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Identities and generalized derivatives of quasigroups

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Abstract

We associate a partial (autostrophical) identity with every generalized derivative. We research when a quasigroup that satisfies an autostrophic identity has a unit (left or/and right or/and middle).

Keywords: quasigroup, quasigroup derivative, generalized quasigroup derivative, partial identity, right unit, left unit, middle unit.

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1 Introduction

This paper is a prolongation of research about units in generalized derivatives of quasigroups started in [14], [16], [20].

Notice, that the biggest part of the results is of combinatorial character, because we have used Prover 9, Mace 4 [17], [18], and standard quasigroup calculations.

Notice, that Prover 9 is one of the best automated reasoning tools and Mace 4 is one of the best constructors of counterexamples. Automated reasoning is the area of computer science that is concerned with applying reasoning in the form of logic to computing systems.

Problem 1 (Belousov's Problem #18 [1]). How to recognize identities which force quasigroups satisfying them to be loops?

The prominent role is played by the newly introduced notion of derivative operation, generalizing Belousov's notions of left/right derivative operations for quasigroups.

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Partial solutions to Belousov's Problem # 18 and their generalizations are obtained.

We would like to remember, that quasigroups have wide applications in cryptology (stream ciphers, cryptcodes, hash functions, secret-sharing schemes, El Gamal signature schemes, etc.) [20].

In order to make reading of this paper more comfortable, we repeat some concepts and definitions from [14].

1.1 Quasigroup

Definition 1. *Garrett Birkhoff [7], [9] has defined an equational quasigroup as an algebra with three binary operations $(Q, \cdot, /, \backslash)$ that satisfies the following six identities:*

$$x \cdot (x \backslash y) = y, \tag{1}$$

$$(y/x) \cdot x = y, \tag{2}$$

$$x \backslash (x \cdot y) = y, \tag{3}$$

$$(y \cdot x)/x = y, \tag{4}$$

$$x/(y \backslash x) = y, \tag{5}$$

$$(x/y) \backslash x = y. \tag{6}$$

Definition 2. *[8], [9], [13]. A groupoid $(Q, \cdot)$ is called a quasigroup if on the set Q there exist operations "$\backslash$" and "$/$" such that in the algebra $(Q, \cdot, \backslash, /)$ identities (1)–(4) are fulfilled.*

1.2 Parastrophes

Definition 3. *An n-ary groupoid (Q, A) with n-ary operation A such that in the equality $A(x_1, x_2, \dots, x_n) = x_{n+1}$ the fact of knowing any n elements of the set $\{x_1, x_2, \dots, x_n, x_{n+1}\}$ uniquely specifies the remaining one element, is called an n-ary quasigroup [3].*

If we put $n = 2$, then we obtain one more definition of a binary quasigroup.

Definition 4. *From Definition 3 it follows that with a given binary quasigroup (Q, A) it is possible to associate $(3! - 1)$ other so-called parastrophes of quasigroup (Q, A):*

$$\begin{array}{ll}
1. & A(x_1, x_2) = x_3 \Longleftrightarrow \\
2. & A^{(12)}(x_2, x_1) = x_3 \Longleftrightarrow \\
3. & A^{(13)}(x_3, x_2) = x_1 \Longleftrightarrow \\
4. & A^{(23)}(x_1, x_3) = x_2 \Longleftrightarrow \\
5. & A^{(123)}(x_2, x_3) = x_1 \Longleftrightarrow \\
6. & A^{(132)}(x_3, x_1) = x_2
\end{array}$$

[21, p. 230], [1, p. 18], [4].

1.3 Translations

The following table (Table 1) shows for each kind of translation the equivalent one in each of the (six) parastrophes of a quasigroup $(Q, \cdot)$. In fact, Table 1 is a rewritten form of results on three kinds of translations from [2]. See also [12], [19].

Table 1. Translations of quasigroup parastrophes.

	ε	(12)	(13)	(23)	(123)	(132)
R	R	L	R^{-1}	P	P^{-1}	L^{-1}
L	L	R	P^{-1}	L^{-1}	R^{-1}	P
P	P	P^{-1}	L^{-1}	R	L	R^{-1}
R^{-1}	R^{-1}	L^{-1}	R	P^{-1}	P	L
L^{-1}	L^{-1}	R^{-1}	P	L	R	P^{-1}
P^{-1}	P^{-1}	P	L	R^{-1}	L^{-1}	R

From Table 1 it follows, for example, that $R^{(132)} = L^{-1} = L^{(23)} = P^{(13)} = (R^{-1})^{(12)} = (P^{-1})^{(123)}$.

1.4 Unit elements

Suppose we have a quasigroup $(Q, \cdot)$.

Definition 5. *The fact that an element $f \in Q$ is a left identity element (left unit) for quasigroup $(Q, \cdot)$ means that $f \cdot x = x$ for all $x \in Q$.*

The fact that an element $e \in Q$ is a right identity element (right unit) for quasigroup $(Q, \cdot)$ means that $x \cdot e = x$ for all $x \in Q$.

The fact that an element $s \in Q$ is a middle identity element (middle unit) for quasigroup $(Q, \cdot)$ means that $s = x \cdot x$ for all $x \in Q$.

1.5 Generalized derivatives

By the letter T we denote the set of all quasigroup translations of a fixed quasigroup $(Q, \cdot)$ and their inverses relatively one fixed element, say, relatively element a.

