1$ and $f(x) = 2$ otherwise, is an OITDIDF of G of weight $2n - \Delta(G) - 1 < 2n - \Delta(G)$, which is a contradiction. Therefore, each vertex in X has exactly one neighbor in $N(v)$ because G is connected. Hence $\deg(w_i) = 1$ for each $1 \leq i \leq s$. If some u_i has two neighbors in X, say w_1, w_2, then the function f defined on G by $f(u_i) = 3$, $f(w_1) = f(w_2) = 0$, $f(u_1) = \cdots = f(u_t) = 1$, and $f(x) = 2$ otherwise, is an OITDIDF of G of weight less than $2n - \Delta(G)$, which is a contradiction. Thus, each u_i is adjacent to at most one vertex in X. Assume, without loss of generality, that $w_i u_i \in E(G)$ for each $1 \leq i \leq s$. If $s = t$, then the function f defined on G by $f(v) = 1$, $f(u_1) = \cdots = f(u_t) = 3$, and $f(x) = 0$ otherwise, is an OITDIDF of G of weight less than $2n - \Delta(G)$, a contradiction. Hence $t > s$. If $\deg(u_i) \geq 2$ for some $i \in \{s+1, \ldots, t\}$, say $i = t$, then the function f defined on G by $f(v) = 2$, $f(u_1) = \cdots = f(u_s) = 3$, $f(u_i) = 1$ for $s + 1 \leq i \leq t - 1$, and $f(x) = 0$ otherwise, is an OITDIDF of G of weight less than $2n - \Delta(G)$, a contradiction. Thus $\deg(u_i) = 1$ for each $i \in \{s+1, \ldots, t\}$. If $t - s \geq 2$, then the function f defined on G by $f(v) = 3$, $f(u_1) = \cdots = f(u_s) = 3$, and $f(x) = 0$ otherwise, is an OITDIDF of G of weight less than $2n - \Delta(G)$, a contradiction, yielding $t = s + 1$. Thus, $G = F \circ K_1$, where $F = G - \{w_1, w_2, \ldots, w_s, u_t\}$ and that $\Delta(F) = n(F) - 1$. This completes the proof. □

Next we present an upper bound on $\gamma^{oi}_{tdI}(G)$ in terms of $\gamma_{oidI}(G)$.

Theorem 16. *If G is a graph without isolated vertices, then*

$$\gamma^{oi}_{tdI}(G) \leq \left\lfloor \frac{1}{2}(3\gamma_{oidI}(G) - 1) \right\rfloor.$$

The bound is sharp for the stars $K_{1,p}$ for $p \geq 2$, and the complete bipartite graphs $K_{2,q}$ for $q \geq 3$.

Proof. Let $f = (V_0, V_1, V_2, V_3)$ be a $\gamma_{oidI}(G)$-function, and assume that $V_1 \cup V_2 \cup V_3 = \{v_1, v_2, \ldots, v_t\}$. If $V_0 = \emptyset$, then $\gamma^{oi}_{tdI}(G) = \gamma_{oidI}(G) \leq \left\lfloor \frac{1}{2}(3\gamma_{oidI}(G) - 1) \right\rfloor$, as desired. Hence assume that $V_0 \neq \emptyset$. Now let $H_1, H_2, \ldots, H_r$ be the connected components of the subgraph $G[V_1 \cup V_2 \cup V_3]$, and let $w \in V_0$. Assume, without loss of generality, that $N(w) \cap V(H_i) \neq \emptyset$ for each i with $1 \leq i \leq s$. We observe that $s \leq r \leq t$, $\sum_{i=s+1}^{r} \sum_{x \in V(H_i)} f(x) \geq 2(r-s)$ and $\sum_{i=1}^{s} \sum_{x \in V(H_i)} f(x) \geq 3$. Therefore, it follows that

