Total Italian domatic number of graphs

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Abstract

Let G be a graph with vertex set $V(G)$. An *Italian dominating function* (IDF) on a graph G is a function $f : V(G) \longrightarrow \{0, 1, 2\}$ such that every vertex v with $f(v) = 0$ is adjacent to a vertex u with $f(u) = 2$ or to two vertices w and z with $f(w) = f(z) = 1$. An IDF f is called a *total Italian dominating function* if every vertex v with $f(v) \geq 1$ is adjacent to a vertex u with $f(u) \geq 1$. A set $\{f_1, f_2, \ldots, f_d\}$ of distinct total Italian dominating functions on G with the property that $\sum_{i=1}^{d} f_i(v) \leq 2$ for each vertex $v \in V(G)$, is called a *total Italian dominating family* (of functions) on G. The maximum number of functions in a total Italian dominating family on G is the *total Italian domatic number* of G, denoted by $d_{tI}(G)$. In this paper, we initiate the study of the total Italian domatic number and present different sharp bounds on $d_{tI}(G)$. In addition, we determine this parameter for some classes of graphs.

Keywords: Total Italian domination number, Total Italian domatic number.

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1 Introduction

For definitions and notations not given here we refer to [10]. We consider simple graphs G with vertex set $V = V(G)$ and edge set $E = E(G)$. The *order* of G is $n = n(G) = |V(G)|$. The *open neighborhood* of a vertex v is the set $N(v) = N_G(v) = \{u \in V(G) \mid uv \in E(G)\}$ and its *closed neighborhood* is the set $N[v] = N_G[v] = N(v) \cup \{v\}$. The *degree* of vertex $v \in V(G)$ is $d(v) = d_G(v) = |N(v)|$. The *maximum degree* and *minimum degree* of G are denoted by $\Delta = \Delta(G)$ and

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$\delta = \delta(G)$, respectively. The *complement* of a graph G is denoted by $\overline{G}$. A *leaf* is a vertex of degree one, and its neighbor is called a *support vertex*. An edge incident with a leaf is called a *pendant edge*. We write P_n for the path of order n, C_n for the cycle of length n, and K_n for the complete graph of order n. The *corona* $H \circ K_1$ of a grah H is that graph obtained from H by adding a pendant edge to each vertex of H.

A set $S \subseteq V(G)$ is a *(total) dominating set* of G if every vertex of $(V(G))$ $V(G) - S$ is adjacent to a vertex in S. The *(total) domination number* of a graph G is the cardinality of a smallest (total) dominating set of G and is denoted $(\gamma_t(G))$ $\gamma(G)$. The (total) domatic number of G, $(d_t(G))$ $d(G)$ is the maximum number of classes of a partition of $V(G)$ such that each class is a (total) dominating set of G.

Cockayne, Dreyer, S.M. Hedetniemi, and S.T. Hedetniemi [8] introduced the concept of *Roman domination* in graphs, and since then a lot of related variations and generalizations have been studied (see [4]–[7]). In this paper, we continue the study of Roman and Italian dominating functions in graphs G. If $f : V(G) \longrightarrow \{0, 1, 2\}$ is a function, then let (V_0, V_1, V_2) be the ordered partition of $V(G)$ induced by f, where $V_i = \{v \in V(G) \mid f(v) = i\}$ for $i \in \{0, 1, 2\}$. There is 1-1 correspondence between the function f and the ordered partition (V_0, V_1, V_2). So, we also write $f = (V_0, V_1, V_2)$.

A function $f : V(G) \longrightarrow \{0, 1, 2\}$ is a *Roman dominating function* (RDF) on G, if every vertex v with $f(v) = 0$ is adjacent to a vertex u with $f(u) = 2$. The *Roman domination number* $\gamma_R(G)$ is the minimum weight of an RDF on G.

A *total Roman dominating function* (TRDF) on a graph G without isolated vertices is defined in [11] as a Roman dominting function f on G with the property that the subgraph induced by $V_1 \cup V_2$ has no isolated vertex. The *total Roman domination number* $\gamma_{tR}(G)$ is the minimum weight of a TRDF on G.