Definition 6. *Quasigroup $(Q, \star) = (Q, \cdot)(\alpha, \beta, \gamma)$, where $(Q, \star)$ is isostrophic image of quasigroup $(Q, \cdot)$, i.e., $\cdot \in \{A, A^{(12)}, A^{(13)}, A^{(23)}, A^{(123)}, A^{(132)}\}$, $\alpha, \beta, \gamma \in T$, and in every case one of the translations α, β, γ is an identity permutation, is called an isostrophic (generalized) derivative of quasigroup $(Q, \cdot)$ with respect to element a [14].*

2 Some results

2.1 Autostrophies of quasigroups from generalized derivatives

From Definition 6 it follows the following

Definition 7. *Quasigroup $(Q, \star) = (Q, \cdot)(\alpha, \beta, \gamma)$, where $(Q, \star)$ is one of parastrophes of quasigroup $(Q, \cdot)$, $\alpha, \beta, \gamma \in T$, and in every case one of the translations α, β, γ, is an identity permutation, is called a derivative autostrophy of quasigroup $(Q, \cdot)$ with respect to element a.*

We can name cortege $[(\alpha, \beta, \gamma), (Q, \star)]$ as nuclear autostrophy of a type $(Q, \star)$.

For example, we can name cortege $\left[(L_a, P_a^{-1}, \varepsilon), x\backslash y\right]$ as middle nuclear autostrophy of type $x\backslash y$.

It is clear that a quasigroup can have or cannot have a derivative autostrophy.

As in the case of derivatives which are connected with autotopies, we can look at generalized derivatives

(i) as at autostrophies of quasigroup $(Q, \cdot)$;

(ii) as at identity with some fixed elements [5], [6];

(iii) as at a special nuclear autostrophy [11];

(iv) as at a nuclear identity [11].

All these approaches to derivative autostrophies are presented in [10]. Notice, it is possible to look at derivatives from the point of view

(v) of some functional equations which are defined on quasigroups or on the related groupoids [15].

2.2 Theorems and examples

Theorem 1. *Quasigroup $(Q, \cdot, /, \backslash)$ with autostrophy $((\varepsilon, L_a, R_a), x \cdot y)$ has right unit element (Subtable 15, first row).*

Proof. We can re-write autostrophy $((\varepsilon, L_a, R_a), x \cdot y)$ in the form

$$(x \cdot y) \cdot a = x \cdot (a \cdot y). \tag{7}$$

In equality (7) we substitute the term $x\backslash y$ instead of variable y and obtain

$$(x \cdot (x\backslash y)) \cdot a \overset{(1)}{=} y \cdot a = x \cdot (a \cdot (x\backslash y)). \tag{8}$$

i.e.,

$$y \cdot a = x \cdot (a \cdot (x\backslash y)). \tag{9}$$

Further, we multiply both sides of equality (9) from the left by the term $x\backslash t$ and, using identity (3), we obtain

$$x\backslash(y \cdot a) = x\backslash(x \cdot (a \cdot (x\backslash y))) \overset{(3)}{=} a \cdot (x\backslash y), \tag{10}$$

i.e.,

$$x\backslash(y \cdot a) = a \cdot (x\backslash y). \tag{11}$$

If we put $x = y$ in equality (11), then we obtain

$$x\backslash(x \cdot a) \overset{(3)}{=} a = a \cdot (x\backslash x). \tag{12}$$

From equality (12), we have

$$x\backslash x = a\backslash a \tag{13}$$

and finally, we have

$$x \cdot (a\backslash a) = x. \tag{14}$$

$\square$

Example 1. *We demonstrate that quasigroup $(Q, \cdot, /, \backslash)$ with partial identity (7) has no left and middle unit.*

$\cdot$	0	1	2	3	4	5
0	1	0	4	5	2	3
1	0	1	3	2	5	4
2	4	2	5	1	3	0
3	5	3	2	0	4	1
4	2	4	0	3	1	5
5	3	5	1	4	0	2

Example 2. *The cortege $\left[(L_a, P_a^{-1}, \varepsilon), x\backslash y\right]$ means autostrophy of the form: $L_a x \cdot P_a^{-1} y = x\backslash y$.*

Using table of translations (Table 1), we can re-write the last equality in the following form:

$$(a \cdot x) \cdot (a/y) = x\backslash y. \tag{15}$$

We have an identity of two variables and one fixed element on quasigroup $(Q, \cdot, /, \backslash)$.

From Table 2, it follows that quasigroup $(Q, \cdot)$ with equality (15) has no left (right, middle) identity element, i.e., it is a constructed quasigroup $(Q, \cdot, /, \backslash)$ with partial identity (15) which has no left (right, middle) identity element for some fixed element a. We give the needed example:

$\cdot$	0	1	2	3	4
0	1	3	0	2	4
1	3	0	2	4	1
2	0	2	4	1	3
3	2	4	1	3	0
4	4	1	3	0	2

Example 3. *The cortege* $\big[(\varepsilon, P_a^{-1}, P_a), xy\big]$ *means the following autostrophy (autostrophic derivative) of quasigroup* $(Q, \cdot)$*:* $x \cdot P_a^{-1} y = P_a(xy)$.

Using Table 1, we can re-write the last equality in the following form:

$$(xy)\backslash a = x \cdot (a/y). \tag{16}$$

From Table 2, it follows that quasigroup with equality (16) has middle identity element (see Theorem 2) and has no left and right identity element. See counterexample below. In this counterexample $a = 0$.

