Computer Science Journal of Moldova, vol.**3**, no.3(9), 1995

On graphs containing a given graph as median

A.Poshtaru R.Boliac

Abstract

P.J.Slater demonstrated that any connected graph G is the median of a graph G^* and indicated the construction of G^*. His construction requires $O(n^2)$ new vertices, where n is the number of vertices of G. In this paper the same problem is considered and some new constructions of G^* are presented. The number of new vertices added to G is considerably improved. From our constructions it follows that in the general case n new vertices are necessary.

1 Introduction

Let G be a connected, simple graph [2], i.e. G is finite, nonoriented, loopless and without multiple edges. We denote by $V(G)$ the set of vertices of G and by $E(G)$ is the set of its edges. Let us define on the set $V(G)$ the function

$$\sigma_G(v) = \sum_{w \in V(G)} d(v,w) \ ,$$

where by $d(v,w)$ is denoted the distance between the vertices v and w in G (the distance between v and w is the minimal number of edges in a v to w path).

Definition 1 *Median of G is called the subgraph generated by the subset of vertices:*

$$\{v \in V(G) : \sigma_G(v) = \min_{w \in V(G)} \sigma_G(w)\}.$$

Let us denote the median of G by $Med(G)$.

Consider the simple graph G with $|V(G)| = n$, $|E(G)| = m$. Let $r_i = \deg v_i,\ i = \overline{1,n},\ R = \max_{i=\overline{1,n}} r_i,\ r = \min_{i=\overline{1,n}} r_i$.

In this paper the following problem is considered. To determine if for the simple graph G there exists a simple graph G^* so that $Med(G^*) \simeq G$. In [1] Slater gave an algorithm that solves positively this problem for any simple graph G. It should be mentioned that Slater's algorithm requires $5n^2 - 2m + 4(k+1)$ new vertices (k is a natural number, $k < n$). In our paper we propose some algorithms that require less new vertices than Slater's algorithm does.

2 The graphs containing the given graph as median (Algorithms)

Algorithm 1.

We shall construct the graph G^* according to the following algorithm.

1. Add to G $\frac{n(n-1)}{2} - m$ new vertices. Denote these vertices by $v'_j,\ j = \overline{1, \frac{n(n-1)}{2} - m}$. Construct on these vertices a complete subgraph $K_{\frac{n(n-1)}{2}-m}$. Connect every new vertex with exactly two vertices of G: the vertex v'_j is connected with vertices v_i and v_k from $V(G)$ iff $(v_i, v_k) \notin E(G)$. Denote the obtained graph by G_1. Obviously, $|V(G_1)| = n + \frac{n(n-1)}{2} - m$. In G_1 we have:

$$\sigma_{G_1}(v_i) = n^2 - 2m - 1, \quad i = \overline{1,n},$$
$$\sigma_{G_1}(v'_j) = \tfrac{n(n-1)}{2} + 2n - m - 3, \quad j = \overline{1, \tfrac{n(n-1)}{2} - m}.$$

Let us observe that the inequality

$$\sigma_{G_1}(v_i) < \sigma_{G_1}(v'_j)$$

is true if and only if

$$m \geq \frac{n(n-3)}{2} + 3. \tag{1}$$

In this case we have that $Med(G_1) \simeq G$ and the problem is solved: $G^* = G_1$. Note that the number of new vertices added to G in this case is $N(G) \le n-3$. If (1) is not verified for G, then we continue to construct the graph G^*.

2. Add to G_1 l^* subsets of new vertices, each of them containing n vertices: v_i^l, $i=\overline{1,n}$, $l=\overline{1,l^*}$. Connect each vertex v_i^l ($l=\overline{1,l^*}$) by means of an edge with the vertex $v_i \in V(G)$. Construct a complete subgraph K_n on each of these l^* subsets of vertices. Denote the obtained graph by G_2. Let $\overline{v}$ be a vertex of G_2 so that $\sigma_{G_2}(\overline{v}) = \min\{\sigma_{G_2}(v_i^l),\ i=\overline{1,n},\ l=\overline{1,l^*}\}$. Note that $\overline{v}$ is connected in G_2 with the vertex $v_i \in V(G)$ with $\deg v_i = r$. In G_2 we have:

$$\begin{array}{l}\sigma_{G_2}(v_i) = n^2-2m-1+l^*(2n-1),\\ \sigma_{G_2}(v_j') = \frac{n(n-1)}{2}+2n-m+l^*(3n-2)-3,\\ \sigma_{G_2}(\overline{v}) = \frac{3n(n-1)}{2}+2n-3m+r-1+(l^*-1)(3n-1),\\ i=\overline{1,n},\ j=\overline{1,\frac{n(n-1)}{2}-m},\ l=\overline{1,l^*}.\end{array}$$

