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On the Family of Conditional Embedded Implicational Dependencies

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Abstract

Certain second-order sentences, called conditional embedded implicational dependencies, about relations in a relational database, are defined and studied.

This class of dependencies includes some of previously defined dependencies as special cases. Thus, the family of implicational embedded dependencies introduced and studied by R. Fagin can be included in the family of conditional embedded implicational dependencies.The conditional-functional dependencies defined by P. De Bra and J. Paredaens are special cases of new dependencies. The family of conditional embedded implicational dependencies, also contains the family of conditional implicational dependencies, defined by the author. A characterisation of a conditional embedded implicational dependency implied by a given class in this family is given.

The existence of Armstrong models for a class of conditional embedded implicational dependencies with a single relational symbol is shown.

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Key Words and **Phrases**: Armstrong relations, functional dependency, conditional dependency, conditional implicational dependency, implicational dependency, conditional embedded implicational dependency, relational database, logical consequence, mathematical logic.

1 Introduction

The purpose of this paper is to investigate a large family of dependencies. Using second-order logic, we define the family of conditional embedded implicational dependency constraints. This family contains the class of conditional functional dependencies introduced by P. De Bra and J. Paredaens [3] and used by them for the horizontal decomposition of a relation schemes. This family also contains the family of implicational dependencies studied by R.Fagin [4]. The family of conditional implicational dependencies [6] is included in this family.

In this paper, we give a characterisation of a conditional embedded implicational dependency implied by a given class in this family. We study the existence of Armstrong models for the class of conditional embedded implicational dependencies with a single relational symbol constant by means of the results obtained by Fagin [4] with respect to Armstrong relations.

2 Preliminaries

In this section we introduce some of the notations used throughout the paper.

Let U be a finite nonempty set of distinct attributes, called the universe of attributes: $U = \{A_1, \ldots, A_n\}$. For each $A \in U$, a set denoted by $dom(A)$, is associated. $dom(A)$ is called *the domain for A*. The domain mapping, denoted by φ, is defined thus: $\varphi(A) = dom(A)$, $\forall A \in U$.

A relation over U is a domain mapping over U along with a set of tuples for U and φ.

In the definition 3.1., embedded implicational dependency statements appear. We consider there formulas defined as in [4]. Let R be a relation symbol, that corresponds to a relation r over U. Assume that there exists a nonempty set of individual variables for every attribute A. An atomic formula is either of the form $Rz_1 \ldots z_n$ (z_i is an individual variable) or of the form $x = y$ (where x and y are individual variables). The atomic formulas are typed; that means for the formula $Rz_1 \ldots z_n$, z_i is a variable associated to the attribute A_i ($1 \leq i \leq n$) and in the formula $x = y$, the variables x, y are associated to the same A_k. If M_i and M_j are the sets of individual variables corresponding to A_i and A_j, respectively, then $M_i \cap M_j = \emptyset$, for $i \neq j$. The atomic formula $Rz_1 \ldots z_d$ will be denoted also under the form $R(z)$, where $z = z_1 \ldots z_d$.

An implicational dependency has the form:

$$(\forall x_1 \dots x_m)((B_1 \wedge \dots \wedge B_k) \Rightarrow C)$$

where each B_i is of the form $Rz_1 \dots z_n$ and C is atomic.

An embedded implicational dependency (or EID) is a typed sentence of the form

$$(\forall x_1 \dots x_m)((B_1 \wedge \dots \wedge B_k) \Rightarrow (\exists y_1 \dots y_r)(C_1 \wedge \dots \wedge C_s))$$

where each B_i is a relational formula and each C_i is atomic. We assume that each of x_j's appears in at least one of the B_i's and that $k \geq 1$. Moreover, we assume that $n \geq 0$ and $s \geq 1$.
For the details about implicational dependencies and embedded implicational dependencies see [4].

Let X be a subset of U and r be a relation over U. A set S of tuples in r is called X-*complete* in r if for all $t_1 \in S$, $t_2 \notin S$, we have $t_1[X] \neq t_2[X]$. In particular, the empty set of tuples is X-*complete* for every set X of attributes.
A conditional-functional dependency [3] is denoted by $X \to Y \supset\!\!- X \to Z$. We said that the relation r obeys $X \to Y \supset\!\!- X \to Z$ if in every X-*complete* set of tuples of r, in which the functional dependency (fd) $X \to Y$ holds, the fd $X \to Y$ must hold too.