$\cdot$	0	1	2	3
0	1	0	2	3
1	0	1	3	2
2	2	3	1	0
3	3	2	0	1

Theorem 2. *Quasigroup* $(Q, \cdot, /, \backslash)$ *with partial identity (16) has middle identity element.*

Proof. From equality (16), substituting $x\backslash y$ instead of y, we have

$$(x \cdot (x\backslash y)\backslash a \overset{(1)}{=} y\backslash a = x \cdot (a/(x\backslash y)). \tag{17}$$

If we substitute $y = a$ in (17), then we have

$$a\backslash a = x \cdot (a/(x\backslash a)) \overset{(5)}{=} x \cdot x \tag{18}$$

for all $x \in Q$. Then $x \cdot x = a\backslash a$ for all $x \in Q$. $\square$

In Table 2, the cortege $[(L_a, L_a, \varepsilon), x/y]$ means the following autostrophy of quasigroup $(Q, \cdot)$: $L_a x \cdot L_a y = x/y$.

Using table of translations (Table 1), we can re-write the last equality in the following form:

$$(a \cdot x) \cdot (a \cdot y) = x/y. \tag{19}$$

Theorem 3. *Quasigroup $(Q, \cdot, /, \backslash)$ with identity (19) has left and middle identity element.*

Proof. Firstly, we prove that any quasigroup with this partial identity has the left identity element. We substitute term $y \backslash x$ in identity (19) for variable y. We have

$$(a \cdot x) \cdot (a \cdot (y \backslash x)) = x/(y \backslash x) \overset{(5)}{=} y. \tag{20}$$

We substitute term xy in identity (19) for variable x. We have

$$(a \cdot xy) \cdot (ay) = (xy)/y \overset{(4)}{=} x. \tag{21}$$

If we substitute $y = a$ in equality (20), then we have

$$(a \cdot x) \cdot (a \cdot (a \backslash x)) = a. \tag{22}$$

But by identity (4), $a \cdot (a \backslash x) = x$. Therefore,

$$(a \cdot x) \cdot x = a. \tag{23}$$

If in equality (23) we substitute term $a \backslash x$ for variable x, we have

$$(a \cdot (a \backslash x)) \cdot (a \backslash x) = a. \tag{24}$$

Taking into consideration that $a \cdot (a \backslash x) \overset{(1)}{=} x$, further we have

$$x \cdot (a \backslash x) = a. \tag{25}$$

If we substitute the term $a \backslash x$ for variable y in equality (21), then we have

$$(a \cdot x(a \backslash x)) \cdot (a(a \backslash x)) = x. \tag{26}$$

If we apply equalities (25) ($x(a \backslash x) = a$) and identity (1) to the left part of the last equality, then we obtain

$$(a \cdot a) \cdot x = x, \tag{27}$$

i.e., quasigroup with identity (19) has left identity element.

Secondly. We prove that this quasigroup has middle unit element. We denote element $a \cdot a$ as the left unit f. In the equality (21), we substitute $x = f$ and we obtain

$$(ay)(ay) = f. \tag{28}$$

If in the equality (28) we substitute the term $a\backslash y$ for the variable y, then we have

$$yy = f. \tag{29}$$

Therefore, quasigroup with identity (19) has the middle unit. $\square$

Example 4. *The following example demonstrates that quasigroup with identity (19) cannot have the right unit.*

$\cdot$	0	1	2	3
0	1	0	3	2
1	0	1	2	3
2	2	3	1	0
3	3	2	0	1

Here $a = 0$.

Example 5. *In Table 2 (Subtable 107, row 5), the cortege* $\big[(\varepsilon, P_a^{-1}, P_a), y/x\big]$ *means the following autostrophy of quasigroup* $(Q, \cdot)$*:* $x \cdot P_a^{-1} y = P_a(y/x)$*. Using table of translations (Table 1), we can rewrite the last equality in the following form:* $(y/x)\backslash a = x \cdot (a/y)$*.*

From Table 2, it follows that if quasigroup $(Q, \cdot, /, \backslash)$ *has the autostrophy* $\big[(\varepsilon, P_a^{-1}, P_a), y/x\big]$*, then quasigroup* $(Q, \cdot)$ *has right unit (see Theorem 4), and in general, this quasigroup cannot have left and middle unit (see counter-example below).*

$\cdot$	0	1	2	3
0	0	2	3	1
1	1	3	2	0
2	2	0	1	3
3	3	1	0	2

Theorem 4. *Quasigroup* $(Q,\cdot,/,\backslash)$ *with partial identity*

$$(y/x)\backslash a = x\cdot(a/y) \tag{30}$$

has right unit.

Proof. We substitute $y = a$ in partial identity (30) and we have

$$(a/x)\backslash a \overset{(6)}{=} x = x\cdot(a/a). \tag{31}$$

From identity (31), we have that $x = x\cdot(a/a)$, i.e., element (a/a) is a right unit in quasigroup $(Q,\cdot,/,\backslash)$.

□

Example 6. *We can re-write autostrophy* $[(\varepsilon, P_a^{-1}, P_a^{-1}), y/x]$ *(Subtable 108, 5 row) in the following form*

$$x\cdot(a/y) = a/(y/x). \tag{32}$$

We demonstrate that quasigroup $(Q,\cdot,/,\backslash)$ *has no left, right and middle unit element.*

$\cdot$	0	1	2	3	4	5
0	1	0	4	5	2	3
1	0	1	3	2	5	4
2	5	3	2	1	4	0
3	4	2	1	3	0	5
4	3	5	0	4	1	2
5	2	4	5	0	3	1

2.3 Table

We collect the obtained results in the following Table 2.

When filling out Table 2, we have used Prover 9, Mace 4 [17], [18] and standard quasigroup calculations.

In fact, Table 2 contains formulations of 1944 Theorems and counterexamples about the existence of units (left, right, middle) in quasigroups with identities that are obtained from the generalized derivatives of a quasigroup $(Q,\cdot,/,\backslash)$.