$$\gamma_{oidI}(G) = \sum_{i=1}^{t} f(v_i) = \sum_{i=1}^{s} \sum_{x \in V(H_i)} f(x) + \sum_{i=s+1}^{r} \sum_{x \in V(H_i)} f(x) \geq 3 + 2(r-s),$$

and thus

$$(r-s) \leq \frac{1}{2}(\gamma_{oidI}(G) - 3).$$

Now let, without loss of generality, $H_{s+1}, H_{s+2}, \ldots, H_k$ $(k \leq r)$ be exactly the components of order one. Since G is a graph without isolated vertices, we can choose a vertex $w_i \in V_0$ for each $s+1 \leq i \leq k$ such that w_i has a neighbor in H_i. Then the function g defined by $g(w) = g(w_i) = 1$ for each $s+1 \leq i \leq k$ and $g(x) = f(x)$ otherwise is an OITDIDF on G, and thus

$$\gamma^{oi}_{tdI}(G) \leq \omega(g) \leq \omega(f) + (k-s) + 1 \leq \omega(f) + (r-s) + 1 \leq \frac{1}{2}(3\gamma_{oidI}(G) - 1).$$

This leads to the desired bound.

Proposition 8 implies $\gamma^{oi}_{tdI}(K_{1,p}) = 4$ and $\gamma^{oi}_{tdI}(K_{2,q}) = 5$. Since $\gamma_{oidI}(K_{1,q}) = 3$ and $\gamma_{oidI}(K_{2,q}) = 4$, we deduce that

$$\gamma^{oi}_{tdI}(K_{1,q}) = 4 = \left\lfloor \frac{1}{2}(3\gamma_{oidI}(K_{1,q}) - 1) \right\rfloor$$

and

$$\gamma_{tdI}^{oi}(K_{2,q}) = 5 = \left\lfloor \frac{1}{2}(3\gamma_{oidI}(K_{2,q}) - 1) \right\rfloor.$$

□

In what follows we establish some lower bounds on $\gamma_{tdR}^{oi}(G)$. The proof of the next result is similar to the proof of Theorem 3.1 in [1].

Theorem 17. *Let G be a graph of order n with minimum degree $\delta \geq 1$ and maximum degree Δ. Then*

$$\gamma_{tdR}^{oi}(G) \geq \left\lfloor \frac{\delta n}{\Delta + \delta - 1} \right\rfloor + 1,$$

and this bound is sharp.

Proof. Let $f = (V_0, V_1, V_2, V_3)$ be a $\gamma_{tdI}^{oi}(G)$-function. Since V_0 is an independent set, every vertex of V_0 has at least δ neighbors in $V_1 \cup V_2 \cup V_3$. In addition, every vertex of $V_1 \cup V_2 \cup V_3$ has at most $\Delta - 1$ neighbors in V_0. Therefore, it follows that

$$\delta(n - |V_1| - |V_2| - |V_3|) = \delta|V_0| \leq (\Delta - 1)(|V_1| + |V_2| + |V_3|),$$

and thus

$$\frac{\delta n}{\Delta + \delta - 1} \leq |V_1| + |V_2| + |V_3| = \gamma_{tdI}^{oi}(G) - |V_2| - 2|V_3|.$$

If $V_2 \cup V_3 \neq \emptyset$, then the last inequality chain leads to the desired bound. If $V_2 \cup V_3 = \emptyset$, then each vertex of V_1 is adjacent to at least two vertices of V_1, and we obtain analogously

$$\gamma_{tdI}^{oi}(G) \geq \frac{\delta n}{\Delta + \delta - 2} > \frac{\delta n}{\Delta + \delta - 1}.$$

Since $\gamma_{tdI}^{oi}(G)$ is an integer, we deduce the desired bound also in this case.