An *Italian dominating function* (IDF) on a graph G is defined in [3] as a function $f : V(G) \longrightarrow \{0, 1, 2\}$ such that $f(N(v)) \geq 2$ for every vertex v with $f(v) = 0$. The weight of an IDF f is the value $\omega(f) = f(V(G)) = \sum_{u \in V(G)} f(u)$. The *Italian domination number* $\gamma_I(G)$ is the minimum weight of an IDF on G. In [3], the authors called the Italian domination number the Roman $\{2\}$-domination number.

A *total Italian dominating function* (TIDF) on a graph G without isolated vertices is defined in [1] as an Italian dominating function f on G with the property that the subgraph induced by $V_1 \cup V_2$ has no isolated vertex. The *total Italian domination number* $\gamma_{tI}(G)$ is the minimum weight of a TIDF on G. A TIDF on G with weight $\gamma_{tI}(G)$ is called a $\gamma_{tI}(G)$-function.

A set $\{f_1, f_2, \ldots, f_d\}$ of distinct total Roman dominating functions on a graph G without isolated vertices with the property that $\sum_{i=1}^{d} f_i(v) \leq 2$ for each vertex $v \in V(G)$, is called in [2] a *total Roman dominating family* (of functions) on G. The maximum number of functions in a total Roman dominating family on G is the *total Roman domatic number* $d_{tR}(G)$ of G.

A set $\{f_1, f_2, \ldots, f_d\}$ of distinct total Italian dominating functions on a graph G without isolated vertices with the property that $\sum_{i=1}^{d} f_i(v) \leq 2$ for each vertex $v \in V(G)$, is called a *total Italian dominating family* (of functions) on G. The maximum number of functions in a total Italian dominating family on G is the *total Italian domatic number* $d_{tI}(G)$ of G. Italian domatic number has been studied in [12], [14].

If G is a graph without isolated vertices, then $\gamma_{tI}(G) \leq \gamma_{tR}(G)$ and $d_{tR}(G) \leq d_{tI}(G)$. On the other hand, if $S_1 \cup S_2 \cup \cdots \cup S_{d_t}$ is a partition of $V(G)$ such that each class is a total dominating set of G, then the family $\{f_1, f_2, \ldots, f_{d_t}\}$ of functions, where f_i is defined on G by $f_i(x) = 2$ for $x \in S_i$ and $f(x) = 0$ otherwise, is a total Italian dominating family (of functions) on G and so $d_t(G) \leq d_{tI}(G)$.

In this paper, we initiate the study of the total Italian domatic number, and we present different sharp bounds on $d_{tI}(G)$. In particular, we prove the Nordhaus-Gaddum type result $d_{tI}(G) + d_{tI}(\overline{G}) \leq n$ for graphs G of order $n \geq 4$ with $\delta(G) \geq 1$ and $\delta(\overline{G}) \geq 1$. In addition, we determine the total Italian domatic number for some classes of graphs.

We make use of the following known results.

Proposition 1 ([1]). *If $n \geq 3$, then $\gamma_{tI}(P_n) = \lceil \frac{2n+2}{3} \rceil$ and $\gamma_{tI}(C_n) = \lceil \frac{2n}{3} \rceil$.*

Proposition 2 ([1]). *Let G be a connected graph of order $n \geq 2$. Then $\gamma_{tI}(G) \leq n$, with equality if and only if G is the corona $F \circ K_1$ of some* *connected graph F or $G = P_3$.*

Proposition 3 ([1]). *If G is a graph without isolated vertices of order n, then $\gamma_{tI}(G) \geq 2$. We have $\gamma_{tI}(G) = 2$ if and only if there exist two vertices u and v with $d(u) = d(v) = n - 1$.*

Proposition 4 ([1]). *If G is a graph without isolated vertices of order n, then*

$$\gamma_{tI}(G) \geq \left\lceil \frac{2n}{\Delta(G) + 1} \right\rceil.$$

Proposition 5 ([2]). *If $t \geq s \geq 1$ are integers, then $d_{tR}(K_{t,s}) = s$.*

2 Bounds and Properties

In this section, we present sharp bounds on the total Italian domatic number and investigate its basic properties. In addition, we determine this parameter for some classes of graphs.

