The Domination Parameters on a kind of the regular honeycomb structure

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Abstract

The honeycomb mesh, based on hexagonal structure, has enormous applications in chemistry and engineering. A major challenge in this field is to understand the unique properties of honeycomb structures, which depend on their properties of topology.

One of the important concepts in graph theory is the domination number which can be used for network control and monitoring. In this paper, we investigate the domination number of the honeycomb network. For this purpose, the domination number, the total domination number, the independent domination number, the connected domination number and the doubly connected domination number of the honeycomb are obtained. Finally, in some honeycomb structures of real models, we obtain the exact amount of these parameters.

Keywords: Honeycomb structure, Total domination number, Independent domination number, Connected domination number, Doubly connected domination number.

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1 Introduction

The honeycomb mesh is a network that is convenient for modeling and designing some engineering models and/or chemical structures. Using honeycomb structures based on the geometry of a honeycomb in engineering sciences allows minimizing the amount of used material to reach minimal weight and minimal material cost. The geometry of

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honeycomb structures can vary widely but the common feature of all such structures is a set of hollow cells formed between thin vertical walls. The cells are often columnar and hexagonal in shape [1]. The honeycomb networks have also been recognized as crucial as a representation of benzenoid hydrocarbons in chemistry. These networks had found widespread applications in various fields such as architecture, mechanical engineering, chemistry, transportation, nanofabrication and biomedicine [2].

A major challenge is to understand the unique properties of honeycomb structures based on the properties of their topologies. So, the study on the properties of the topology of honeycomb structures has been considered. Honeycomb networks are better in terms of degree, diameter, and total number of links, cost, and bisection width than mesh connected planar graphs [3]. Stojmenovic [4] has studied the topological properties of honeycomb networks, routing in honeycomb networks and honeycomb torus networks.

In [5], the degree diameter problem on honeycomb networks is studied. Manuel et al. determined the minimum metric bases for hexagonal and honeycomb networks [6]. In [7], an approximation algorithm is proposed to obtain the harmonious chromatic number of honeycomb. An algorithm for finding a perfect packing of honeycomb networks is proposed in [8].

One of the important and well-known concepts in graph theory is the study of the dominating sets in a graph. The studies of domination set are important in the control of engineering systems. The dominating set has already been applied to the control or design of different types of engineering systems, which include mobile computing [9], computer communication networks [10], computational biology and biomedical analysis [11].

Recall that for a simple graph G with the vertices set V and the edges set E, the dominating set D of the vertices subset of the graph G is such that every vertex is either in D or adjacent to a vertex in D. Domination in graphs has been extensively researched as one of the branches in graph theory and has many applications in science and technology [12]. A survey of several advanced topics of domination is given in the book by Haynes et al. [13]. The domination number of graph G, denoted by $\gamma(G)$, is the minimum size of a dominating set of G.

The minimum dominating set is classified as NP-Completeness and in general cannot be solved exactly in polynomial time [14], [15]. It means that there is no theoretically efficient algorithm that finds the exact smallest dominating set for a given graph. Therefore, many heuristic and approximation algorithms are proposed to find the minimum dominating set of a graph. Some proposed algorithms for selecting the minimum dominating set of a given graph can be found in [16]–[18]. In this paper, we determine the minimum dominating set of a honeycomb network and obtain the exact formula for the domination number based on the parameters of the honeycomb structure.

There are several parameters of domination that can be used to simulate some properties of networks and chemical graphs [19]–[21]. A dominating set D is a total dominating set of G if every vertex of the graph is adjacent to at least one vertex in D. The total domination number of G, denoted by $\gamma_t(G)$ is the minimum size of a total dominating set of G. A dominating set D is called an independent dominating set if D is an independent set. The independent domination number of G denoted by $\gamma_i(G)$ is the minimum size of an independent dominating set of G. Obviously, for each graph G, $\gamma(G) \leq \gamma_i(G)$ [13]. The subset D of the set of vertices $V(G)$ is a connected dominating set in G if D is a dominating set and the subgraph induced by D is connected. The minimum cardinality of any connected dominating set in G is called the connected domination number of G and it is denoted by $\gamma_c(G)$ [22]. A set $D \subseteq V(G)$ is a doubly connected dominating set of G if it is dominating and both induced subgraphs D and $V(G) \setminus D$ are connected. The cardinality of a minimum doubly connected dominating set of G is the doubly connected domination number of G and is denoted by $\gamma_{cc}(G)$ [23].

In this paper, we obtain the total domination number, the independent domination number, the connected domination number and the doubly connected domination number of a given honeycomb network as one of the properties of their topologies. In Section 3, applied examples based on the modeling of honeycomb structures are included to obtain the validity and applicability of the domination parameters.