For each integer $p \geq 3$, let H_{3p} be the graph obtained from a cycle C_p by adding $2p$ new vertices and joining each new vertex to all vertices of C_p. Then we observe that $n(H_{3p}) = 3p$, $\Delta(H_{3p}) = 2p+2$, $\delta(H_{3p}) = p$ and $\gamma_{tdI}^{oi}(H_{3p}) = p = \left\lfloor \frac{3p^2}{3p+1} \right\rfloor + 1$. □

Theorem 18. *If G is a connected graph of order $n \geq 2$, then $\gamma_{tdI}^{oi}(G) \geq \beta(G)$. Furthermore, this bound is sharp for the complete t-partite graph $G = K_{n_1,n_2,\ldots,n_t}$ with $n_1 \leq n_2 \leq \ldots \leq n_t$ and $t \geq 4$, or $t = 3$ and $n_1 \geq 2$.*

Proof. Let $f = (V_0, V_1, V_2, V_3)$ be an OITDIDF on G. Then $|V_0| \leq \alpha(G)$, and therefore

$$\beta(G) = n - \alpha(G) \leq n - |V_0| = |V_1| + |V_2| + |V_3|.$$

Thus

$$\gamma_{tdI}^{oi}(G) = |V_1| + 2|V_2| + 3|V_3| \geq |V_1| + |V_2| + |V_3| \geq \beta(G),$$

as desired.

Observation 9 shows that

$$\gamma_{tdI}^{oi}(K_{n_1,n_2,\ldots,n_t}) = n - n_t = \beta(K_{n_1,n_2,\ldots,n_t})$$

if $n_1 \leq n_2 \leq \ldots \leq n_t$ and $t \geq 4$, or $t = 3$ and $n_1 \geq 2$. □

Theorem 19. *If G is a connected graph of order $n \geq 2$, then $\gamma_{tdI}^{oi}(G) \leq 2\gamma_{tI}^{oi}(G) - 1$. The bound is sharp for any graph G with $\gamma_{tI}^{oi}(G) = 2$.*

Proof. If $f = (V_0, V_1, V_2)$ is a $\gamma_{tI}^{oi}(G)$-function, then $\gamma_{tI}^{oi}(G) = |V_1| + 2|V_2|$. If $V_1 = \emptyset$, then $|V_2| \geq 2$ and the function $(V_0, \emptyset, \emptyset, V_2)$ is an OITDIDF on G of weight $3|V_2|$, and so

$$\gamma_{tdI}^{oi}(G) \leq 3|V_2| \leq 4|V_2| - 2 = 2\gamma_{tI}^{oi}(G) - 2.$$

Assume that $V_1 \neq \emptyset$ and let $w \in V_1$. Now the function $(V_0, \{w\}, V_1 \setminus \{w\}, V_2)$ is an OITDIDF on G. If $V_2 \neq \emptyset$, then

$$\gamma_{tdI}^{oi}(G) \leq 1 + 2(|V_1| - 1) + 3|V_2| < 2|V_1| + 4|V_2| - 1 = 2\gamma_{tI}^{oi}(G) - 2.$$

Suppose that $V_2 = \emptyset$. Then we have

$$\gamma_{tdI}^{oi}(G) \leq 1 + 2(|V_1| - 1) \leq 2|V_1| - 1 = 2\gamma_{tI}^{oi}(G) - 1.$$

□

4 Nordhaus-Gaddum type inequalities

In this section, we present Nordhaus-Gaddum type inequalities for the outer independent total double Italian domination number. Let $\mathcal{G}$ be a family of graphs G such that G is obtained from a complete graph K_p, $(p \geq 4)$, an empty graph $\overline{K_s}$, where $s \geq \left\lceil \frac{3p}{p-3} \right\rceil$ and a new vertex u, by joining u to every vertex of K_p and joining each vertex of $\overline{K_s}$ to at least three vertices of K_p such that each vertex of K_p is non-adjacent to at least three vertices of $\overline{K_s}$. It is clear from the construction of G that $G \in \mathcal{G}$ if and only if $\overline{G} \in \mathcal{G}$. The proof of the following result can be found in [3].