If l^* is taken according to the formula:

$$l^* = \max\left\{\left[\frac{n(n-3)-2m+4}{2(n-1)}\right],\left[\frac{-n(n-5)+2m-2r-2}{2n}\right]\right\}+1 \qquad (2)$$

then the following inequalities are true:

$$\begin{array}{l}\sigma_{G_2}(v_i) < \sigma_{G_2}(v_j'),\ i=\overline{1,n},\ j=\overline{1,\frac{n(n-1)}{2}-m},\\ \sigma_{G_2}(v_i) < \sigma_{G_2}(\overline{v}),\ i=\overline{1,n}.\end{array}$$

Thus if l^* is given by (2) then $Med(G_2) \simeq G$ and $G^* = G$, and so the problem is solved. Mention that $\mid V(G^*) \mid = n + \frac{n(n-1)}{2} - m + nl^*$. This algorithm proves the following theorem.

Theorem 1 *For any simple graph G there exists a simple graph G^* so that $Med(G^*) \simeq G$. The graph G^* is obtained from G by adding to it $N(G) = \frac{n(n-1)}{2} - m + nl^*$ new vertices, where $l^* = O(n)$. If $m \ge \frac{n(n-3)}{2}+3$, then $l^*=0$.*

Algorithm 2.

Let $\alpha = R - r + 1$. The construction of G^* follows.

1. Add to G α new vertices v_i^* $(i = \overline{1,\alpha})$. Connect these vertices with the vertices of G as follows: Connect with vertex v_1^* all vertices from G and denote the obtained graph by $\overline{G_1}$. In the general case, connect with v_i^* $(2 \leq i \leq \alpha)$ all those vertices from G that have the degree less than $R+1$ in $\overline{G_{i-1}}$ and denote the obtained graph by $\overline{G_i}$. Construct a complete subgraph K_α on vertices $v_1, v_2, \ldots, v_\alpha$. Denote the obtained graph by G_1. Obviously, $\sigma_{G_1}(v_1^*) \leq \sigma_{G_1}(v_i^*),\ i = \overline{1,\alpha}$. In this graph

$$\sigma_{G_1}(v_i) = 2n + R - 2r - 1, \quad i = \overline{1,n},$$
$$\sigma_{G_1}(v_1^*) = n + R - r.$$

It is evident that $\sigma_{G_1}(v_1^*) \leq \sigma_{G_1}(v_i)$.

2. Add l new vertices v_j^{**} $(j = \overline{1,l})$ to G_1 and connect each of them with all vertices $v_i \in V(G)$. Denote the obtained graph by G_2. In this graph

$$\sigma_{G_2}(v_i) = 2n + R - 2r + l - 1, \quad i = \overline{1,n},$$
$$\sigma_{G_2}(v_1^*) = n + R - r + 2l,$$
$$\sigma_{G_2}(v_j^{**}) = n + 2R - 2r + 2l, \quad j = \overline{1,l}.$$

If we take $l = n - r$, then the following inequalities are true

$$\sigma_{G_2}(v_i) < \sigma_{G_2}(v_1^*),$$
$$\sigma_{G_2}(v_i) < \sigma_{G_2}(v_j^{**}).$$

Thus if $l = n - r$ then $Med(G_2) \simeq G$ and $G^* = G_2$. It should be observed that G^* has $|V(G^*)| = 2n + R - 2r + 1$ vertices.

This algorithm proves the following theorem.

Theorem 2 *For any simple graph G there exists a simple graph G^* so that $Med(G^*) \simeq G$ and this graph is obtained from G by adding to it $N(G) = n + R - 2r + 1$ new vertices.*

Algorithm 3.

Let G be a simple irregular graph, i.e. $R \neq r$. We shall construct the graph G^* according to the following algorithm.

1. Add to G a new vertex x_1 and connect it with all vertices $v_i \in V(G)$ for which $r_i < R$. Then add a new vertex x_2 and connect it with all vertices $v_i \in V(G)$ for which $r_i + 1 < R$. In the general case, add a new vertex x_l and connect it with all vertices $v_i \in V(G)$ for which $r_i + l - 1 < R$ $(l \geq 1)$. Obviously, adding $R-r$ new verticces $x_1, x_2, \dots,$ x_{R-r} we obtain that the new degrees of vertices $v_i \in V(G)$ are equal to R.