Theorem 1. *Let G be a graph of order $n \geq 3$ without isolated vertices. If G has $2 \leq p \leq n$ vertices of degree $n - 1$, then $d_{tI}(G) \geq p$.*

Proof. Let $\{v_1, v_2, \ldots, v_n\}$ be the vertex set of G, and let, without loss of generality, $v_1, v_2, \ldots, v_p$ be the vertices of degree $n - 1$. If $p \geq 3$, then define the functions f_i by $f_i(v_i) = f_i(v_{i+1}) = 1$ and $f_i(x) = 0$ for $x \neq v_i, v_{i+1}$ for $1 \leq i \leq p$, where $v_{p+1} = v_1$. Then $f_1, f_2, \ldots, f_p$ are distinct TIDF on G such that $\sum_{i=1}^{p} f_i(x) \leq 2$ for each $x \in V(G)$. Therefore, $\{f_1, f_2, \ldots, f_p\}$ is a total Italian dominating family on G and, thus, $d_{tI}(G) \geq p$. If $p = 2$, then define f_1 by $f_1(v_1) = f_1(v_2) = 1$ and $f_1(x) = 0$ for $x \neq v_1, v_2$. Moreover, define f_2 by $f_2(v_i) = 1$ for all $1 \leq i \leq n$. Since $n \geq 3$, it follows that $\{f_1, f_2\}$ is total dominating family on G, and so $d_{tI}(G) \geq 2 = p$ also in this case. □

Theorem 2. *If G is a graph of order n without isolated vertices, then*

$$\gamma_{tI}(G) \cdot d_{tI}(G) \leq 2n.$$

Moreover, if we have the equality $\gamma_{tI}(G) \cdot d_{tI}(G) = 2n$, then for each total Italian dominating family $\{f_1, f_2, \ldots, f_d\}$ with $d = d_{tI}(G)$, each f_i is a $\gamma_{tI}(G)$-function and $\sum_{i=1}^{d} f_i(v) = 2$ for all $v \in V(G)$.

Proof. Let $\{f_1, f_2, \ldots, f_d\}$ be a total Italian dominating family on G with $d = d_{tI}(G)$. Then

$$d \cdot \gamma_{tI}(G) = \sum_{i=1}^{d} \gamma_{tI}(G) \leq \sum_{i=1}^{d} \sum_{v \in V(G)} f_i(v) =$$

$$= \sum_{v \in V(G)} \sum_{i=1}^{d} f_i(v) \leq \sum_{v \in V(G)} 2 = 2n.$$

If $\gamma_{tI}(G) \cdot d_{tI}(G) = 2n$, then the two inequalities occuring in the proof become equalities. Hence, for the total Italian dominating family $\{f_1, f_2, \ldots, f_d\}$ on G and for each i, $\sum_{v \in V(G)} f_i(v) = \gamma_{tI}(G)$. Thus, each f_i is a $\gamma_{tI}(G)$-function and $\sum_{i=1}^{d} f_i(v) = 2$ for all $v \in V(G)$. □

Proposition 3 and Theorem 2 imply the next result immediately.

Corollary 1. *If G is a graph of order n without isolated vertices, then $d_{tI}(G) \leq n$.*

Theorem 3. *If G is a graph of order n without isolated vertices, then*

$$d_{tI}(G) \leq \delta(G) + 1.$$

Moreover, if $F = \{f_1, f_2, \ldots, f_{d_{tI}(G)}\}$ is a total Italian dominating family with $d_{tI}(G) = \delta(G) + 1$, then for any minimum degree vertex v, the following statements must be held:

(a) $\sum_{u \in N[v]} f_i(u) = 2$ *for each* $f_i \in F$ *and* $\sum_{i=1}^{d} f_i(u) = 2$ *for each* $u \in N[v]$.