2 Main Results

In this section, we consider the kind of regular honeycomb structure called the general structure [24]. This structure of honeycomb contains an array of hollow cells with the number of $n \geq 2$ cells that are formed between $k \geq 1$ thin vertical walls of the cells, which have a hexagonal shape. So, k is an odd number (see Figure 1).

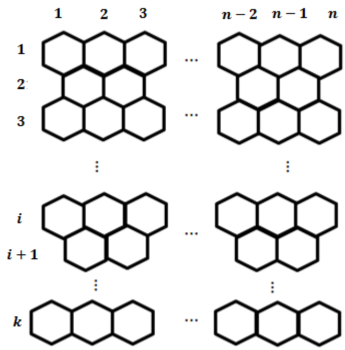

Figure 1. Honeycomb structure with n cells in rows and k cells in columns.

However, the configuration of Figure 1 can be changed with different geometric transformations [25] and structure graph HC is shown as Figure 2.

According to Figure 1, the number of the vertices of this honeycomb structure (HC) is $(2n+1)(k+1)$. The vertex degrees are 2 and 3 such that $2(n-1)+(2n-1)(k-1)$ vertices have degree 3, and the remained vertices have degree 2. According to Figure 2, in any subgraph G_i for $1 \leq i \leq \lceil \frac{k}{2} \rceil$, vertices are labeled as $\{1, 2, \ldots, 2n+1\}$ in path P_1 and $\{2n+2, 2n+3, \ldots, 4n+2\}$ in path P_2. Also, in any subgraph H_i, for $1 \leq i \leq \lfloor \frac{k}{2} \rfloor$, vertices are labeled as $\{2n+3, 2n+4, 2n+5, \ldots, 4n+1, 2, 3, 4, 5, \ldots, 2n-1\}$.

We obtain the number of domination parameters of the graph HC. Also, the minimum set of these parameters is determined.

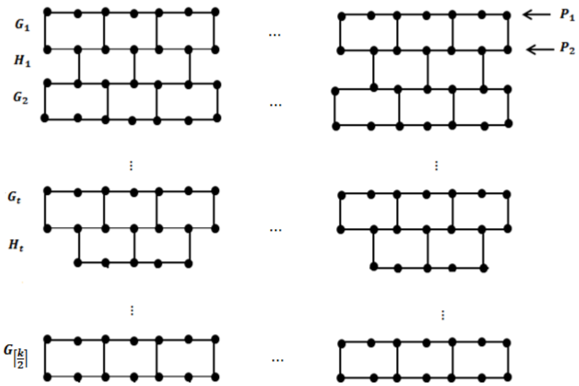

Figure 2. Graph HC obtained from Figure 1.

Theorem 1. *Let the graph HC be a honeycomb structure that contains n hexagonal cells in rows and k hexagonal cells in columns for $n \geq 2$ and $k > 1$. Then*

$$\gamma(HC) = (n+1)\left\lceil \frac{k}{2} \right\rceil.$$

Proof. Let D be the dominating set of graph HC that is shown in Figure 2. We consider subgraphs G_i of HC for $1 \leq i \leq \lceil \frac{k}{2} \rceil$. If D_i is the dominating set of graph G_i, then it is easy to see that $D = \bigcup_{i=1}^{\lceil \frac{k}{2} \rceil} D_i$. So, one can select the set D_i as follows:

Case 1: If n is odd, then

$$D_i = \{1, 5, 9, 13, \ldots, 2n-1, 2n+4, 2n+8, 2n+12, \ldots, 4n-2, 4n+2\},$$

such that $|D_i| = n+1$.

Case 2: If n is even, then

$$D_i = \{1, 5, 9, 13, \ldots, 2n-3, 2n+1, 2n+4, 2n+8, \ldots, 4n-4, 4n\},$$

and $|D_i| = n+1$.

Therefore, $|D| = (n+1)\lceil \frac{k}{2} \rceil$.