2. Add a new vertex y_1 and connect it with all vertices of G. Denote the obtained graph by G_1. Obviously, $V(G_1) = V(G) \bigcup \{x_1, \dots, x_{R-r}, y_1\}$. If there exist any vertices x_l so that $d_{G_1}(x_l, v_i) > 2$, then connect each of them with y_1. Denote the obtained graph by G_2. It is evident that

$$d_{G_2}(v_i, v_j) \leq 2, \; i, j = \overline{1, n},$$
$$d_{G_2}(y_1, x_l) \leq 2, \; l = \overline{1, R-r}.$$

Also it is clear that $\sigma_{G_2}(y_1) < \sigma_{G_2}(x_l)$. Denote by k the number of vertices x_l for which $d_{G_2}(y_1, x_l) = 1$.$(0 \leq k \leq R-r)$ It is clear that

$$\sigma_{G_2}(v_i) = 2n + R - 2r - 1, \quad i = \overline{1, n},$$
$$\sigma_{G_2}(y_1) = n + 2(R-r) - k.$$

3. If $n \geq R - k + 1$ (and only this case is interesting), then $\sigma_{G_2}(v_i) \geq \sigma_{G_2}(y_1)$. In this case add to G_2 s new vertices $y_2, \dots, y_{s+1}$, connecting each of them with all vertices $v_i \in V(G)$. Denote by G_3 the obtained graph. Obviously,

$$\sigma_{G_3}(y_j) < \sigma_{G_3}(x_l),$$
$$\sigma_{G_3}(v_i) = 2n + R + s - 2r - 1,$$
$$\sigma_{G_3}(y_j) = n + 2R + 2s - 2r - k,$$
$$i = \overline{1, n}, \; j = \overline{1, s+1}, \; l = \overline{1, R-r}.$$

Hence, if $s \geq n + k - R - 1$, then $\sigma_{G_3}(v_i) \leq \sigma_{G_3}(y_j), \; i = \overline{1, n}, \; j = \overline{1, s+1}$.

Taking $s = n + k - R$ we obtain:

$$\sigma_{G_3}(v_i) = 3n + k - 2r - 1, \quad i = \overline{1, n},$$
$$\sigma_{G_3}(y_j) = 3n + k - 2r, \quad j = \overline{1, s+1},$$

and $\sigma_{G_3}(v_i) < \sigma_{G_3}(y_j)$, $i = \overline{1, n}$, $j = \overline{1, s+1}$.

Thus G_3 with $s = n + k - R$ contains G as median: $Med(G_3) \simeq G$. And this graph is obtained from G by adding to it $N(G) = n + k - r + 1$ new vertices. Because $0 \leq k \leq R - r$, the above algorithm proves the following theorem.

Theorem 3 *For any simple graph G there exists a simple graph G^* with $Med(G^*) \simeq G$. This graph is obtained from G adding to it $N(G) = n + k - r + 1$ new vertices.*

Remark 1 *If $R = r$, then algorithm 3 still applies.*

3 Comparison of algorithms

Let us construct the graph G^* with $Med(G^*) \simeq G$ using the above algorithms. Denote by $N_1(G)$ the number of new vertices added to G by the algorithm 1, by $N_2(G)$ – the number of new vertices added to G by the algorithm 2, and by $N_3(G)$ – the number of new vertices added to G by the algorithm 3.

As we saw above, $N_1(G) = \frac{n(n-1)}{2} - m + nl^*$, where l^* is given by (2), $N_2(G) = n + R - 2r + 1$, $N_3(G) = n + k - r + 1$.

In the general case, the algorithm 2 is "better" than the algorithm 1, i.e. $N_2(G) \leq N_1(G)$. It should be observed that if for G the conditions: $m \geq \frac{n(n-3)}{2} + 3$ and $R > 2r - 4$ are verified, then $N_1(G) < N_2(G)$, i.e. the algorithm 1 adds to G less new vertices than the algorithm 2 does.

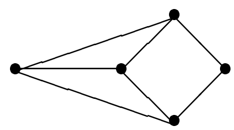
Because $0 \leq k \leq R - r$, the number of new vertices added to G by the algorithm 3 verifies the inequalities:

$$n - r + 1 \leq N_3(G) \leq n + R - 2r + 1.$$

Obviously, $N_3(G) \le N_2(G)$. We should observe that if $k = R - r$ then $N_2(G) = N_3(G) = n+R-2r+1$. Also, if $m \ge \frac{n(n-3)}{2}+3$ and $k > r-4$, then $N_1(G) < N_3(G)$, i.e. the algorithm 1 adds to G less new vertices than the algorithm 3 does.

