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The aproximation of functions in generalized Hölder spaces

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Abstract

The theorems of approximation of complex functions determined on the closed arbitrary contour Γ of the complex plane by means of Lagrange interpolating polynomials in generalized Hölder spaces H_ω were obtained.

The main purpose of the paper is to prove the theorems of the approximation of functions of complex variable defined on arbitrary smooth closed contours of the complex plane by interpolating Lagrange polinomial in generalized Hölder spaces. These theoretical rezults have a large application in the theory of numerical methods for solving functional equations.

Let Γ be arbitrary smooth closed contour bounding a simply connected region of the complex plane.

By $\omega(\sigma)(\sigma \in (0,l],\ l = \max_{t',t''\in\Gamma} |t'-t''|)$ we shall denote the arbitrary module of continuity and by $H(\omega)$ the generalized Hölder space [1] with the norm

$$\| g \|_\omega = \| g \|_c + H(g;\omega); \tag{1}$$

$$\| g \|_c = \max_{t\in\Gamma} |g(t)|, H(g;\omega) = \sup_{\sigma\in(0,l]} \frac{\omega(g;\sigma)}{\omega(\sigma)},$$

here the $\omega(g;\sigma)$ is the module of continuity of function $g(t)$ on Γ. We consider only the spaces $H(\omega)$ with the modules of continuity satisfying the Bari-Stechkin conditions [2]:

$$\int_0^h \frac{\omega(\xi)}{\xi} < \infty, \tag{2}$$

$$\int_0^\delta \frac{\omega(\xi)}{\xi} d\xi + \delta \cdot \int_\delta^h \frac{\omega(\xi)}{\xi^2} d\xi = O(\omega(\delta)), \delta \to 0. \quad (3)$$

By $H^{(r)}(\omega)$, $r \geq 0 (H^{(0)}(\omega) = H(\omega))$ we denote the space of r-times continuous-differentiable functions.The r -order derivatives of these functions are elements of space $H(\omega)$. The norm on $H^{(r)}(\omega)$ is defined by

$$\| g \|_{\omega,r} = \sum_{k=0}^{r} \| g^{(k)} \|_c + H(g^{(r)}; \omega).$$

Recall that if $\omega(\delta) = \delta^\alpha$, $\alpha \in (0,1]$, then $H(\omega) = H_\alpha$ is the Hölder space with exponent α.

The space $H(\omega)$ is a Banach nonseparable space. So the approximation of the whole class of functions $H(\omega)$ by norm (1) with the help of finite-dimensional approximation is impossible. But for some subset from $H(\omega)$ the problem can be solved in the affirmative.

Let $t_j (j = \overline{0, 2n})$ be a set of distinct points on Γ.By U_n we denote the operator, which maps any function $g(t)$ defined on Γ into its interpolating Lagrange polinomial defined by using the nodes t_j :

$$(U_n g)(t) = \sum_{j=0}^{2n} g(t_j) l_j(t), t \in \Gamma, \quad (4)$$

$$l_j(t) = \prod_{k=0, k \neq j}^{2n} \frac{t - t_k}{t_j - t_k} \cdot \left(\frac{t_j}{t} \right)^n \equiv \sum_{r=-n}^{n} \Lambda_r^{(j)} \cdot t^r, j = \overline{0, 2n}.$$

The following theorem gives the deviation of the Lagrange polinomial from generalized Hölder function.

Theorem 1 *Let $\omega_1(\sigma)$ and $\omega_2(\sigma) (\sigma \in (0, l])$ be modules of continuity satisfying (2) or (3) such that the function $\Phi(\sigma) = \omega_1 / \omega_2$ is nontecreasing on (0,l] and the points $t_j, j = (\overline{0, 2n})$ make a system of Fejer nodes on Γ:*

$$t_j = \psi(exp \frac{2\pi i}{2n+1} (j - n)), j = (\overline{0, 2n}), \quad (5)$$

here $z = \psi(W)$ *is Riemann function for contour* Γ. *Then for any function* $g(t) \in H^r(\omega_1), r = 0, 1, 2, ...$ *the following estimate holds:*

$$\| g - U_n g \|_{\omega_2} \leq (d_1 + d_2 ln(n)) \cdot \frac{1}{n^r} \cdot \Phi\left(\frac{1}{n}\right) H(g^{(r)}; \omega_1) \qquad (6)$$

($d_k (k = 1, 2, \ldots)$ *denote certain constants not depending on* n*).*

The proof of this theorem for $r = 0$ is given in [3]. The proof for $r = 0$ from [3] cannot be adopted for $r \geq 1$ as well as the proof for $r \geq 1$ does not permit to obtain the theorem for $r = 0$.