Assume that there exists a dominating set D of the graph HC with $|D| \leq (n+1)\lceil \frac{k}{2} \rceil - 1$. According to the structure of the graph HC

shown in Figure 2, at least one of the vertices $\{1, 2n+1\}$ and one of the vertices $\{2n+2, 4n+2\}$ in any subgraph G_i for $1 \leq i \leq \lceil \frac{k}{2} \rceil$ must be selected in D. Because otherwise, the set D must contain the vertices $\{2, 2n, 2n+3, 4n+1\}$, where instead of two vertices, we have to select 4 vertices. Therefore, at least $2\lceil \frac{k}{2} \rceil$ vertices of the set D have the degree 2. So, the number of vertices of degree 3 is at most

$$(n+1)\left\lceil \frac{k}{2} \right\rceil - 1 - 2\left\lceil \frac{k}{2} \right\rceil = (n-1)\left\lceil \frac{k}{2} \right\rceil - 1.$$

Since any vertices with degree 2 in D dominate two vertices in a graph HC and any vertices with degree 3 dominate three vertices, thus, the number of vertices of HC dominated by D is at most $4n\lceil \frac{k}{2} \rceil - 4$. With a simple computation, for $n \geq 2$ and $k > 1$, we can obtain $(2n+1)(k+1) > 4n\lceil \frac{k}{2} \rceil - 4$. So, it is a contraction and we have $\gamma(HC) = (n+1)\lceil \frac{k}{2} \rceil$. □

Theorem 2. *Let the graph HC be a honeycomb structure containing n hexagonal cells in rows and k hexagonal cells in columns for $n \geq 2$ and $k > 1$. Then*

$$\gamma_t(HC) = 2(n+1) + (k-1)n.$$

Proof. Let TD be the total dominating set of the graph HC. According to Figure 2 and by considering subgraphs G_1 and $G_{\lceil \frac{k}{2} \rceil}$, one can select the total dominating sets TD_1 and $TD_{\lceil \frac{k}{2} \rceil}$ as the following sets on the paths P_1 and P_2 from subgraphs G_1 and $G_{\lceil \frac{k}{2} \rceil}$, respectively.

Case a) If n is even, then

$$TD_1 = \big\{2, 3, 6, 7, 10, 11, 14, 15, \ldots, 2n-2, 2n-1, 2n\big\},$$

with $|TD_1| = n+1$ and

$$TD_{\lceil \frac{k}{2} \rceil} = \big\{2n+3, 2n+4, 2n+7, 2n+8, \ldots, 4n-1, 4n, 4n+1\big\},$$

where $\left|TD_{\lceil \frac{k}{2} \rceil}\right| = n+1$. Also, the total dominating set TD_i from subgraphs H_i, $1 \leq i \leq \lfloor \frac{k}{2} \rfloor$, is considered as follows:

$$TD_i = \big\{2, 4, \ldots, 2n, 2n+3, 2n+5, \ldots, 4n+1\big\},$$

with $|TD_i| = 2n$.

Thus, we can consider the set $TD = \bigcup_{i=1}^{\lfloor \frac{k}{2} \rfloor} TD_i \bigcup \{TD_1, TD_{\lceil \frac{k}{2} \rceil}\}$ as the total dominating set for the graph HC that $|TD| = 2(n+1)+2\lfloor \frac{k}{2} \rfloor n$. Since k is odd, then $|TD| = 2(n+1) + (k-1)n$.

Case b) If n is odd, we select

$$TD_1 = \{2, 3, 6, 7, 10, 11, 14, 15, \ldots, 2n-2, 2n, 2n+1\},$$

and

$$TD_{\lceil \frac{k}{2} \rceil} = \{2n+3, 2n+4, 2n+6, 2n+7, \ldots, 4n+1, 4n+2\},$$

where $|TD_1| = \left|TD_{\lceil \frac{k}{2} \rceil}\right| = n+1$. For $1 \leq i \leq \lfloor \frac{k}{2} \rfloor$, we consider

$$TD_i = \{2, 4, \ldots, 2n, 2n+3, 2n+5, \ldots, 4n+1\}.$$

Therefore, $|TD| = 2(n+1) + (k-1)n$. Thus, we have $\gamma_t(HC) \leq 2(n+1) + (k-1)n$.

Assume that the set TD be the total dominating set of the graph HC and $|TD| \leq 2(n+1) + (k-1)n - 1$. According to the structure of the graph HC and the degrees of vertices of the graph, there are the following cases.

Case 1: There are $4\lceil \frac{k}{2} \rceil$ vertices of degree 2 such that at least any two vertices are adjacent to each other and the remained vertices of TD have degree 3. The number of vertices of degree 3 is $2(n+1) + (k-1)n - 1 - 4\lceil \frac{k}{2} \rceil$. Since any vertex of degree 2 dominates at least one vertex of the graph HC, the number of dominated vertices of HC is equal to $8\lceil \frac{k}{2} \rceil$.