3 The Family of Conditional Embedded Implicational Dependencies

Definition 3.1 *A Conditional Embedded Implicational Dependency (CEID) is a universal sentence in second-order logic of the form:*

$$\alpha(R, X) \equiv (\forall S)\,[\varphi_1(S, R) \wedge \psi_1(S, R, X) \wedge \sigma_1(S) \wedge \dots \\ \dots \wedge \sigma_h(S) \Rightarrow \sigma(S)] \qquad (1)$$

where:

a) R is a predicate symbol, which represents a relation r over U.

b) S is a predicate symbol, which is universally quantified and represents a subset s of r.

c) $\varphi_1(S,R)$ is a first-order formula of the form:

$$(\forall u)[S(u) \Rightarrow R(u)]$$

This means that whenever a tuple t belongs to s, it also belongs to r.

d) X is a fixed subset of U.

e) $\psi_1(S,R,X)$ is a formula of the form:

$$(\forall x)(\forall t)\left[S(t_1,x) \wedge R(t_2,x) \Rightarrow S(t_2,x)\right],$$

where x is a tuple of variables corresponding to X and t_1, t_2 are tuples of variables for all attributes in $U - X$. t is a term, which contains both the variables from t_1 and t_2.

*The pair (s,r), where s and r are relations over U, satisfies $\psi_1(S,R,X)$ and $\varphi_1(S,R)$ if and only if s is X-*complete *in r.*

f) $\sigma_j(S)$ is an embedded implicational dependency, which contains S as an unique relational symbol, $1 \leq j \leq h$. $\sigma(S)$ is an embedded implicational dependency which contains on right-hand side of the implication just one atomic formula.

Let us consider the following formula:

$$\alpha_1(R) \equiv [\sigma_1(R) \wedge \ldots \wedge \sigma_h(R) \Rightarrow \sigma(R)] \tag{2}$$

Let r be a relation over U and D a family of relations over U. Using the semantics of second-order logic [7], we obtain:

$$\begin{array}{l} r \text{ satisfies } \alpha_1(R) \text{ iff } (r,D) \text{ satisfies } \alpha(R,\emptyset), \\ \text{for every family } D \text{ of relations over } U \end{array} \tag{3}$$

The simbol $\emptyset$ denotes the empty set.

Since (3) is true, we may include the sentence $\alpha_1(R)$ in the class of conditional embedded implicational dependencies as the formula $\alpha(R, \emptyset)$.

When we consider in formula $\alpha_1(R)$, h equal to 1 and $\sigma_1(R)$ a tautology, then $\alpha_1(R)$ is equivalent to $\sigma(R)$, which is an embedded implicational dependency.
It means that conditional embedded implicational dependencies are generalizations of embedded implicational dependencies.

When in $\alpha(R, X)$ we consider the value 1 for h and $\sigma_1(R)$, $\sigma(R)$ are functional dependencies (which are the embedded implicational dependencies), then we obtain a formula corresponding to a conditional-functional dependency. The class of conditional-functional dependencies has been defined and studied by P. De Bra and J. Paredaens [3].

Example 3.1 *Let* $U = \{A, B, C, D, E\}$ *and* $m_1 : AB \twoheadrightarrow C/D$ *a multivalued dependency. The following formula, denoted* $\alpha_{m_1}(R)$ *represents* m_1*:*

$$\alpha_{m_1}(R) \equiv (\forall abc_1d_1e_1c_2d_2e_2)[Rabc_1d_1e_1 \wedge Rabc_2d_2e_2 \Rightarrow (\exists e_3)Rabc_1d_2e_3]$$

Similarly, for $m_2 : AB \twoheadrightarrow C/E$*, we have the formula:*

$$\alpha_{m_2}(R) \equiv (\forall abc_1d_1e_1c_2d_2e_2)[Rabc_1d_1e_1 \wedge Rabc_2d_2e_2 \Rightarrow (\exists d_3)Rabc_1d_3e_2]$$

We consider $\eta : AB \twoheadrightarrow C/D \supset\!- AB \twoheadrightarrow C/E$ *as a conditional-multivalued dependency. A relation* r *over* U *obeys this dependency if in every* AB*-complete set of tuples in* r*, in which* m_1 *holds,* m_2 *must hold too. For the dependency* η*, the associated formula is:*

$$\alpha(R, AB) \equiv (\forall S)\, [\varphi_1(S, R) \wedge \psi_1(S, R, AB) \wedge \sigma_1(S) \Rightarrow \sigma(S)]$$

where

$\varphi_1(S, R) \equiv (\forall u_1u_2u_3u_4)\, [Su_1u_2u_3u_4 \Rightarrow Ru_1u_2u_3u_4]$

$\psi_1(S, R, AB) \equiv (\forall x_1x_2c_1d_1c_2d_2)\, [Sx_1x_2c_1d_1 \wedge Rx_1x_2c_2d_2 \Rightarrow$
$\Rightarrow Sx_1x_2c_2d_2]$

$\sigma_1(S) \equiv \alpha_{m_1}(S)$, $\sigma(S) \equiv \alpha_{m_2}(S)$

The simbol "≡" denotes the logical equivalence.