(b) *There are exactly $\delta(G) - 1$ Italian dominating functions such that $f_i(v) = 0$, and exactly two TIDFs such that $f_i(v) = 1$.*

(c) *If $f_i(v) = 1$, then $f_i(u) = 0$ for each neighbor of v but exactly one which is assigned 1 under f_i.*

Proof. Let $\{f_1, f_2, \ldots, f_d\}$ be a total Italian dominating family on G with $d = d_{tI}(G)$. Assume that v is a vertex of minimum degree. It follows from the definitions that $\sum_{x\in N[v]} f_i(x) \geq 2$ for each $i \in \{1, 2, \ldots, d\}$. Therefore, we deduce that

$$2d \leq \sum_{i=1}^{d} \sum_{x\in N[v]} f_i(x) = \sum_{x\in N[v]} \sum_{i=1}^{d} f_i(x) \leq \sum_{x\in N[v]} 2 = 2(\delta(G) + 1) \quad (1)$$

and so, $d_{tI}(G) = d \leq \delta(G) + 1$.

Assume that the equality holds, that is $d_{tI}(G) = \delta(G)+1$. Then the inequalities occurring in (1) become equalities which gives the properties given in the statement (a).

Without loss of generality, assume that $f_1, f_2, \ldots, f_{d'}$ are the TIDFs such that $f_i(v) = 0$ (for some d'). For each i such that $f_i(v) = 0$, we must have $\sum_{x\in N(v)} f_i(x) \geq 2$. Therefore,

$$2d' \leq \sum_{i=1}^{d'} \sum_{x\in N(v)} f_i(x) = \sum_{x\in N(v)} \sum_{i=1}^{d'} f_i(x) \leq \sum_{x\in N(v)} 2 = 2\delta(G). \quad (2)$$

If the equality holds in (2), that is $d' = \delta(G)$, then we must have $\sum_{i=1}^{d'} f_i(x) = 2$ for each $x \in N(v)$. It follows from $2 = \sum_{i=1}^{d'} f_i(x) \leq \sum_{i=1}^{d} f_i(x) \leq 2$ that $f_d(x) = 0$ for each $x \in N(v)$, which contradicts the totality of f_d. Thus, there are at most $\delta(G)-1$ total Italian dominating functions such that $f_i(v) = 0$. Since there are at most two Italian dominating functions such that $f_i(v) \geq 1$, we deduce that there are exactly $\delta(G) - 1$ Italian dominating functions such that $f_i(v) = 0$, and exactly two TIDFs such that $f_i(v) = 1$. Thus, the statement (b) holds.

(c) immediately comes from $\sum_{u\in N[v]} f_i(u) = 2$ (see (a)). This completes the proof. □

For regular graphs, we can use the statements about vertices of minimum degree at equality to every vertex, so that if $d = d_{tI}(G) = \delta(G) + 1$, and $F = \{f_1, f_2, \ldots, f_d\}$ is a family of Italian dominating functions, then this implies each Italian dominating function is a function $f_i : V(G) \to \{0, 1\}$. So we can consider the Italian dominating functions as indicator functions, and in what follows, it will be convenient to restate the property that $d_{tI}(G) = \delta(G)+1$ for a regular graph G in terms of a family of sets. The proof of next result is essentially similar to the proof of Lemma 1 in [12].

Corollary 2. *Let G be a δ-regular graph, where $\delta \geq 1$. Then $d_{tI}(G) = \delta + 1$ if and only if there are distinct sets $S_1, S_2, \ldots, S_{\delta+1}$, $S_i \subseteq V(G)$, that satisfy the following:*

(a) *Every vertex of G appears in exactly two sets S_i.*

(b) *Each set S_i induces a perfect matching, i.e., the induced subgraph $G[S_i]$ is 1-regular.*

(c) *For any vertex $v \notin S_i$, $|N(v) \cap S_i| = 2$.*

(d) *For each i, $|S_i| = \frac{2n}{\delta+1} = \gamma_{tI}(G)$.*

Proof. Suppose that there exist sets $S_1, S_2, \ldots, S_{\delta+1} \subseteq V(G)$ satisfying (a),(b),(c) and (d). Let f_i be the characteristic function of S_i for each i. By Conditions (b) and (c), each f_i is a total Italian dominating function, and by Condition (a), these functions form a total Italian dominating family with $\delta + 1$ total Italian dominating functions. Since $d_{tI}(G) \leq \delta + 1$, we get $d_{tI}(G) = \delta + 1$.