Also, the vertices of degree 3 dominate at least one vertex of the graph HC; the number of them is as follows:

$$2\Big(2(n+1) + (k-1)n - 1 - 4\lceil \frac{k}{2} \rceil\Big).$$

Therefore, the number of dominated vertices of HC is as follows

$$2\Big(2(n+1) + (k-1)n - 1\Big).$$

Since the number of vertices of the graph HC is $(2n+1)(k+1)$, some vertices of the graph HC cannot dominate by TD. Therefore, it is a contradiction that TD is the total dominating set.

Case 2: Assume that the set TD contains $2(n+2)+4\Big(\lceil\frac{k}{2}\rceil-2\Big)$ vertices of degree 2 such that in the neighborhood of its any vertex exists at least one vertex of degree 3. So, the number of vertices of degree 3 of the graph HC is $(k-1)n-4\lceil\frac{k}{2}\rceil+5$ in TD. Since any vertex of degree 2 and 3 at least dominates one vertex of HC, the number of dominated vertices of the graph by TD is $2(k-1)n+4(n+2)-6$. For $n\geq 2$ and $k\geq 1$, $(2n+1)(k+1) > 2(k-1)n+4(n+2)-6$. Thus, at least one vertex of honeycomb HC cannot dominate by vertices of TD and it is a contradiction.

Case 3: If all of the vertices of degree 3 are in the set TD. Then the number of them is $4(n-1)+2n\lceil\frac{k}{2}\rceil$ and the remained vertices of the set TD have the degree 2 with size $2(n+1)+(k-1)n-1-4(n-1)-n\lceil\frac{k}{2}\rceil$. For $n\geq 2$ and $k\geq 1$, one can obtain that the number of degree 2 in the set TD is negative by a simple calculation. So, it is a contradiction in this case.

Case 4: Let TD contains $2n+4\lceil\frac{k}{2}\rceil$ vertices of HC such that one of two vertices of any two adjacent vertices in TD has degree 2 and the other vertex has degree 3. The number of remaining vertices of TD in HC equals $(k-1)n-4\lceil\frac{k}{2}\rceil-1$. Since $(k-1)n-4\lceil\frac{k}{2}\rceil-1<0$ for $n\geq 2$ and $k\geq 1$, thus this case is a contradiction.

Therefore, in all of the above cases and similar cases, for the total dominating set with $|TD|\leq 2(n+1)+(k-1)n-1$, we obtain a contradiction. So, the result follows. □

Theorem 3. *Let HC be a honeycomb structure containing n hexagonal cells in rows and k hexagonal cells in columns for $n\geq 2$ and $k>1$. Then*

$$\gamma_i(HC)=(n+1)\lceil\frac{k}{2}\rceil.$$

Proof. Similar to Theorem 1, one can obtain the independent dominating set ID of HC as follows.

Let ID_i be the independent dominating set of the graph G_i shown in Figure 2. For the two following cases, we get

Case 1: If n is even, then we can choose

$$ID_i = \big\{1, 5, 9, 13, \ldots, 2n-3, 2n+1, 2n+4, \ldots, 4n-4, 4n\big\},$$

Case 2: If n is odd, then

$$ID_i = \big\{1, 5, 9, 13, \ldots, 2n-1, 2n+4, 2n+8, \ldots, 4n-2, 4n+2\big\}.$$

For both cases, $|ID_i| = n+1$, where $1 \leq i \leq \lceil \frac{k}{2} \rceil$. Therefore, we have $ID = \bigcup_{i=1}^{\lceil \frac{k}{2} \rceil} ID_i$ and $|ID| = (n+1)\lceil \frac{k}{2} \rceil$. Since the set ID is a dominating and independent set, then $\gamma_i(HC) \leq \lceil \frac{k}{2} \rceil (n+1)$.

On the other hand, using Theorem 1,

$$\left\lceil \frac{k}{2} \right\rceil (n+1) = \gamma(HC) \leq \gamma_i(HC) = \left\lceil \frac{k}{2} \right\rceil (n+1).$$

Therefore, the result completes. □

Theorem 4. *Let HC be a honeycomb structure containing n hexagonal cells in rows and k hexagonal cells in columns for $n \geq 4$ and $k > 1$. Then*

$$\gamma_c(HC) = 2\left\lceil \frac{k}{2} \right\rceil (2n-3) + 1.$$

Proof. Let CD be the connected dominating set of the graph HC. One can select the connected dominating set CD as follows.