For every case a theorem is proved (usually by Prover) or there is constructed a counterexample (usually by Mace). Every sign "+"means that quasigroup with respective autostrophy (partial identity) has a corresponding unit element.

In the next papers, we plan to give human proof for the largest part of the theorems.

Table 2: Units in quasigroup that is a generalized derivative.

Avtstr.	**f**	**e**	**s**	**Avtstr.**	**f**	**e**	**s**	**Avtstr.**	**f**	**e**	**s**
1.(L_a, L_a, ε)	f	e	s	2.$(L_a, L_a^{-1}, \varepsilon)$	f	e	s	3.(L_a, R_a, ε)	f	e	s
xy	+	-	-	xy	+	-	-	xy	-	-	-
yx	-	+	-	yx	-	+	-	yx	+	+	-
$x\backslash y$	-	-	-	$x\backslash y$	+	-	-	$x\backslash y$	-	-	-
$y\backslash x$	-	-	-	$y\backslash x$	-	-	+	$y\backslash x$	-	-	-
y/x	-	-	-	y/x	-	+	-	y/x	-	-	-
x/y	+	-	+	x/y	-	-	+	x/y	-	-	-
4.$(L_a, R_a^{-1}, \varepsilon)$	f	e	s	5.(L_a, P_a, ε)	f	e	s	6.$(L_a, P_a^{-1}, \varepsilon)$	f	e	s
xy	-	-	-	xy	+	-	-	xy	+	-	-
yx	-	+	-	yx	-	+	-	yx	-	+	-
$x\backslash y$	-	-	-	$x\backslash y$	-	-	-	$x\backslash y$	-	-	-
$y\backslash x$	-	-	-	$y\backslash x$	+	-	-	$y\backslash x$	-	-	-
y/x	+	-	-	y/x	-	-	-	y/x	-	-	-
x/y	-	-	-	x/y	-	-	-	x/y	+	-	-
7.(L_a, ε, L_a)	f	e	s	8.$(L_a, \varepsilon, L_a^{-1})$	f	e	s	9.(L_a, ε, R_a)	f	e	s
xy	+	-	-	xy	+	-	-	xy	+	-	-
yx	+	+	-	yx	-	-	-	yx	-	-	-
$x\backslash y$	+	-	-	$x\backslash y$	-	-	-	$x\backslash y$	-	-	-
$y\backslash x$	-	-	+	$y\backslash x$	-	-	-	$y\backslash x$	-	-	-
y/x	-	+	+	y/x	-	-	+	y/x	+	-	+
x/y	-	-	-	x/y	-	-	-	x/y	-	-	-
10.$(L_a, \varepsilon, R_a^{-1})$	f	e	s	11.(L_a, ε, P_a)	f	e	s	12.$(L_a, \varepsilon, P_a^{-1})$	f	e	s
xy	+	-	-	xy	-	-	-	xy	-	-	-
yx	+	-	-	yx	-	-	-	yx	-	-	-
$x\backslash y$	-	-	-	$x\backslash y$	-	-	-	$x\backslash y$	-	-	-
$y\backslash x$	-	-	-	$y\backslash x$	-	-	-	$y\backslash x$	+	-	-
y/x	-	-	+	y/x	-	-	+	y/x	-	-	+
x/y	-	-	-	x/y	+	-	-	x/y	-	-	-
13.(ε, L_a, L_a)	f	e	s	14.$(\varepsilon, L_a, L_a^{-1})$	f	e	s	15.(ε, L_a, R_a)	f	e	s
xy	-	-	-	xy	-	-	-	xy	-	+	-
yx	+	+	-	yx	-	+	-	yx	+	-	-
$x\backslash y$	-	-	-	$x\backslash y$	-	-	-	$x\backslash y$	-	-	+
$y\backslash x$	-	+	-	$y\backslash x$	-	-	-	$y\backslash x$	+	-	-
y/x	-	-	-	y/x	-	-	-	y/x	-	-	+
x/y	+	-	+	x/y	-	-	+	x/y	-	+	+
16.$(\varepsilon, L_a, R_a^{-1})$	f	e	s	17.(ε, L_a, P_a)	f	e	s	18.$(\varepsilon, L_a, P_a^{-1})$	f	e	s
xy	-	+	-	xy	-	-	+	xy	-	-	+
yx	-	-	-	yx	-	-	-	yx	-	-	-
$x\backslash y$	-	-	-	$x\backslash y$	-	-	-	$x\backslash y$	-	+	-
$y\backslash x$	-	-	-	$y\backslash x$	-	-	-	$y\backslash x$	-	-	-
y/x	-	-	-	y/x	-	+	-	y/x	-	-	-
x/y	-	-	+	x/y	-	-	+	x/y	-	-	+
19. $(L_a^{-1}, L_a, \varepsilon)$	f	e	s	20. $(L_a^{-1}, L_a^{-1}, \varepsilon)$	f	e	s	21. $(L_a^{-1}, R_a, \varepsilon)$	f	e	s
xy	+	-	-	xy	+	-	-	xy	-	-	-
yx	-	-	-	yx	-	+	-	yx	+	-	-
$x\backslash y$	-	-	-	$x\backslash y$	+	-	-	$x\backslash y$	-	-	-
$y\backslash x$	-	+	-	$y\backslash x$	-	+	+	$y\backslash x$	-	+	-
y/x	-	-	-	y/x	-	+	-	y/x	-	-	-