Theorem 20. *If G and $\overline{G}$ are connected graphs of order $n \geq 4$, then*

$$\gamma_{oidI}(G) + \gamma_{oidI}(\overline{G}) \geq n - 1,$$

with equality if and only if $G \in \mathcal{G}$.

Since for any graph G without isolated vertices, $\gamma^{oi}_{tdI}(G) \geq \gamma_{oidI}(G)$, we get the following result.

Theorem 21. *If G and $\overline{G}$ are connected graphs of order $n \geq 4$, then*

$$\gamma^{oi}_{tdI}(G) + \gamma^{oi}_{tdI}(\overline{G}) \geq n - 1,$$

with equality if and only if $G \in \mathcal{G}$.

Theorem 22. *Let G and $\overline{G}$ be graphs without isolated vertices of order n. If $\delta(G) = \delta(\overline{G}) = 1$, then*

$$\gamma^{oi}_{tdI}(G) + \gamma^{oi}_{tdI}(\overline{G}) \leq 2n + 4.$$

The equality holds if and only if $G = P_4$.

If $\delta(G) \geq 2$ and $\delta(\overline{G}) \geq 2$, then

$$\gamma^{oi}_{tdI}(G) + \gamma^{oi}_{tdI}(\overline{G}) \leq 2n.$$

This bound is sharp for C_5.

If $\delta(G) \geq 2$ or $\delta(\overline{G}) \geq 2$, then

$$\gamma^{oi}_{tdI}(G) + \gamma^{oi}_{tdI}(\overline{G}) \leq 2n + \left\lfloor \frac{n}{2} \right\rfloor.$$

This bound is sharp for C_4.

Proof. If $\delta(G) = \delta(\overline{G}) = 1$, then it follows from Theorem 15 that

$$\begin{aligned} \gamma_{tdI}^{oi}(G) + \gamma_{tdI}^{oi}(\overline{G}) &\leq (2n - \Delta(G)) + (2n - \Delta(\overline{G})) \\ &= (2n - (n-2)) + (2n - (n-2)) = 2n + 4. \end{aligned}$$

If $G = P_4$, then $\overline{G} = P_4$, and Proposition 6 implies that $\gamma_{tdI}^{oi}(G) + \gamma_{tdI}^{oi}(\overline{G}) = 6 + 6 = 2n + 4$. Conversely, let $\gamma_{tdI}^{oi}(G) + \gamma_{tdI}^{oi}(\overline{G}) = 2n + 4$. Then we deduce from the above inequality chain that $\gamma_{tdI}^{oi}(G) = 2n - \Delta(G)$ and $\gamma_{tdI}^{oi}(\overline{G}) = 2n + \Delta(\overline{G})$. Theorem 15 implies that $G = F \circ K_1$ and $\overline{G} = F' \circ K_1$, where F and F' are connected graphs with $\Delta(F) = n(F) - 1$ and $\Delta(F') = n(F') - 1$. If $\Delta(F) \geq 2$, then we have $\delta(\overline{G}) \geq 2$ which is a contradiction. Hence $\Delta(F) = 1$, and so $F = K_2$. Thus $G = P_4$.

Proposition 10 implies $\gamma_{tdI}^{oi}(G) + \gamma_{tdI}^{oi}(\overline{G}) \leq 2n$ immediately when $\delta(G) \geq 2$ and $\delta(\overline{G}) \geq 2$. According to Proposition 6, this bound is sharp for C_5.