Case 1: Assume that n is even. According to Figure 2, we consider all of the vertices from P_1 in subgraph G_1 except $\{1, 2n+1\}$ in CD. Thus, the number of these vertices is $2n-1$. The remained vertices selected from P_2 in subgraph G_1 are as follows:

$$Big\{2n+3, 2n+4, \ldots, 2n + \left\lceil \frac{2n+1}{2} \right\rceil - 1, 2n + \left\lceil \frac{2n+1}{2} \right\rceil + 3, \\ 2n + \left\lceil \frac{2n+1}{2} \right\rceil + 4, \ldots, 4n+1\Big\},$$

where the number of these vertices is $2n-4$.

For $2 \leq i \leq \lceil \frac{k}{2} \rceil$, we consider the following vertices of subgraphs G_i in the set CD

$$\Big\{2, 3, 4, \ldots, \left\lceil \frac{2n+1}{2} \right\rceil - 1, \left\lceil \frac{2n+1}{2} \right\rceil + 2, \left\lceil \frac{2n+1}{2} \right\rceil + 3, \ldots, 2n, \\ 2n+3, 2n+4, \ldots, 2n + \left\lceil \frac{2n+1}{2} \right\rceil, 2n + \left\lceil \frac{2n+1}{2} \right\rceil + 3, \ldots, 4n+1\Big\},$$

where the number of these vertices is $4n-6$.

So, $|CD| = (2n-1)+(2n-4)+\left(\left\lceil\frac{k}{2}\right\rceil-1\right)(4n-6) = 2\left\lceil\frac{k}{2}\right\rceil(2n-3)+1$.

Case 2: Let n be odd. Similar to case 1, select the following vertices of subgraphs G_i for $1 \leq i \leq \lceil\frac{k}{2}\rceil$:

In path P_1 of subgraph G_1, select all of the vertices except $\{1, 2n+1\}$ in CD.

In path P_2 of subgraph G_1, we consider the following vertices:

$$\Big\{2n+3, 2n+4, \dots, 2n+\left\lceil\frac{2n+1}{3}\right\rceil+1, 2n+\left\lceil\frac{2n+1}{3}\right\rceil+5, \\ 2n+\left\lceil\frac{2n+1}{3}\right\rceil+6, \dots, 4n+1\Big\}.$$

For $2 \leq i \leq \lceil\frac{k}{2}\rceil$, we consider the following vertices of subgraphs G_i in the set CD:

$$\Big\{2, 3, 4, \dots, \left\lceil\frac{2n+1}{3}\right\rceil+1, \left\lceil\frac{2n+1}{3}\right\rceil+4, \dots, 2n, 2n+3, 2n+4, \dots, \\ 2n+\left\lceil\frac{2n+1}{3}\right\rceil+2, 2n+\left\lceil\frac{2n+1}{3}\right\rceil+4, \dots, 4n+1\Big\}.$$

So, $|CD| = 2\left\lceil\frac{k}{2}\right\rceil(2n-3)+1$. Thus, we have $\gamma_c(HC) \leq 2\left\lceil\frac{k}{2}\right\rceil(2n-3)+1$.

Let CD be the connected dominating set, where $|CD| \leq 2\left\lceil\frac{k}{2}\right\rceil(2n-3)$. If the set CD contains only vertices of degree 2 or degree 3, then it contradicts the definition of the connected dominating set for the set CD. Thus, the vertices in CD have both degrees 2 and 3.

Let CD contain at least $2\lceil\frac{k}{2}\rceil$ vertices of degree 2, and the remained vertices are of degree 3. Thus, $2\left\lceil\frac{k}{2}\right\rceil(2n-4)$ vertices of degree 3 are dominated by some vertices of the graph HC.

According to the structure HC and the definition of the connected dominating set CD in HC, any vertex with degree 2 dominates at least another vertex such that one of its adjacent vertices must be in CD. So, the number of dominated vertices in HC by these vertices is $2\left\lceil\frac{k}{2}\right\rceil$. Since CD is the connected set, any vertex with degree 3 dominates at least two vertices of a graph HC such that at least one of its adjacent vertices must be in CD. Therefore, the number of dominated vertices in HC is at most

$$2\left\lceil\frac{k}{2}\right\rceil + 2\left\lceil\frac{k}{2}\right\rceil(2n-4).$$

Using easy computing, one can obtain that the number of dominated vertices in graph HC is less than the number of vertices in $V \setminus D$. So, it is a contradiction. Therefore, the result is complete. □

Theorem 5. *Let HC be a honeycomb structure containing n hexagonal cells in rows and k hexagonal cells in columns for $k > 1$.*