The proof of theorem for $r \geq 1$**.** From the definition of the norm in $H(\omega_2)$ we obtain

$$\| g - U_n g \|_{\omega_2} = \| g - U_n g \|_c + H(g - U_n g; \omega_2) = S_1 + S_2$$

For the first term S_1 applying the Jackson theorem [4] and the estimation of norm $\| U_n \|_c$ from [5] we deduce

$$S_1 \leq (1 + \| U_n \|_c) \cdot E_n(g; \Gamma) \leq (1 + \| U_n \|_c) \cdot \frac{d_3}{n^r} \omega(g^{(r)}; \frac{1}{n}) \leq$$

$$\leq \frac{d_3 \cdot (d_4 + d_5 ln(n))}{n^r} \cdot \omega(g^{(r)}; \frac{1}{n}) =$$

$$= \frac{d_6 + d_7 ln(n)}{n^r} \cdot \frac{\omega(g^{(r)}; \frac{1}{n})}{\omega_1(\frac{1}{n})} \cdot \omega_1(\frac{1}{n}) \leq$$

$$\leq \frac{d_6 + d_7 ln(n)}{n^r} \cdot \omega_1(\frac{1}{n}) \cdot H(g^{(r)}; \omega_1), (d_6 = d_3 \cdot d_4; d_7 = d_3 \cdot d_5),$$

where $E_n(g; \Gamma)$ is the best uniform approximation of function $g(t)$ on Γ by polynomial of the type $\sum_{k=-n}^{n} r_k t^k, t \in \Gamma$ where r_k are arbitrary numbers.

Let estimate the second term:

$$S_2 \leq \max \left\{ \sup_{|t' - t''| > \frac{1}{n}} \frac{|g(t') - (U_n g)(t') - g(t'') + (U_n g)(t'')|}{\omega_2(|t' - t''|)}; \right.$$

$$\left.\sup_{0<|t'-t''|<\frac{1}{n}} \frac{|g(t')-(U_ng)(t')-g(t'')+(U_ng)(t'')|}{\omega_2(|t'-t''|)}\right\} = \max\{T_1;T_2\}.$$

Now we can estimate T_1,knowing the estimation of S_1 and

$$T_1 \leq 2\cdot \parallel g-U_ng \parallel_c \cdot \frac{1}{\omega_2(\frac{1}{n})} \leq 2\cdot \frac{d_6+d_7 ln(n)}{n^r}\Phi(\frac{1}{n})H(g^{(r)};\omega_1).$$

Since $lim_{n\to\infty} \parallel g-U_ng \parallel_c = 0$, the difference $g(t)-U_ng(t)$ can be written as follows:

$$g(t)-U_ng(t) = \sum_{k=1}^{\infty} R_k(t),$$

$$R_k(t) = U_{2^k n}g(t) - U_{2^{k-1}n}g(t)$$

Then

$$g(t')-U_ng(t')-g(t'')+U_ng(t'') = \sum_{k=1}^{\infty}\left[R_k(t')-R_k(t'')\right]. \qquad (7)$$

For the contour Γ we have the relation

$$t'\breve{t}'' \leq d_8|t'-t''|, \forall t', t'' \in \Gamma,$$

where $t'\breve{t}''$ is the lenght of the smallest arc between the points t' and t''. Therefore

$$|R_k(t')-R_k(t'')| = |\int_{t'}^{t''} R_k^{'}(\tau)d\tau| \leq \parallel R_k' \parallel_c \cdot t'\breve{t}'' \leq d_8 \parallel R_k' \parallel_c \cdot|t'-t''|$$

According to [5,p.43],

$$\parallel R_k^{'} \parallel_c \leq d_9\cdot 2^k \cdot n\cdot \parallel R_k \parallel_c.$$

Hence

$$|R_k(t')-R_k(t'')| \leq d_8\cdot d_9\cdot 2^k\cdot n\cdot \parallel R_k \parallel_c \cdot|t'-t''| \qquad (8)$$

Since

$$\parallel R_k \parallel_c \leq \parallel g-U_{2^k n} \parallel_c + \parallel g-U_{2^{k-1}n} \parallel_c$$

then using the estimation of S_1 we have

$$\| R_k \|_c \leq \frac{d_6 + d_7 \cdot ln(2^k n)}{(2^k \cdot n)^r} \cdot \omega(g^{(r)}; \frac{1}{2^k n}) +$$

$$+ \frac{d_6 + d_7 \cdot ln(2^{k-1} n)}{(2^{k-1} \cdot n)^r} \cdot \omega(g^{(r)}; \frac{1}{2^{k-1} n})$$