Conversely, assume that $d_{tI}(G) = \delta+1$ and let $F = \{f_1, f_2, \ldots, f_{\delta+1}\}$ be a total Italian dominating family. Since G is δ-regular, we deduce from Theorem 3-(b) that $f_i(v) \leq 1$ for each i and each $v \in V(G)$. For each f_i, define $S_i = \{v \in V(G) \mid f_i(v) = 1\}$. Note that $\omega(f) = |S_i|$. Clearly, (a) and (c) come from Theorem 3-(b). Also (b) follows from Theorem 3-(c). Now we prove (d). Using Proposition 4 and noting that G is δ-regular, we obtain $\lceil \frac{2n}{\delta+1} \rceil (\delta+1) \leq \sum_{i=1}^{\delta+1} |S_i| = 2n \leq \lceil \frac{2n}{\delta+1} \rceil (\delta+1)$. Equality is possible only if $2n$ is divisible by $\delta + 1$, and $|S_i| = \frac{2n}{\delta+1}$ for each i. □

Corollary 3. *Let G be a graph of order $n \geq 3$ without isolated vertices. Then $d_{tI}(G) = n$ if and only if $G = K_n$.*

Proof. If $G = K_n$, then Theorem 1 and Corollary 1 imply $d_{tI}(G) = n$.

Conversely, assume that $d_{tI}(G) = n$. If $\delta(G) \leq n-2$, then Theorem 3 yields the contradiction $d_{tI}(G) \leq \delta(G) + 1 \leq n - 1$. Therefore, $\delta(G) = n - 1$ and, thus, $G = K_n$. □

The Cartesian product of two graphs G and H, denoted $G\square H$, is a graph whose vertex set is $V(G) \times V(H) = \{(x, y) \mid x \in V(G) \text{ and } y \in V(H)\}$ and two vertices (x_1, y_1) and (x_2, y_2) of $G\square H$ are adjacent if and only if either $x_1 = x_2$ and $y_1y_2 \in E(H)$ or $y_1 = y_2$ and $x_1x_2 \in E(G)$. It is shown that, for any two graphs G and H without isolated vertices, $d_t(G\square H) \geq \max\{d(G), d(H)\}$ [9].

Corollary 4. *If $n \geq 2$, then $d_{tI}(K_n\square K_2) = n$.*

Proof. Since $d(K_n) = n$, we have $d_{tI}(K_n\square K_2) \geq d_t(K_n\square K_2) \geq \max\{d(K_n), d(K_2)\} = n$. On the other hand, one can easily see that $\gamma_{tI}(K_n\square K_2) = 4$ and so by Theorem 2 we have $d_{tI}(K_n\square K_2) \leq \frac{4n}{4} = n$. Thus, $d_{tI}(K_n\square K_2) = n$. □

Theorem 4. *Let C_n be a cycle of length $n \geq 3$. Then $d_{tI}(C_n) = 3$ when $n \equiv 0 \pmod 3$ and $d_{tI}(C_n) = 2$ when $n \equiv 1, 2 \pmod 3$.*

Proof. Let $n \equiv 0 \pmod 3$, and let $C_n = v_1v_2 \dots v_nv_1$ with $n = 3p$ for an integer $p \geq 1$. Define the functions f, g, and h by $f(v_{3i-2}) = f(v_{3i-1}) = 1$ and $f(v_{3i}) = 0$, $g(v_{3i-1}) = g(v_{3i}) = 1$ and $g(v_{3i-2}) = 0$, and $h(v_{3i}) = h(v_{3i-2}) = 1$ and $h(v_{3i-1}) = 0$ for $1 \leq i \leq p$. Then f, g, and h are total Italian dominating functions on C_n such that $f(x) + g(x) + h(x) = 2$ for each vertex $x \in V(C_n)$. Therefore, $\{f, g, h\}$ is a total Italian dominating family on C_n and, thus, $d_{tI}(C_n) \geq 3$. Theorem 3 yields to $d_{tI}(C_n) \leq 3$ and so $d_{tI}(C_n) = 3$ in this case.