Continued on next page

Table 2 – *Continued from previous page*

Avtstr.	**f**	**e**	**s**	**Avtstr.**	**f**	**e**	**s**	**Avtstr.**	**f**	**e**	**s**
x/y	-	-	-	x/y	+	-	+	x/y	-	-	-
22. $(L_a^{-1}, R_a^{-1}, \varepsilon)$	f	e	s	23. $(L_a^{-1}, P_a, \varepsilon)$	f	e	s	24. $(L_a^{-1}, P_a^{-1}, \varepsilon)$	f	e	s
xy	-	-	-	xy	+	-	-	xy	+	-	-
yx	-	-	-	yx	-	-	-	yx	-	-	-
$x\backslash y$	-	-	-	$x\backslash y$	-	-	-	$x\backslash y$	-	-	-
$y\backslash x$	-	+	-	$y\backslash x$	+	+	-	$y\backslash x$	-	+	-
y/x	+	-	-	y/x	-	-	-	y/x	-	-	-
x/y	-	-	-	x/y	-	-	-	x/y	+	-	-
25. $(L_a^{-1}, \varepsilon, L_a)$	f	e	s	26. $(L_a^{-1}, \varepsilon, L_a^{-1})$	f	e	s	27. $(L_a^{-1}, \varepsilon, R_a)$	f	e	s
xy	+	-	-	xy	+	-	-	xy	+	-	-
yx	-	+	-	yx	+	+	-	yx	-	-	-
$x\backslash y$	+	-	-	$x\backslash y$	-	-	-	$x\backslash y$	-	-	-
$y\backslash x$	-	-	+	$y\backslash x$	-	-	-	$y\backslash x$	-	-	-
y/x	-	+	-	y/x	-	-	-	y/x	+	-	-
x/y	-	-	+	x/y	-	-	+	x/y	-	-	+
28. $(L_a^{-1}, \varepsilon, R_a^{-1})$	f	e	s	29. $(L_a^{-1}, \varepsilon, P_a)$	f	e	s	30. $(L_a^{-1}, \varepsilon, P_a^{-1})$	f	e	s
xy	+	-	-	xy	-	-	-	xy	-	-	-
yx	+	-	-	yx	-	-	-	yx	-	-	-
$x\backslash y$	-	-	-	$x\backslash y$	-	-	-	$x\backslash y$	-	-	-
$y\backslash x$	-	-	-	$y\backslash x$	-	-	-	$y\backslash x$	+	-	-
y/x	-	-	-	y/x	-		-	y/x	-	-	-
x/y	-	-	+	x/y	+	-	+	x/y	-	-	+
31.$(\varepsilon, L_a^{-1}, L_a)$	f	e	s	32$(\varepsilon, L_a^{-1}, L_a^{-1})$	f	e	s	33.$(\varepsilon, L_a^{-1}, R_a)$	f	e	s
xy	-	-	-	xy	-	-	-	xy	-	+	-
yx	-	-	-	yx	+	+	-	yx	+	-	-
$x\backslash y$	-	-	-	$x\backslash y$	-	-	-	$x\backslash y$	-	-	+
$y\backslash x$	-	+	-	$y\backslash x$	-	-	-	$y\backslash x$	+	-	-
y/x	-	-	+	y/x	-	-	+	y/x	-	-	+
x/y	-	-	-	x/y	+	-	+	x/y	-	+	-
34. $(\varepsilon, L_a^{-1}, R_a^{-1})$	f	e	s	35. $(\varepsilon, L_a^{-1}, P_a)$	f	e	s	36. $(\varepsilon, L_a^{-1}, P_a^{-1})$	f	e	s
xy	-	+	-	xy	-	-	+	xy	-	-	+
yx	-	-	-	yx	-	-	-	yx	-	-	-
$x\backslash y$	-	-	-	$x\backslash y$	-	-	-	$x\backslash y$	-	+	-
$y\backslash x$	-	-	-	$y\backslash x$	-	-	-	$y\backslash x$	-	-	-
y/x	-	-	+	y/x	-	+	+	y/x	-	-	+
x/y	-	-	-	x/y	-	-	-	x/y	-	-	-
37.(R_a, L_a, ε)	f	e	s	38.$(R_a, L_a^{-1}, \varepsilon)$	f	e	s	39.(R_a, R_a, ε)	f	e	s
xy	+	+	-	xy	+	+	-	xy	-	+	-
yx	-	-	-	yx	-	+	-	yx	+	-	-
$x\backslash y$	-	-	-	$x\backslash y$	+	-	-	$x\backslash y$	-	+	+
$y\backslash x$	-	-	-	$y\backslash x$	-	-	+	$y\backslash x$	-	-	-
y/x	-	-	-	y/x	-	+	-	y/x	-	-	-
x/y	-	-	-	x/y	-	-	+	x/y	-	-	-
40.$(R_a, R_a^{-1}, \varepsilon)$	f	e	s	41.(R_a, P_a, ε)	f	e	s	42.$(R_a, P_a^{-1}, \varepsilon)$	f	e	s
xy	-	+	-	xy	-	+	-	xy	-	+	-
yx	-	-	-	yx	-	-	-	yx	-	-	-
$x\backslash y$	-	-	-	$x\backslash y$	-	-	-	$x\backslash y$	-	-	-
$y\backslash x$	-	-	-	$y\backslash x$	+	-	-	$y\backslash x$	-	-	-
y/x	+	-	-	y/x	-	-	-	y/x	-	-	-
x/y	-	-	-	x/y	-	-	-	x/y	+	-	-
43.(R_a, ε, L_a)	f	e	s	44.$(R_a, \varepsilon, L_a^{-1})$	f	e	s	45.(R_a, ε, R_a)	f	e	s
xy	+	-	-	xy	+	-	-	xy	-	-	-
yx	-	+	-	yx	-	-	-	yx	+	+	-
$x\backslash y$	+	-	+	$x\backslash y$	-	-	+	$x\backslash y$	-	+	+
$y\backslash x$	-	-	+	$y\backslash x$	-	-	-	$y\backslash x$	-	-	-
y/x	-	+	-	y/x	-	-	-	y/x	+	-	-
x/y	-	-	+	x/y	-	-	-	x/y	-	-	-
46.$(R_a, \varepsilon, R_a^{-1})$	f	e	s	47.(R_a, ε, P_a)	f	e	s	48.$(R_a, \varepsilon, P_a^{-1})$	f	e	s