Finally assume, without loss of generality, that $\delta(\overline{G}) \geq 2$. We deduce from Propositions 12 and 10 that

$$\gamma_{tdI}^{oi}(G) + \gamma_{tdI}^{oi}(\overline{G}) \leq n + \left\lfloor \frac{n}{2} \right\rfloor + n = 2n + \left\lfloor \frac{n}{2} \right\rfloor .$$

If $\overline{G} = C_4$, then $G = 2P_2$, and we observe that $\gamma_{tdI}^{oi}(G) + \gamma_{tdI}^{oi}(\overline{G}) = 4 + 6 = 2n + \lfloor \frac{n}{2} \rfloor$. □

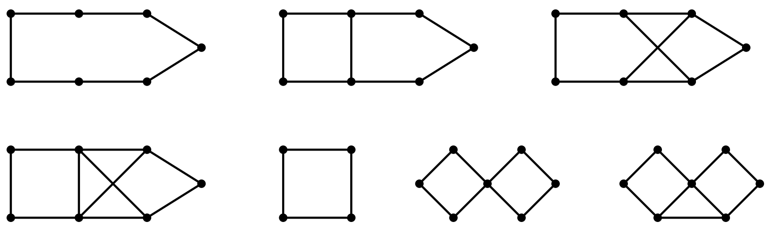

Figure 1. Graphs in family $\mathcal{A}$

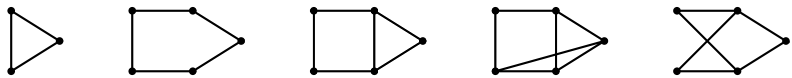

Figure 2. Graphs in family $\mathcal{B}$

Using Theorem 14, we improve Theorem 22 for $n \geq 12$. We recall the definition of some families of graphs. Let $\mathcal{A}$ be the collections of graphs in Figure 1 and $\mathcal{B}$ be the collections of graphs in Figure 2. Let

$$\mathcal{G}_1 = \{C_4\} \cup \{G \mid G = H \circ K_1, \text{ where } H \text{ is connected}\}$$

and

$$\mathcal{G}_2 = \mathcal{A} \cup \mathcal{B} - \{C_4\}.$$

For any graph H, let $\mathcal{S}(H)$ denote the set of connected graphs, each of which can be formed from $H \circ K_1$ by adding a new vertex x and edges joining x to one or more vertices of H and define

$$\mathcal{G}_3 = \cup_H \mathcal{S}(H),$$

where the union is taken over all graphs H. Let y be a vertex of a copy of C_4 and for $G \in \mathcal{G}_3$, let $\theta(G)$ be the graph obtained by joining G to C_4 with the single edge xy, where x is the new vertex added in forming G. Define

$$\mathcal{G}_4 = \{\theta(G) \mid G \in \mathcal{G}_3\}.$$

Next, let uvw be a path P_3. For any graph H, let $\mathcal{P}(H)$ be the set of connected graphs which may be formed from $H \circ K_1$ by joining each of u and w to one or more vertices of H. Then define

$$\mathcal{G}_5 = \bigcup_H \mathcal{P}(H).$$

Let H be a graph $X \in \mathcal{B}$. Let $\mathcal{R}(H, X)$ be the set of connected graphs which may be obtained from $H \circ K_1$ by joining each vertex of $U \subseteq V(X)$ to one or more vertices of H such that no set with fewer than $\gamma(X)$ vertices of X dominates $V(X) - U$. Then define

$$\mathcal{G}_6 = \bigcup_{H,X} \mathcal{R}(H, X).$$

The proof of the following result can be found in [14], [20].

Theorem 23. *A connected graph G satisfies* $\gamma(G) = \lfloor \frac{n(G)}{2} \rfloor$ *if and only if* $G \in \cup_{i=1}^{6} \mathcal{G}_i$.

Lemma 24. *If* $G \in \cup_{i=1}^{6} \mathcal{G}_i$ *and* $n(G) \geq 12$, *then* $\gamma_{tdI}^{oi}(\overline{G}) \leq n - 2$.