Let now $n \equiv 1, 2 \pmod 3$ and $C_n = v_1v_2 \dots v_nv_1$. Theorem 2 and Proposition 1 imply

$$d_{tI}(C_n) \leq \frac{2n}{\gamma_{tI}(C_n)} = \frac{2n}{\lceil \frac{2n}{3} \rceil} < 3$$

and, hence, $d_{tI}(C_n) \leq 2$. Define the functions f and g by $f(v_i) = 1$ for $1 \leq i \leq n$ and $g(v_1) = 0$ and $g(v_i) = 1$ for $2 \leq i \leq n$. Then f and g are total Italian dominating functions on C_n such that $f(x) + g(x) \leq 2$ for each vertex $x \in V(C_n)$. Therefore, $\{f, g\}$ is a total Italian dominating family on C_n and, thus, $d_{tI}(C_n) \geq 2$. This leads to $d_{tI}(C_n) = 2$ in this case. □

Proposition 6. *If P_n is a path of order $n \geq 5$, then $d_{tI}(P_n) = 2$.*

Proof. Let $P_n = v_1 v_2 \ldots v_n$. Define the functions f and g by $f(v_i) = 1$ for $1 \leq i \leq n$ and $g(v_3) = 0$ and $g(v_i) = 1$ for $1 \leq i \leq n$ with $i \neq 3$. Then f and g are total Italian dominating functions on P_n such that $f(x) + g(x) \leq 2$ for each vertex $x \in V(P_n)$. Therefore, $\{f, g\}$ is a total Italian dominating family on P_n and, thus, $d_{tI}(P_n) \geq 2$. Theorem 3 implies $d_{tI}(P_n) \leq 2$ and so we obtain $d_{tI}(P_n) = 2$. □

The proof of the next proposition is identical to the proof of Proposition 5 and is, therefore, omitted.

Proposition 7. *If $t \geq s \geq 1$ are integers, then $d_{tI}(K_{t,s}) = s$.*

Theorem 5. *Let G be a connected graph of order $n \geq 2$. Then $d_{tI}(G) = 1$ if and only if every vertex of G is a leaf or a support vertex.*

Proof. Let G contain a vertex w which is neither a leaf nor a support vertex. Since w is not a leaf, w has at least two neighbors, and since w is not a support vertex, $G - w$ has no isolated vertex. Therefore, the function f with $f(w) = 0$ and $f(x) = 1$ for $x \in V(G) \setminus \{w\}$ is a TIDF on G. In addition, the function g with $f(x) = 1$ for all $x \in V(G)$ is also a TIDF on G with the property that $f(x) + g(x) \leq 2$ for all $x \in V(G)$. Therefore, $\{f, g\}$ is a total Italian dominating family on G and, thus, $d_{tI}(G) \geq 2$.

Conversely, assume that each vertex of G is a leaf or a support vertex. Theorem 3 implies $d_{tI}(G) \leq 2$. Suppose that $\{f, g\}$ is a total Italian dominating family on G. If v is a support vertex, then the definitions lead to $f(v), g(v) \geq 1$. If $f(v) = 2$, then the condition $f(v) + g(v) \leq 2$ yields the contradiction $g(v) = 0$. Thus, $f(v) = g(v) = 1$ for all support vertices v. If follows that $f(u) = g(u) = 1$ for all leaves u, a contradiction to the condition that f and g are distinct. Consequently, $d_{tI}(G) = 1$, and the proof is complete. □

Theorem 6. *Let G be a connected graph of order $n \geq 3$. Then*

$$\gamma_{tI}(G) + d_{tI}(G) \leq n + 2,$$

with equality if and only if $G = K_n$.