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Table 2 – *Continued from previous page*

Avtstr.	**f**	**e**	**s**	**Avtstr.**	**f**	**e**	**s**	**Avtstr.**	**f**	**e**	**s**
xy	-	-	-	xy	-	-	+	xy	-	-	+
yx	+	-	-	yx	-	-	-	yx	-	-	-
$x\backslash y$	-	-	+	$x\backslash y$	-	-	+	$x\backslash y$	-	-	+
$y\backslash x$	-	-	-	$y\backslash x$	-	-	-	$y\backslash x$	+	-	-
y/x	-	-	-	y/x	-	-	-	y/x	-	-	-
x/y	-	-	-	x/y	+	-	-	x/y	-	-	-
49.(ε, R_a, L_a)	f	e	s	50.$(\varepsilon, R_a, L_a^{-1})$	f	e	s	51.(ε, R_a, R_a)	f	e	s
xy	-	+	-	xy	-	+	-	xy	-	+	-
yx	-	-	-	yx	-	+	-	yx	+	+	-
$x\backslash y$	-	-	-	$x\backslash y$	-	-	-	$x\backslash y$	-	-	+
$y\backslash x$	-	+	+	$y\backslash x$	-	-	+	$y\backslash x$	+	-	+
y/x	-	-	-	y/x	-	-	-	y/x	-	-	+
x/y	-	-	-	x/y	-	-	-	x/y	-	+	-
52.$(\varepsilon, R_a, R_a^{-1})$	f	e	s	53.(ε, R_a, P_a)	f	e	s	54.$(\varepsilon, R_a, P_a^{-1})$	f	e	s
xy	-	+	-	xy	-	-	-	xy	-	-	-
yx	-	-	-	yx	-	-	-	yx	-	-	-
$x\backslash y$	-	-	-	$x\backslash y$	-	-	-	$x\backslash y$	-	+	-
$y\backslash x$	-	-	+	$y\backslash x$	-	-	+	$y\backslash x$	-	-	+
y/x	-	-	-	y/x	-	+	-	y/x	-	-	-
x/y	-	-	-	x/y	-	-	-	x/y	-	-	-
55. $(R_a^{-1}, L_a, \varepsilon)$	f	e	s	56. $(R_a^{-1}, L_a^{-1}, \varepsilon)$	f	e	s	57. $(R_a^{-1}, R_a, \varepsilon)$	f	e	s
xy	+	+	-	xy	+	+	-	xy	-	+	-
yx	+	-	-	yx	+	+	-	yx	+	-	-
$x\backslash y$	-	-	+	$x\backslash y$	+	-	+	$x\backslash y$	-	-	+
$y\backslash x$	+	-	-	$y\backslash x$	+	-	+	$y\backslash x$	+	-	-
y/x	-	-	+	y/x	-	+	+	y/x	-	-	+
x/y	-	+	-	x/y	-	+	+	x/y	-	+	-
58. $(R_a^{-1}, R_a^{-1}, \varepsilon)$	f	e	s	59. $(R_a^{-1}, P_a, \varepsilon)$	f	e	s	60. $(R_a^{-1}, P_a^{-1}, \varepsilon)$	f	e	s
xy	-	+	-	xy	-	+	-	xy	-	+	-
yx	+	-	-	yx	+	-	-	yx	+	-	-
$x\backslash y$	-	+	+	$x\backslash y$	-	-	+	$x\backslash y$	-	-	+
$y\backslash x$	+	-	-	$y\backslash x$	+	-	-	$y\backslash x$	+	-	-
y/x	+	-	+	y/x	-	-	+	y/x	-	-	+
x/y	-	+	-	x/y	-	+	-	x/y	+	+	-
61.$(R_a^{-1}, \varepsilon, L_a)$	f	e	s	62$(R_a^{-1}, \varepsilon, L_a^{-1})$	f	e	s	63.$(R_a^{-1}, \varepsilon, R_a)$	f	e	s
xy	+	-	-	xy	+	-	-	xy	-	-	-
yx	-	+	-	yx	-	-	-	yx	-	-	-
$x\backslash y$	+	-	-	$x\backslash y$	-	-	-	$x\backslash y$	-	-	-
$y\backslash x$	-	-	+	$y\backslash x$	-	-	+	$y\backslash x$	-	-	+
y/x	-	+	-	y/x	-	-	-	y/x	+	-	-
x/y	-	-	+	x/y	-	-	-	x/y	-	-	-
64. $(R_a^{-1}, \varepsilon, R_a^{-1})$	f	e	s	65. $(R_a^{-1}, \varepsilon, P_a)$	f	e	s	66. $(R_a^{-1}, \varepsilon, P_a^{-1})$	f	e	s
xy	-	-	-	xy	-	-	+	xy	-	-	+
yx	+	+	-	yx	-	-	-	yx	-	-	-
$x\backslash y$	-	+	+	$x\backslash y$	-	-	-	$x\backslash y$	-	-	-
$y\backslash x$	-	-	+	$y\backslash x$	-	-	+	$y\backslash x$	+	-	+
y/x	-	-	-	y/x	-	-	-	y/x	-	-	-
x/y	-	-	-	x/y	+	-	-	x/y	-	-	-
67. $(\varepsilon, R_a^{-1}, L_a)$	f	e	s	68. $(\varepsilon, R_a^{-1}, L_a^{-1})$	f	e	s	69. $(\varepsilon, R_a^{-1}, R_a)$	f	e	s
xy	-	+	-	xy	-	+	-	xy	-	+	-
yx	-	-	-	yx	-	+	-	yx	+	-	-
$x\backslash y$	-	-	+	$x\backslash y$	-	-	+	$x\backslash y$	-	-	+
$y\backslash x$	-	+	-	$y\backslash x$	-	-	-	$y\backslash x$	+	-	-
y/x	-	-	-	y/x	-	-	-	y/x	-	-	+
x/y	-	-	-	x/y	-	-	-	x/y	-	+	-
70. $(\varepsilon, R_a^{-1}, R_a^{-1})$	f	e	s	71. $(\varepsilon, R_a^{-1}, P_a)$	f	e	s	72. $(\varepsilon, R_a^{-1}, P_a^{-1})$	f	e	s
xy	-	+	-	xy	-	-	-	xy	-	-	-