Proof. Let $G \in \cup_{i=1}^{6} \mathcal{G}_i$. Since $n(G) \geq 12$, we have $G \notin \mathcal{G}_2$. Let $G \in \mathcal{G}_1$ and let $G = H \circ K_1$. We deduce from $n(G) \geq 12$ that $n(H) \geq 6$. If $z \in V(H)$ and z' is a leaf adjacent to z in G, then the function f defined on $V(\overline{G})$ by $f(z) = f(z') = 0$ and $f(s) = 1$ otherwise, is an OITDIDF on $\overline{G}$ of weight $n - 2$, and so $\gamma_{tdI}^{oi}(\overline{G}) \leq n - 2$, the required bound.

If $G \in \mathcal{G}_3 \cup \mathcal{G}_4 \cup \mathcal{G}_5 \cup \mathcal{G}_6$, then G has at least four leaves and the function f defined above leads to $\gamma_{tdI}^{oi}(\overline{G}) \leq n - 2$. □

Theorem 25. *If G and* $\overline{G}$ *are graphs without isolated vertices of order* $n \geq 12$, *then*

$$\gamma_{tdI}^{oi}(G) + \gamma_{tdI}^{oi}(\overline{G}) \leq 2n + \left\lfloor \frac{n}{2} \right\rfloor - 2.$$

Proof. If $\delta(G) = \delta(\overline{G}) = 1$ or $\delta(G) \geq 2$ and $\delta(\overline{G}) \geq 2$, then the desired result follows from Theorem 22. Let now, without loss of generality, $\delta(G) = 1$ and $\delta(\overline{G}) \geq 2$. If $\delta(\overline{G}) = 2$, then $\Delta(G) = n - 3$, and thus we deduce from Theorem 15 and the hypothesis $n \geq 12$ that

$$\begin{aligned} \gamma_{tdI}^{oi}(G) + \gamma_{tdI}^{oi}(\overline{G}) &\leq (2n - \Delta(G)) + (2n - \Delta(\overline{G}) - 1) \\ &= (2n - (n-3)) + (2n - (n-2) - 1) \\ &= 2n + 4 \leq 2n + \left\lfloor \frac{n}{2} \right\rfloor - 2. \end{aligned}$$

Assume that $\delta(\overline{G}) \geq 3$. If $G \in \cup_{i=1}^{6} \mathcal{G}_i$, then by Lemma 24 and Proposition 12, we obtain

$$\gamma_{tdI}^{oi}(G) + \gamma_{tdI}^{oi}(\overline{G}) \leq n + \left\lfloor \frac{n}{2} \right\rfloor + n - 2 = 2n + \left\lfloor \frac{n}{2} \right\rfloor - 2.$$

If $G \notin \cup_{i=1}^{6} \mathcal{G}_i$, then using Theorems 23, 14 and Proposition 12, we obtain

$$\gamma_{tdI}^{oi}(G) + \gamma_{tdI}^{oi}(\overline{G}) \leq n + \left\lfloor \frac{n}{2} \right\rfloor - 1 + (n - 1) = 2n + \left\lfloor \frac{n}{2} \right\rfloor - 2.$$

□

References

[1] N. A. Abd Aziz, H. Kamrulhaili, F. Azviv, and N. Jafari Rad, "On the outer-independent double Italian domination number," *Electron. J. Graph Theory Appl.*, vol. 10, pp. 365–374, 2022.

[2] H. Abdollahzadeh Ahangar, M. Chellali, S. M. Sheikholeslami, and J. C. Valenzuela-Tripodoro, "On the outer independent total double Roman domination in graphs," *Mediterr. J. Math.*, vol. 20, Article ID 171, 2023.

[3] F. Azvin, N. Jafari Rad, and L. Volkmann, "Bounds on the outer-independent double Italian domination number," *Commun. Comb. Optim.*, vol. 6, pp. 123–136, 2021.

[4] M. Benatallah, "Outer-independent double Italian domination in graphs," Manuscript, 2021.

[5] M. Chellali, T. W. Haynes, S.T. Hedetniemi, and A. MacRae, "Roman {2}-domination," *Discrete Appl. Math.*, vol. 204, pp. 22–28, 2016.