Hence, by this estimation and by the inequalites (8) and

$$\omega(g^{(r)}; \frac{1}{2^{k-1} n}) \leq \omega(g^{(r)}; \frac{1}{n}), k = 1, 2, \dots,$$

it follows that

$$\sum_{k=1}^{n} |R_k(t') - R_k(t'')| \leq d_8 \cdot d_9 \cdot n \cdot |t' - t''| \cdot \frac{1}{n^r} \sum_{k=1}^{\infty} \left[\frac{d_6 + d_7 \cdot ln(2^k n)}{2^{(k-1)r}} + \right.$$

$$\left. + \frac{d_6 + d_7 \cdot ln(2^{k-1} n)}{2^{(k-2)r}} \right] \cdot \omega(g^{(r)}; \frac{1}{n}).$$

It is proved in [5,p.51-53] that the last sum does not excede the value $(d_{10} + d_{11} ln(n))$. Therefore

$$\sum_{k=1}^{\infty} |R_k(t') - R_k(t'')| \leq \frac{d_{12} + d_{13} \cdot ln(n)}{n^r} \cdot n \cdot |t' - t''| \cdot \omega(g^{(r)}; \frac{1}{n}),$$

$$d_{12} = d_8 \cdot d_9 \cdot d_{10}; \qquad d_{13} = d_8 \cdot d_9 \cdot d_{11}.$$

From this inequality, (7) and the definition of T_2 we get

$$T_2 \leq \frac{d_{12} + d_{13} \cdot ln(n)}{n^r} \cdot n \cdot \omega(g^{(r)}; \frac{1}{n}) \cdot \sup_{0 < |t' - t''| \leq \frac{1}{n}} \frac{|t' - t''|}{\omega_2(|t' - t''|)} =$$

$$= \frac{d_{12} + d_{13} \cdot ln(n)}{n^r} \cdot n \cdot \omega(g^{(r)}; \frac{1}{n}) \cdot \sup \left\{ \Phi(|t' - t''|) \cdot \frac{|t' - t''|}{\omega_1(|t' - t''|)} \right\}.$$

According to [1,p.50], if $\omega(\sigma)$ is arbitrary module of continuity, then for $\sigma_1 \leq \sigma_2$ the following inequality holds

$$\frac{\sigma_1}{\omega(\sigma_1)} \leq 2 \cdot \frac{\sigma_2}{\omega(\sigma_2)}.$$

In our case $|t' - t''| \leq \frac{1}{n}$, so

$$\frac{|t' - t''|}{\omega_1(|t' - t''|)} \leq 2 \cdot \frac{\frac{1}{n}}{\omega_1(\frac{1}{n})},$$

Hence

$$T_2 \leq 2 \cdot \frac{d_{12} + d_{13} \cdot ln(n)}{n^r} \cdot n \cdot \omega(g^{(r)}; \frac{1}{n}) \cdot \frac{\frac{1}{n}}{\omega_1(\frac{1}{n})} \sup_{|t'-t''| \leq \frac{1}{n}} \Phi(|t' - t''|) \leq$$

$$\leq 2 \cdot \frac{d_{12} + d_{13} \cdot ln(n)}{n^r} \Phi(\frac{1}{n}) H(g^{(r)}; \omega_1).$$

From this inequality and from estimation of T_1 we obtain (6).The theorem is proved. The analogous theorem for the case of classical Hölder spaces on the field of real numbers was proved in [6].

References

[1] Gusejnov A.I., Muhtarov H.Sh. Introduction in the theory of nonlinear integral equations, Moscow, Nauka, 1980 (Russian).

[2] Bari N.K. Trigonometrical series , Moscow, GIFML, 1961 (Russian).

[3] V.Zolotarevski, G.Andriesh. Aproximate solutions of nonelliptic singular integral equations in generalized Hölder spaces, Bilanţul activităţii ştiinţifice a U.S.M. pe anii 1993-1994, 23–27 martie 1995, Chişinău,U.S.M., 1995 (Romanian).

[4] Dziadik V.K. Introduction to the theory of the uniform approximation of functions by polinomials, Moscow, Nauka, 1977 (Russian).

[5] Zolotarevski V.A. Finite dimensional methods of solving singular integral equations on closed contours of integration, Kishinev, Shtiintsa, 1991 (Russian).

[6] Gabdulhaev B.G. About the approximation by the trigonometrical polinomials and the errors of the quadrature formules for singular integrals, Kazan, Uchennie zapiski Kazanskogo universiteta, 1967 (Russian).

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