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Table 2 – *Continued from previous page*

Avtstr.	**f**	**e**	**s**	**Avtstr.**	**f**	**e**	**s**	**Avtstr.**	**f**	**e**	**s**
yx	+	+	-	yx	-	-	-	yx	-	-	-
$x\backslash y$	-	-	+	$x\backslash y$	-	-	+	$x\backslash y$	-	+	+
$y\backslash x$	-	-	-	$y\backslash x$	-	-	-	$y\backslash x$	-	-	-
y/x	-	-	-	y/x	-	+	-	y/x	-	-	-
x/y	-	-	-	x/y	-	-	-	x/y	-	-	-
73.(P_a, L_a, ε)	f	e	s	74.$(P_a, L_a^{-1}, \varepsilon)$	f	e	s	75.(P_a, R_a, ε)	f	e	s
xy	+	-	-	xy	+	-	-	xy	-	+	-
yx	-	-	-	yx	-	+	-	yx	+	-	-
$x\backslash y$	-	+	-	$x\backslash y$	+	+	-	$x\backslash y$	-	+	-
$y\backslash x$	-	-	-	$y\backslash x$	-	-	+	$y\backslash x$	-	-	-
y/x	-	-	-	y/x	-	+	-	y/x	-	-	-
x/y	-	-	-	x/y	-	-	+	x/y	-	-	-
76.$(P_a, R_a^{-1}, \varepsilon)$	f	e	s	77.(P_a, P_a, ε)	f	e	s	78.$(P_a, P_a^{-1}, \varepsilon)$	f	e	s
xy	-	+	-	xy	-	-	-	xy	-	-	-
yx	-	-	-	yx	-	-	-	yx	-	-	-
$x\backslash y$	-	+	-	$x\backslash y$	-	+	+	$x\backslash y$	-	+	-
$y\backslash x$	-	+	-	$y\backslash x$	+	-	-	$y\backslash x$	-	-	-
y/x	+	-	-	y/x	-	-	-	y/x	-	-	-
x/y	-	-	-	x/y	-	-	-	x/y	+	-	-
79.(P_a, ε, L_a)	f	e	s	80.$(P_a, \varepsilon, L_a^{-1})$	f	e	s	81.(P_a, ε, R_a)	f	e	s
xy	+	-	+	xy	+	-	+	xy	-	-	+
yx	-	+	-	yx	-	-	-	yx	-	-	-
$x\backslash y$	+	-	-	$x\backslash y$	-	-	-	$x\backslash y$	-	-	-
$y\backslash x$	-	-	+	$y\backslash x$	-	-	-	$y\backslash x$	-	-	-
y/x	-	+	-	y/x	-	-	-	y/x	+	-	-
x/y	-	-	+	x/y	-	-	-	x/y	-	-	-
82.$(P_a, \varepsilon, R_a^{-1})$	f	e	s	83.(P_a, ε, P_a)	f	e	s	84.$(P_a, \varepsilon, P_a^{-1})$	f	e	s
xy	-	-	+	xy	-	-	+	xy	-	-	+
yx	+	-	-	yx	-	-	-	yx	-	-	-
$x\backslash y$	-	-	-	$x\backslash y$	-	-	-	$x\backslash y$	-	-	-
$y\backslash x$	-	-	-	$y\backslash x$	-	-	-	$y\backslash x$	+	-	-
y/x	-	-	-	y/x	-	-	-	y/x	-	-	-
x/y	-	-	-	x/y	+	-	-	x/y	-	-	-
85.(ε, P_a, L_a)	f	e	s	86.$(\varepsilon, P_a, L_a^{-1})$	f	e	s	87.(ε, P_a, R_a)	f	e	s
xy	-	-	+	xy	-	-	+	xy	-	+	+
yx	-	-	+	yx	-	+	+	yx	+	-	+
$x\backslash y$	-	+	-	$x\backslash y$	-	+	-	$x\backslash y$	-	+	+
$y\backslash x$	-	+	-	$y\backslash x$	-	+	-	$y\backslash x$	+	+	-
y/x	+	-	-	y/x	+	-	-	y/x	+	-	+
x/y	+	-	-	x/y	+	-	-	x/y	+	+	-
88.$(\varepsilon, P_a, R_a^{-1})$	f	e	s	89.(ε, P_a, P_a)	f	e	s	90.$(\varepsilon, P_a, P_a^{-1})$	f	e	s
xy	-	+	-	xy	-	-	-	xy	-	-	-
yx	-	-	+	yx	-	-	+	yx	-	-	+
$x\backslash y$	-	-	-	$x\backslash y$	-	-	-	$x\backslash y$	-	+	-
$y\backslash x$	-	-	-	$y\backslash x$	-	-	-	$y\backslash x$	-	-	-
y/x	-	-	-	y/x	-	+	-	y/x	-	-	-
x/y	-	-	-	x/y	-	-	-	x/y	-	-	-
91. $(P_a^{-1}, L_a, \varepsilon)$	f	e	s	92. $(P_a^{-1}, L_a^{-1}, \varepsilon)$	f	e	s	93. $(P_a^{-1}, R_a, \varepsilon)$	f	e	s
xy	+	-	-	xy	-	-	-	xy	-	-	-
yx	-	-	-	yx	-	-	-	yx	+	-	-
$x\backslash y$	-	-	-	$x\backslash y$	+	-	-	$x\backslash y$	-	-	-
$y\backslash x$	-	-	-	$y\backslash x$	-	-	-	$y\backslash x$	-	-	-
y/x	-	+	-	y/x	-	+	-	y/x	-	+	-
x/y	-	-	-	x/y	-	-	-	x/y	-	-	-
94. $(P_a^{-1}, R_a^{-1}, \varepsilon)$	f	e	s	95. $(P_a^{-1}, P_a, \varepsilon)$	f	e	s	96. $(P_a^{-1}, P_a^{-1}, \varepsilon)$	f	e	s
xy	-	-	-	xy	-	-	-	xy	-	-	-
yx	-	-	-	yx	-	-	-	yx	-	-	-
$x\backslash y$	-	-	-	$x\backslash y$	-	-	-	$x\backslash y$	-	-	-
$y\backslash x$	-	-	-	$y\backslash x$	+	-	-	$y\backslash x$	-	-	-
y/x	+	+	-	y/x	-	+	-	y/x	-	+	-