4 The Main Result

In this section we give a characterisation of a conditional embedded implicational dependency logically implied by a given class of conditional embedded implicational dependencies.

Let M be a class of CEIDs and $\alpha(R, X)$ be a CEID of the form (1). Let us denote by $\alpha_L(S, R, X)$ the following formula:

$$\alpha_L(S, R, X) \equiv \varphi_1(S, R) \wedge \psi_1(S, R, X) \wedge \sigma_1(S) \wedge \ldots \wedge \sigma_h(S)$$

Definition 4.1 *Let F be a set of formulas in second-order logic and $\models$ the logical implication. Then, let us denote by F^*, the set $F^* = \{\alpha | F \models \alpha\}$.*

We need to define recursively the set of formulas, denoted by $T_i(M, \alpha_L(S, R, X))$, which correspond to the class M and to the formula $\alpha_L(S, R, X)$.

Definition 4.2

$$T_0(M, \alpha_L(S, R, X)) = \{\varphi_1(S, R), \psi_1(S, R, X), \sigma_1(S), \ldots, \sigma_h(S)\}$$

$T_{i+1}(M, \alpha_L(S, R, X)) = T_i^*(M, \alpha_L(S, R, X)) \cup \{\gamma(S)|$ *there is* $G \in M$, $G \equiv (\forall S')\left[\varphi_2(S', R) \wedge \psi_2(S', R, X') \wedge \gamma_1(S') \wedge \ldots \wedge \gamma_k(S') \Rightarrow \gamma(S')\right]$, *such that* $\varphi_2(S, R) \in T_i^*(M, \alpha_L(S, R, X)), \psi_2(S, R, X') \in$
$\in T_i^*(M, \alpha_L(S, R, X)), \gamma_j(S) \in T_i^*(M, \alpha_L(S, R, X)), j = \overline{1, k}\}$.

$$T(M, \alpha_L(S, R, X)) = \bigcup_{i=0}^{\infty} T_i(M, \alpha_L(S, R, X))$$

We omit S and R in $T_i(M, \alpha_L(S, R, X))$ and $T(M, \alpha_L(S, R, X))$ whenever S and R result from context.

Theorem 4.1 *Let M be a class of CEIDs and $\alpha(R, X)$ be a CEID of the form (1). We have:*

$$M \models \alpha(R, X) \text{ iff } \sigma(S) \in T(M, \alpha_L(S, R, X))$$

Proof: ($\Leftarrow$)
Let $\sigma(S)$ be from $T(M, \alpha_L(S,R,X))$. Let us show that

$$\begin{array}{l} M \cup T_0(M, \alpha_L(S,R,X)) \models \beta(S), \\ \text{ for every } \beta(S) \in T_n(M, \alpha_L(S,R,X)) \\ \text{ and every } n = 0, 1, 2, \ldots \end{array} \qquad (4)$$

We proceed by induction on n:
For $n = 0$, we have $M \cup T_0(M, \alpha_L(S,R,X)) \models T_0(M, \alpha_L(S,R,X))$, hence the relation (4) is true for $n = 0$.
Assume that (4) is true for a natural number n.
Let $\beta(S)$ be from $T_{n+1}(M, \alpha_L(S,R,X)) - T_n(M, \alpha_L(S,R,X))$.
We have two cases:
a) $\beta(S) \in T_n^*(M, \alpha_L(S,R,X)) - T_n(M, \alpha_L(S,R,X))$ and
b) $\beta(S) \in T_{n+1}(M, \alpha_L(S,R,X)) - T_n^*(M, \alpha_L(S,R,X))$
In the case a), we obtain $T_n(M, \alpha_L(S,R,X)) \models \beta(S)$. But, by the induction hypothesis for n, we have:

$$M \cup T_0(M, \alpha_L(S,R,X)) \models T_n(M, \alpha_L(S,R,X)).$$

By the transitivity of relation $\models$, it obtains

$$M \cup T_0(M, \alpha_L(S,R,X)) \models \beta(S)$$

In the case b), there exists a G from M, where
$G \equiv (\forall S')\,[\varphi_2(S',R) \wedge \psi_2(S',R,X') \wedge \gamma_1(S') \wedge \ldots \wedge \gamma_k(S') \Rightarrow \gamma(S')]$,
such that $\beta(S) \equiv \gamma(S)$, $T_n(M, \alpha_L(S,R,X)) \models \varphi_2(S,R)$,

$$T_n(M, \alpha_L(S,R,X)) \models \psi_2(S,R,X'),\ T_n \models \gamma_j(S),\ j = \overline{1,k}. \qquad (5)$$

The relation (4) is true for n, which implies:

$$M \cup T_0(M, \alpha_L(S,R,X)) \models T_n(M, \alpha_L(S,R,X)) \qquad (6)$$

From (5) and (6), it results:

$$\begin{array}{l} M \cup T_0(M, \alpha_L(S,R,X)) \models \varphi_2(S,R),\ \psi_2(S,R,X') \\ M \cup T_0(M, \alpha_L(S,R,X)) \models \gamma_j(S),\ j = \overline{1,k} \end{array} \qquad (7)$$

G being from M and the relations (7) conclude that

$$M \cup T_0(M, \alpha_L(S, R, X)) \models \gamma(S).$$

But $\gamma(S) = \beta(S)$, hence we have $M \cup T_0(M, \alpha_L(S, R, X)) \models \beta(S)$. Thus, by the induction theorem we have the relations (4). When $\sigma(S) \in T(M, \alpha_L(S, R, X))$, there is a natural number n, such that $\sigma(S) \in T_n(M, \alpha_L(S, R, X))$.
From (4) it obtains:

$$M \cup T_0(M, \alpha_L(S, R, X)) \models \sigma(S)$$

which is the same with the following:

$$M \cup \{\varphi_1(S, R), \psi_1(S, R, X), \sigma_1(S), \ldots, \sigma_h(S)\} \models \sigma(S)$$

By the logical implication, we have:

$$M \models [\varphi_1(S, R) \wedge \psi_1(S, R, X) \wedge \sigma_1(S) \wedge \ldots \wedge \sigma_h(S) \Rightarrow \sigma(S)]$$

hence $M \models \alpha(R, X)$.
($\Rightarrow$) Let $M' = \{G | G \in M,\ \exists n$ such that G appears in the construction of
$T_{n+1}(M, \alpha_L(S, R, X))\}$.
Let $M \models \alpha(R, X)$ and assume that $\sigma(S) \notin T(M, \alpha_L(S, R, X))$. We show that $M \not\models \alpha(R, X)$ by constructing a model M_0 which obeys M and does not obey $\alpha(R, X)$, hence a contradiction.
The formula $\sigma(S)$ is an embedded implicational dependency, hence it has the form:

$$\sigma(S) \equiv (\forall x_1 \ldots \forall x_n)\left((B_1 \wedge \ldots \wedge B_k) \Rightarrow (\exists y_1 \ldots \exists y_r)(C)\right)$$

The formula $\sigma(S)$ is equivalent with the following sentence:

$$\sigma(S) \equiv (\forall x_1 \ldots \forall x_n \exists y_1 \ldots \exists y_r (\neg R_{j_1}(t_1) \vee \ldots \vee \neg R_{j_k}(t_k) \vee R_{j_{k+1}}(t_{k+1}))$$

where R_{j_i} is an n-ary predicate symbol ($B_i \equiv R_{j_i}(t_i)$, $i = \overline{1, k}$, $C \equiv R_{j_{k+1}}(t_{k+1})$), t_i, $1 \leq i \leq k+1$ are sequences of terms such that every

x-variable in t_{k+1} is contained in some t_i, $1 \leq i \leq k$ and no t_i for $1 \leq i \leq k$, contains any y-variables. We have $R_{j_1} = \ldots = R_{j_k} = S$, and $R_{j_{k+1}} = S$ or $R_{j_{k+1}}$ is the equality predicate.
The elements of $T(M, \alpha_L(S, R, X))$ are generalized dependencies.
Associated to $\sigma(S)$ and $T(M, \alpha_L(S, R, X))$, J. Grant and B. E. Jacobs [8] have defined a set of atomic formulas, denoted by
$Y(T(M, \alpha_L(S, R, X)), \sigma(S))$.
Let us denote a class of embedded implicational dependencies by ξ, and σ be an embedded implicational dependency.