[6] M. Chellali, N. Jafari Rad, S. M. Sheikholeslami, and L. Volkmann, "Roman domination in graphs," in *Topics in Domination in Graphs*, T. W. Haynes, S. T. Hedetniemi, and M. A. Henning, Eds. Springer, 2020, pp. 365–409.

[7] M. Chellali, N. Jafari Rad, S. M. Sheikholeslami, and L. Volkmann, "Varieties of Roman domination," in *Structures of Domination in Graphs*, T. W. Haynes, S. T. Hedetniemi, and M. A. Henning, Eds. Springer, 2021, pp. 273–307.

[8] M. Chellali, N. Jafari Rad, S. M. Sheikholeslami, and L. Volkmann, "Varieties of Roman domination II," *AKCE J. Graphs Combin.*, vol. 17, pp. 966–984, 2020.

[9] E. J. Cockayne, P. A. Dreyer, S. M. Hedetniemi, and S. T. Hedetniemi, "Roman domination in graphs," *Discrete Math.*, vol. 278, pp. 11–22, 2004.

[10] J. F. Fink, M. S. Jacobson, L. F. Kinch, and J. Roberts, “On graphs having domination number half their order,” *Period. Math. Hungar.*, vol. 16, pp. 287–293, 1985.

[11] T. W. Haynes, S. T. Hedetniemi, and P. J. Slater, *Fundamentals of Domination in Graphs*, New York: Marcel Dekker, Inc., 1998.

[12] D. A. Mojdeh and L. Volkmann, “Roman {3}-domination (double Italian domination),” *Discrete Appl. Math.*, vol. 283, pp. 555–564, 2020.

[13] C. Payan and N. H. Xuong, “Domination-balanced graphs,” *J. Graph Theory*, vol. 6, pp. 23–32, 1982.

[14] B. Randerath and L. Volkmann, “Characterization of graphs with equal domination and covering number,” *Discrete Math.*, vol. 191, pp. 159–169, 1998.

[15] Z. Shao, D.A. Mojdeh, and L. Volkmann, “Total Roman {3}-domination in graphs,” *Symmetry*, vol. 12, Article No. 268, 15 p., 2020.

[16] S. M. Sheikholeslami and L. Volkmann, “New bounds on the outer independent total double Roman domination number,” *Discrete Math. Algorithms Appl.*, to be published.

[17] A. Teymourzadeh and D. A. Mojdeh, “Covering total double Roman domination in graphs,” *Commun. Comb. Optim.*, vol. 8, pp. 115–125, 2023.

[18] I. Stewart, “Defend the Roman Empire!,” *Sci. Amer.*, vol. 281, no. 6, pp. 136–139, 1999.

[19] L. Volkmann, “Restrained double Italian domination in graphs,” *Commun. Comb. Optim.*, vol. 8, pp. 1–11, 2023.

[20] B. Xu, E. J. Cockayne, T. W. Haynes, Stephen T. Hedetniemi, and Z. Shangchao, “Extremal graphs for inequalities involving domination parameters,” *Discrete Math.*, vol. 216, pp. 1–10, 2000.

Outer independent total double Italian domination number

Seyed Mahmoud Sheikholeslami, Lutz Volkmann

Received May 17, 2023
Accepted June 30, 2023

Seyed Mahmoud Sheikholeslami
ORCID: https://orcid.org/0000-0003-2298-4744
Department of Mathematics
Azarbaijan Shahid Madani University
Tabriz, I. R. Iran
E–mail: s.m.sheikholeslami@azaruniv.ac.ir

Lutz Volkmann
ORCID: https://orcid.org/0000-0003-3496-277X
Lehrstuhl C für Mathematik
RWTH Aachen University
52062 Aachen, Germany
E–mail: volkm@math2.rwth-aachen.de