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Table 2 – *Continued from previous page*

Avtstr.	**f**	**e**	**s**	**Avtstr.**	**f**	**e**	**s**	**Avtstr.**	**f**	**e**	**s**
x/y	-	-	-	x/y	-	-	-	x/y	+	-	-
97. $(P_a^{-1}, \varepsilon, L_a)$	f	e	s	98. $(P_a^{-1}, \varepsilon, L_a^{-1})$	f	e	s	99. $(P_a^{-1}, \varepsilon, R_a)$	f	e	s
xy	+	-	+	xy	+	-	+	xy	-	-	+
yx	-	+	+	yx	-	-	+	yx	-	-	+
$x\backslash y$	+	+	-	$x\backslash y$	-	+	-	$x\backslash y$	-	+	-
$y\backslash x$	-	+	+	$y\backslash x$	-	+	-	$y\backslash x$	-	+	-
y/x	+	+	-	y/x	+	-	-	y/x	+	-	-
x/y	+	-	+	x/y	+	-	-	x/y	+	-	-
100. $(P_a^{-1}, \varepsilon, R_a^{-1})$	f	e	s	101. $(P_a^{-1}, \varepsilon, P_a)$	f	e	s	102. $(P_a^{-1}, \varepsilon, P_a^{-1})$	f	e	s
xy	-	-	+	xy	-	-	+	xy	-	-	+
yx	+	-	+	yx	-	-	+	yx	-	-	+
$x\backslash y$	-	+	-	$x\backslash y$	-	+	-	$x\backslash y$	-	+	+
$y\backslash x$	-	+	-	$y\backslash x$	-	+	-	$y\backslash x$	+	+	-
y/x	+	-	-	y/x	+	-	-	y/x	+	-	+
x/y	+	-	-	x/y	+	-	-	x/y	+	-	-
103. $(\varepsilon, P_a^{-1}, L_a)$	f	e	s	104. $(\varepsilon, P_a^{-1}, L_a^{-1})$	f	e	s	105. $(\varepsilon, P_a^{-1}, R_a)$	f	e	s
xy	-	-	+	xy	-	-	+	xy	-	+	+
yx	-	-	-	yx	-	+	-	yx	+	-	-
$x\backslash y$	-	-	-	$x\backslash y$	-	-	-	$x\backslash y$	-	-	+
$y\backslash x$	-	+	-	$y\backslash x$	-	-	-	$y\backslash x$	+	-	-
y/x	-	-	-	y/x	-	-	-	y/x	-	-	+
x/y	-	-	-	x/y	-	-	-	x/y	-	+	-
106. $(\varepsilon, P_a^{-1}, R_a^{-1})$	f	e	s	107. $(\varepsilon, P_a^{-1}, P_a)$	f	e	s	108. $(\varepsilon, P_a^{-1}, P_a^{-1})$	f	e	s
xy	-	+	+	xy	-	-	+	xy	-	-	+
yx	-	-	-	yx	-	-	-	yx	-	-	-
$x\backslash y$	-	-	-	$x\backslash y$	-	-	-	$x\backslash y$	-	+	-
$y\backslash x$	-	-	-	$y\backslash x$	-	-	-	$y\backslash x$	-	-	-
y/x	-	-	-	y/x	-	+	-	y/x	-	-	-
x/y	-	-	-	x/y	-	-	-	x/y	+	-	+

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