4 Examples

Example 1. Consider the graph G (figure 1).

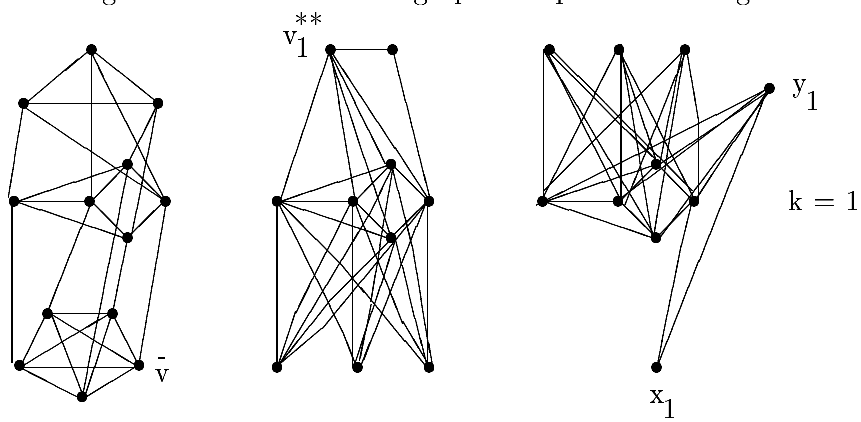


n = 5, m = 7,

R = 3, r = 2.

Fig.1. The graph G

For this graph we construct the graph G^* with $Med(G^*) \simeq G$ using these three algorithms. The obtained graphs are presented in figure 2.

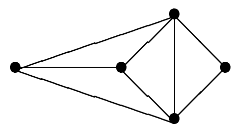


a) b) c)

Fig.2. The graph G^* is obtained using
a) algorithm 1, b) algorithm 2, c) algorithm 3

As we can see, $N_1(G) = 8$, $N_2(G) = 5$, $N_3(G) = 5$. In this case $k = R - r$ and algorithms 2 and 3 add to G the same number of new vertices. Algorithm 1 requires more new vertices than algorithms 2 and 3 do.

Example 2. Let us consider the graph G represented in figure 3.

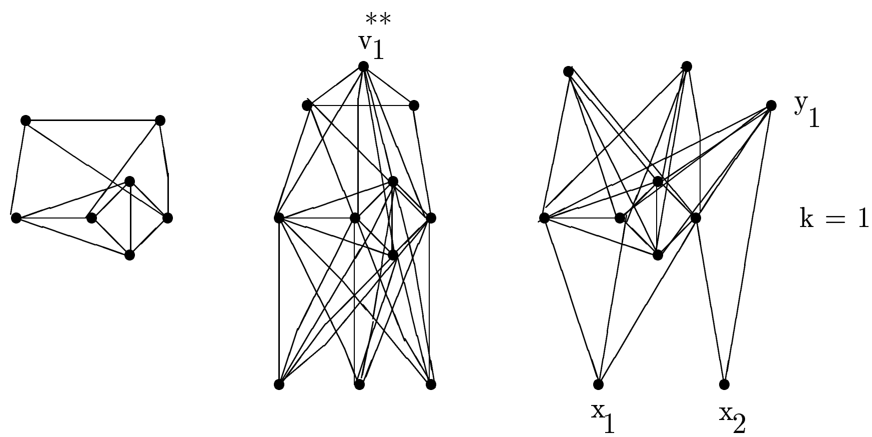


n = 5, m = 8,

R = 4, r = 2.

Fig.3. The graph G

Construct the graph G^* with $Med(G^*) \simeq G$ using the above algorithms. The resulting graphs are presented in figure 4.



a) b) c)

Fig.4. The graph G^* is obtained using
a) algorithm 1, b) algorithm 2, c) algorithm 3

$N_1(G) = 2,\ N_2(G) = 6,\ N_3(G) = 5.$
In this case the inequality (2) is verified and algorithm 1 requires less new vertices than the algorithms 2 and 3 do.

References

[1] P.J.Slater. Medians of arbitrary graphs. Journal of Graph Theory. Vol.4 (1980), 389 – 392.

[2] Zykov A.A. Fundamentals of Graph Theory. Moscow, Nauka, 1987 (Russian).

A.Poshtaru,
Republic of Moldova's State University,
Department of Mathematics and Cybernetics,
A.Mateevici str., 60,
Kishinev, 277003, Moldova

Received 16 November, 1995

R.Boliac,
Institute of Mathematics,
Academy of Sciences of Moldova,
5 Academiei str.,
Kishinev, 277028, Moldova.