Proof. If $d_{tI}(G) = 1$, then Proposition 2 implies $\gamma_{tI}(G)+d_{tI}(G) \leq n+1$. Let next $d_{tI}(G) \geq 2$. It follows from Theorem 2 that

$$\gamma_{tI}(G) + d_{tI}(G) \leq \frac{2n}{d_{tI}(G)} + d_{tI}(G).$$

Using the bounds $2 \leq d_{tI}(G) \leq n$ (see Corollary 1), and the fact that the function $g(x) = \frac{2n}{x} + x$ is decreasing for $2 \leq x \leq \sqrt{2n}$ and increasing for $\sqrt{2n} \leq x \leq n$, we obtain

$$\gamma_{tI}(G) + d_{tI}(G) \leq \frac{2n}{d_{tI}(G)} + d_{tI}(G) \leq \max\{n+2, 2+n\} = n+2, \quad (3)$$

and the bound is proved.

If $G = K_n$, then we deduce from Proposition 3 and Corollary 3 that $\gamma_{tI}(G) + d_{tI}(G) = n + 2$.

Conversely, assume that $\gamma_{tI}(G) + d_{tI}(G) = n + 2$. It follows from (3) that

$$n + 2 = \gamma_{tI}(G) + d_{tI}(G) \leq \frac{2n}{d_{tI}(G)} + d_{tI}(G) \leq n + 2$$

and, therefore, $d_{tI}(G) = 2$ and $\gamma_{tI}(G) = n$ or $d_{tI}(G) = n$ and $\gamma_{tI}(G) = 2$. If $d_{tI}(G) = n$ and $\gamma_{tI}(G) = 2$, then Corollary 3 yields $G = K_n$. If $d_{tI}(G) = 2$ and $\gamma_{tI}(G) = n$, then Proposition 2 implies $G = F \circ K_1$ for a connected graph F or $G = P_3$. But now Theorem 5 leads to the contradiction $d_{tI}(G) = 1$. □

3 Nordhaus-Gaddum type results

Results of Nordhaus-Gaddum type study the extreme values of the sum or product of a parameter on a graph and its complement. In their classical paper [13], Nordhaus and Gaddum discussed this problem for the chromatic number. We establish such inequalities for the total Italian domatic number.

Theorem 7. *If G is a graph of order $n \geq 4$ with $\delta(G) \geq 1$ and $\delta(\overline{G}) \geq 1$, then*

$$d_{tI}(G) + d_{tI}(\overline{G}) \leq n.$$

Proof. Theorem 3 implies

$$d_{tI}(G)+d_{tI}(\overline{G}) \leq (\delta(G)+1)+(\delta(\overline{G})+1) = \delta(G)+1+(n-\Delta(G)-1)+1.$$

If G is not regular, then $\Delta(G) - \delta(G) \geq 1$, and the inequality chain above leads to the desired bound.

Let now G be δ-regular. Then $\overline{G}$ is $\overline{\delta}$-regular with $\overline{\delta} = n - \delta - 1$. Assume, without loss of generality, that $\delta \leq \overline{\delta}$.

If $\delta = 1$, then $G = \frac{n}{2}K_2$ and, thus, $d_{tI}(G) = 1$. According to Corollaries 1 and 3, we observe that $d_{tI}(\overline{G}) \leq n-1$ and, thus, $d_{tI}(G) + d_{tI}(\overline{G}) \leq n$.

Thus, let now $\delta \geq 2$ and $n = p(\delta+1) + r$ with integers $p \geq 1$ and $0 \leq r \leq \delta$. If $r \neq 0, \frac{\delta+1}{2}$, then Corollary 2 implies $d_{tI}(G) \leq \delta$ and, as above, we obtain $d_{tI}(G) + d_{tI}(\overline{G}) \leq n$. Next we discuss the case $r = 0$ or $r = \frac{\delta+1}{2}$.