(i) If $n = 2$, then

$$\gamma_c(HC) = 3\left\lceil\frac{k}{2}\right\rceil,$$

(ii) If $n = 3$, then

$$\gamma_c(HC) = 8\left\lceil\frac{k}{2}\right\rceil + 10.$$

Proof. (i) Assume that $n = 2$ and $k > 1$. Let CD be the connected dominating set of HC. One can select the vertices of the set CD_i of subgraphs G_i for $1 \le i \le \lceil\frac{k}{2}\rceil$ as follows:

$$CD_i = \{2, 3, \ldots, 2n\}.$$

Since $CD = \bigcup_{i=1}^{\lceil\frac{k}{2}\rceil} CD_i$ is the connected dominating set of the graph HC, then $|CD| = 3\left\lceil\frac{k}{2}\right\rceil$. Therefore, $\gamma_c(HC) = 3\left\lceil\frac{k}{2}\right\rceil$. Similar to Theorem 4 and the structure of the graph HC, it is easy to see that the connected dominating set CD has the minimum cardinality between the connected dominating set of HC. Thus, the result follows.

(ii) Similar to case (i) and the proof of Theorem 4, we select the connected domination set CD for the graph HC as follows. The vertices of subgraph G_1 and $G_{\lceil\frac{k}{2}\rceil}$ are considered as

$$CD_1 = CD_{\lceil\frac{k}{2}\rceil} = \{2, 3, 4, 5, 6, 9, 10, 12, 13\}.$$

For $2 \le i \le \lceil \frac{k}{2} \rceil - 1$, we select the vertices of subgraphs G_i

$$CD_i = \{2, 3, 5, 6, 9, 10, 12, 13\}.$$

Since $CD = \bigcup_{i=1}^{\lceil \frac{k}{2} \rceil} CD_i$, we get $|CD| = 2 \times 9 + 8\Big(\Big\lceil \frac{k}{2} \Big\rceil - 1\Big)$.

□

Theorem 6. *Let HC be a honeycomb structure containing n hexagonal cells in rows and k hexagonal cells in columns for $n \ge 3$ and $k > 1$. Then*

$$\gamma_{cc}(HC) = k(2n+1) - 4\left\lceil \frac{k}{2} \right\rceil + 9.$$

Proof. Let CCD be the doubly connected dominating set of the graph HC. One can select the set CCD such that all of the vertices of HC are selected in CCD except for the vertices sets of subgraphs G_i for $1 \le i \le \lceil \frac{k}{2} \rceil$ as follows: in subgraph G_1, the set $\{3, 4, 2n+4, 2n+5\}$; in subgraph $G_{\lceil \frac{k}{2} \rceil}$, the set $\{4, 5, 6, \ldots, 2n-1\}$; in subgraph G_i, the set $\{4, 5, 2n+5, 2n+6\}$ for $2 \le i \le \lceil \frac{k}{2} \rceil - 1$.

Therefore,

$$|CCD| = (2n+1)(k+1) - \left(4\left\lceil \frac{k}{2} \right\rceil + 2n - 8\right) = k(2n+1) - 4\left\lceil \frac{k}{2} \right\rceil + 9.$$

So, $\gamma_{cc}(HC) \le k(2n+1) - 4\left\lceil \frac{k}{2} \right\rceil + 9$.

Let CCD be the doubly connected dominating set such that $\gamma_{cc}(HC) \le k(2n+1) - 4\left\lceil \frac{k}{2} \right\rceil + 8$. Since CCD is the doubly connected domination set, then, by the structure of the graph HC, the vertices $\{1, 2n+1, 2n+2, 4n+2\}$ in any subgraph G_i for $1 \le i \le \lceil \frac{k}{2} \rceil$ must be selected in CCD. So, the number of vertices with degree 2 is at least $4\left\lceil \frac{k}{2} \right\rceil$. Since CCD is the connected set, then the set CCD contains the vertices $\{2, 2n, 2n+3, 4n+1\}$. Thus, the number of these vertices in CCD is at least $8\left\lceil \frac{k}{2} \right\rceil$. Similarly, the number of vertices with degree 3 is at most $k(2n+1) - 8\left\lceil \frac{k}{2} \right\rceil + 8$ in the set CCD such that at least one of its adjacent vertices is in CCD. Therefore, the number of these vertices is equal to $2k(2n+1) - 16\left\lceil \frac{k}{2} \right\rceil + 16$.