Case 1: Let $r = 0$ and, therefore, $n = p(\delta+1)$. We also have $n = (\overline{\delta}+1) + \delta$ with $2 \leq \delta \leq \overline{\delta}$. If $\delta \neq \frac{\overline{\delta}+1}{2}$, then Corollary 2 yields $d_{tI}(\overline{G}) \leq \overline{\delta}$, and we obtain $d_{tI}(G) + d_{tI}(\overline{G}) \leq n$ as above. Let now $\delta = \frac{\overline{\delta}+1}{2}$. Then

$$n = \overline{\delta} + 1 + \frac{\overline{\delta}+1}{2} = \frac{3}{2}(\overline{\delta}+1) = \frac{3}{2}(n-\delta)$$

and so $n = 3\delta$. Hence, $n = p(\delta+1) = 3\delta$ and, thus, $p = 2$. We deduce that $\delta = 2$ and $n = 6$. Consequently, G is a cycle of length 6 or the union of two cycles of length 3. Using Theorem 4 and Proposition 7, it is easy to verify that $d_{tI}(G) + d_{tI}(\overline{G}) = 6 = n$ in both cases.

Case 2: Let $r = \frac{\delta+1}{2}$ and, therefore, $n = p(\delta+1) + \frac{\delta+1}{2}$. As in Case 1, there remains the case that $n = 3\delta$. Hence, $n = 3\delta = (p+\frac{1}{2})(\delta+1)$ and so $p \leq 2$. If $p = 1$, then we obtain the contradiction $\delta = 1$. If $p = 2$, then $\delta = 5$ and $n = 15$, a contradiction to the fact that the number of vertices of odd degree is even. $\square$

Since $d_{tR}(G) \leq d_{tI}(G)$, Theorem 7 leads to the next known Nordhaus-Gaddum bound.

Theorem 8 ([2]). *If G is a graph of order $n \geq 4$ with $\delta(G) \geq 1$ and $\delta(\overline{G}) \geq 1$, then*

$$d_{tR}(G) + d_{tR}(\overline{G}) \leq n.$$

Theorem 9. *If G is a graph of order $n \geq 5$ with $\delta(G) \geq 1$ and $\delta(\overline{G}) \geq 1$, then*

$$d_{tI}(G) + d_{tI}(\overline{G}) \geq 3.$$

Proof. Assume, without loss of generality, that $d_{tI}(G) \leq d_{tI}(\overline{G})$. If $d_{tI}(G) \geq 2$, then we even see that $d_{tI}(G) + d_{tI}(\overline{G}) \geq 4$. So let now $d_{tI}(G) = 1$.

If G is not connected, then the condition $\delta(G) \geq 1$ shows that $\overline{G}$ is connected such that $\delta(\overline{G}) \geq 2$. Therefore, Theorem 5 leads to $d_{tI}(\overline{G}) \geq 2$, and we obtain $d_{tI}(G) + d_{tI}(\overline{G}) \geq 3$.

Let now G be connected. Then it follows from Theorem 5 that each vertex of G is a leaf or a support vertex. Let $S(G) = \{v_1, v_2, \ldots, v_s\}$ be the set of support vertices. Since $\delta(\overline{G}) \geq 1$, we observe that $s \geq 2$. If $s = 2$, then the condition $n \geq 5$ shows that v_1 or v_2, say v_1, is adjacent to more than one leaf. We deduce that v_2 is neither a leaf nor a support vertex of $\overline{G}$. Since $\overline{G}$ is connected, it follows from Theorem 5 that $d_{tI}(\overline{G}) \geq 2$ and, thus, $d_{tI}(G) + d_{tI}(\overline{G}) \geq 3$. If $s \geq 3$, then $\delta(\overline{G}) \geq 2$, and Theorem 5 leads to $d_{tI}(G) + d_{tI}(\overline{G}) \geq 3$ again. □

Since $d_{tI}(P_4) + d_{tI}(\overline{P_4}) = 2$, we observe that the condition $n \geq 5$ in Theorem 9 is necessary.

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