According to the above argument, the cardinality of CCD is at least $2k(2n+1) - 8\left\lceil \frac{k}{2} \right\rceil + 16$. By a simple calculation, for $n \geq 3$ and $k > 1$, the number of vertices in CCD is more than $k(2n+1) - 4\left\lceil \frac{k}{2} \right\rceil + 8$ that it is a contradiction. Therefore, the result completes. □

Theorem 7. *Let the graph HC be a honeycomb structure containing n hexagonal cells in rows and k hexagonal cells in columns for $n = 2$ and $k > 3$. Then*

$$\gamma_{cc}(HC) = 5k - 6\left\lceil \frac{k}{2} \right\rceil + 13.$$

Proof. Let CCD be the doubly connected dominating set of the graph HC for $n = 2$ and $k > 3$. We can select all of the vertices of HC except the following ones: in subgraph G_1, the set $\{7, 8\}$; in subgraph $G_{\lceil \frac{k}{2} \rceil}$, the set $\{2, 3\}$; in subgraph G_i, the set $\{1, 2, 3, 6, 7, 8\}$ for $2 \leq i \leq \lceil \frac{k}{2} \rceil - 1$.

Since the number of vertices of HC is equal to $5(k+1)$, then we have $|CCD| = 5k - 6\left\lceil \frac{k}{2} \right\rceil + 13$.

It is easy to see that the doubly connected dominating set CCD has the minimum cardinality between the doubly connected dominating set of the graph HC. Therefore, the result follows.

□

3 Illustrative examples

In this section, we study the domination number, the total domination number, the independent domination number, the connected domination number and the doubly connected domination number of two applied honeycomb models in engineering and chemistry.

We present a kind of the structure of hydrocarbon as Hexa-peri-hexabenzocoronene (HBC) that contains a central coronene molecule, with an extra benzene ring fused between each adjacent pair of rings around the periphery (see Figure 3). The HBC molecule is interesting to chemists because of its applications and structure [26]. In the following example, we obtain the domination parameters studied in this paper on the structure of HBC.

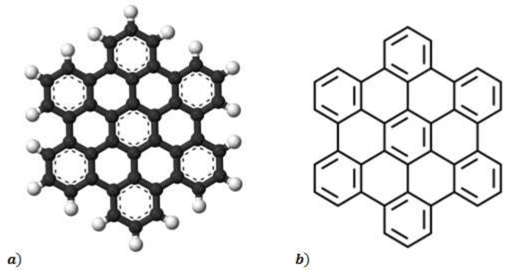

Figure 3. a) Molecular structure of Hexa-peri-hexabenzocoronene, b) The Honeycomb model of Hexa-peri-hexabenzocoronene.

Example 1. Suppose that HBC is the honeycomb structure of Hexa-peri-hexabenzocoronene is shown in Figure 3(b). With geometric transformations, the configuration of Figure 3(b) can be changed as shown in Figure 4. According to the honeycomb structure studied in Figure 2, one can consider the structure HBC containing the graph G that has the honeycomb structure with 4 cells in rows and 3 cells in columns and two paths with labels $\{a, b, c\}$ and $\{d, e, f\}$ added according to Figure 4.

First, we obtain the dominating set and the domination number of graph G. Let D be the dominating set of the structure HBC. According to Figure 4, if D_i is the dominating set of subgraph G_i for $i = 1, 2$, then, using Theorem 1, we have

$$D_i = \{1, 5, 9, 12, 16\}.$$

To dominate the vertices $\{a, b, c, d, e, f\}$, the set of vertices $\{b, e\}$ must belong to D. Therefore, the dominating set of HCB is as $D_1 \cup D_2 \cup \{b, e\}$. Thus, by Theorem 1, we obtain

$$\gamma(HBC) = \gamma(G) + 2 = (4 + 1)\lceil\frac{3}{2}\rceil + 2 = 12.$$

For obtaining the total dominating set and the total domination number of the structure HBC, if TD_i is the total dominating set of subgraph G_i for $i = 1, 2$, then, using Theorem 2, we have

$$TD_1 = \{2, 3, 6, 7, 8, 11, 13, 15, 17\},$$

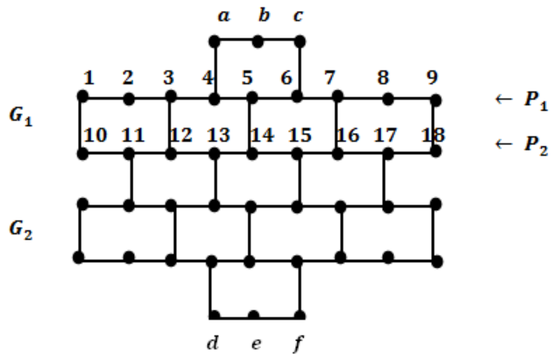

Figure 4. The structure HBC obtained from Figure 3(b)

and

$$TD_2 = \{2, 4, 6, 7, 11, 12, 15, 16, 17\}.$$

On the other hand, with selecting the vertices set $\{a, b, d, e\}$ on paths P_1 and P_2 the total dominating set of the structure HBC is as follows:

$$TD_1 \cup TD_2 \cup \{a, b, d, e\}.$$

Therefore, by Theorem 2, we obtain

$$\gamma_t(HBC) = \gamma_t(G) + 4 = 2(4 + 1) + 4(3 - 1) + 4 = 22.$$

Similar to determining the dominating set and using Theorem 3, the independent dominating set of the structure HBC is as $ID_1 \cup ID_2 \cup \{b, e\}$, where ID_i for $i = 1, 2$ are as follows:

$$ID_1 = ID_2 = \{1, 5, 9, 12, 16\}.$$

Therefore, the independent domination number of HBC is $\gamma_i(G) + 2 = 12$.

We obtain the connected dominating set of G using Theorem 4 as follows:

$$CD_1 = \{2, 3, 4, \ldots, 8, 11, 12, 16, 17\},$$

and

$$CD_2 = \{2, 3, 4, 7, 8, 11, 12, 13, 16, 17\}.$$

Therefore, one can select the connected dominating set of the structure of HBC as follows:

$$CD_1 \cup CD_2 \cup \{a, d, e\}.$$

Thus, the connected domination number of HBC is as $\gamma_c(HBC) = \gamma_c(G) + 3 = 24$.

Finally, we obtain the doubly connected dominating set of the structure HBC by Theorem 6. In this way, the subsets CCD_i of graph G are as follows:

$$CCD_1 = \{1, 2, 5, 6, \ldots, 11, 14, 15, \ldots, 18\},$$

and

$$CCD_2 = \{1, 2, 3, 8, 9, 10, \ldots, 18\}.$$

Therefore, for the doubly connected dominating set of the structure HBC, we get

$$CCD_1 \cup CCD_2 \cup \{b, c, d, e, f\}.$$

Thus, the doubly connected HBC is $\gamma_{cc}(HBC) = \gamma(G) + 5 = 27 - 4\lceil\frac{3}{2}\rceil + 9 + 5 = 32$.

For studying the domination parameters on other molecular structures that are similar to the honeycomb structure studied in this paper, these parameters can be easily computed.

Honeycomb structures are lightweight and flexible cellular structures having enormous applications in the aerospace industry, high-speed automobiles, computers and other electronics equipment bodies. A major challenge in this field is to understand the topological properties of honeycomb structures. In the next example, we study the kind of honeycomb structure as a graded honeycomb structure (GHS) (see Figure 5(a)). We consider the structure studied in [27] that has 6 rows and 15 cells in each row.

We investigate the domination parameters on the structure GHS as one of the topological properties in the honeycomb structure. For this

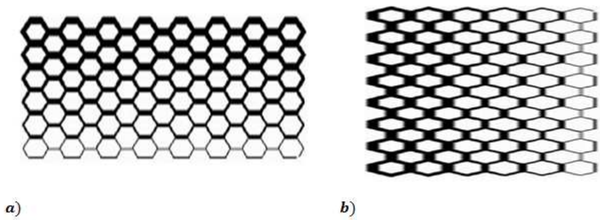

Figure 5. a) The modified graded honeycomb structure (GHS), b) Rotate Figure 5(a) 90 degrees counterclockwise

purpose, we consider the honeycomb structure GHS shown in Figure 5(b), which can be obtained by rotating 90 degrees counterclockwise of Figure 5(a). Based on the change, the structure GHS has $n = 6$ cells in rows and $k = 15$ cells in columns. Therefore, we compute easily the domination parameters of GHS using the theorems in Section 2.

Example 2. Suppose that GHS is the honeycomb structure of graded honeycomb structure is shown in Figure 5(a) with 6 rows and 15 cells in each row. With considering Figure 5(b) and using Theorem 1, the domination number of the structure GHS is obtained as follows:

$$\gamma(GHS) = (6+1)\lceil\frac{15}{2}\rceil = 56.$$

Using Theorem 2, the total domination number of GHS is as follows:

$$\gamma_t(GHS) = 2(6+1) + 6(15-1) = 98.$$

For other domination parameters, using Theorem 3, Theorem 4 and Theorem 6 we get

$$\gamma_i = (6+1)\lceil\frac{15}{2}\rceil = 56.$$

Also, the connected domination number of GHS is equal to

$$\gamma_c(GHS) = 2\lceil\frac{15}{2}\rceil(12-3) + 1 = 145,$$

and the doubly connected domination number of GHS is

$$\gamma_{cc}(GHS) = 15(12+1) - 4\lceil \frac{15}{2} \rceil + 9 = 172.$$

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