$$Y_0(\xi, \sigma) = \{R_{j_i}(t_i) | \neg R_{j_i}(t_i) \text{ is a disjunct in } \sigma\}$$

$$\begin{aligned} Y_{k+1}(\xi, \sigma) = \ & \{R_{l_{m+1}}(t_{m+1}) | \text{ there is a } G \in \xi,\ G \equiv \forall x'_1 \ldots \forall x'_p \exists z'_1 \ldots \exists z'_q \\ & (\neg R_{l_1}(u_1) \vee \ldots \vee \neg R_{l_m}(u_m) \vee R_{l_{m+1}}(u_{m+1})) \text{ and a substi-} \\ & \text{tution } \theta \text{ such that } u_{m+1}\theta = t_{m+1},\ \text{each } z'_i\theta \text{ is a new} \\ & \text{constant symbol, } 1 \leq i \leq q, \text{ and for every negated atomic} \\ & \text{disjunct } \neg R_{l_i}(u_i) \text{ of } G,\ R_{l_i}(u_i)\theta \in Y_k(\xi, \sigma)\} \cup Y_k(\xi, \sigma) \end{aligned}$$

$$Y(\xi, \sigma) = \bigcup_{n=0}^{\infty} Y_n(\xi, \sigma).$$

The process of obtaining $R_{l_{m+1}}(t_{m+1})$ is realized once only for each $\{R_{l_1}(u_1)\theta, \ldots, R_{l_m}(u_m)\theta\}$ and formula G.
J. Grant and B. E. Jacobs in [8] have shown that:

$$\begin{aligned} & \xi \models \sigma \text{ iff these is a substitution } \theta \text{ on the } z\text{-variables of } \sigma, \\ & \text{such that } R_{j_{k+1}}(t_{k+1})\theta \in Y(\xi, \theta). \end{aligned} \tag{8}$$

From $\alpha(S) \notin T(M, \alpha_L(S, R, X))$ it obtains:

$$T(M, \alpha_L(S, R, X)) \not\models \sigma(S) \tag{9}$$

Applying (8) for $T(M, \alpha_L(S, R, X))$ and $\sigma(S)$, and using (9), it results:

$$R_{j_{k+1}}(t_{k+1})\theta \notin Y(T(M, \alpha_L(S, R, X)), \sigma(S))$$

for every substitution θ on the y-variables of $\sigma(S)$.
We construct a model M_0 in the following way:
Let U^{M_0}, the universe of M_0, be the set of strings which are either terms in $\sigma(S)$ or constants in some $G \in T(M, \alpha_L(S, R, X))$. We include $R_i(t)$ in M_0 iff $R_i(t) \in Y(T(M, \alpha_L(S, R, X)), \sigma(S))$, where $R_i(t)$ is an atom.

Now we show that M_0 obeys every element in $T(M, \alpha_L(S, R, X))$.
Let us denote $Y(T(M, \alpha_L(S, R, X)), \sigma(S))$ by $Y(S)$ and
$Y_i(T(M, \alpha_L(S, R, X)), \sigma(S))$ by $Y_i(S)$.
Let G be from $T(M, \alpha_L(S, R, X))$. It results that G has the same form as in definition of $Y_{k+1}(S)$.
Let τ be a substitution such that $R_{l_i}(u_i)\tau \in M_0$, for $1 \leq i \leq m$. By the definition of M_0, we have: $R_{l_i}(u_i)\tau \in Y(S)$, $1 \leq i \leq m$. Since $Y(S) = \bigcup_{i\geq 0} Y_i(S)$, it results that there exist p_i, $1 \leq i \leq m$ such that $R_{l_i}(u_i)\tau \in Y_{p_i}(S)$, $1 \leq i \leq m$. Let p be $\max\{p_1, \ldots, p_m\}$. Since $Y_0(S) \subseteq Y_1(S) \subseteq \ldots$, we obtain that $R_{l_i}(u_i)\tau \in Y_p(S)$, $1 \leq i \leq m$.
As in the process of constructing $R_{l_{m+1}}(t_{m+1})$ in $Y_{k+1}(S)$ we extend the substitution τ for the variables z'_j, $1 \leq j \leq q$, such that $z'_j\tau$ is a new constant symbol, $u_{m+1}\tau = t_{m+1}$ and $R_{l_{m+1}}(t_{m+1}) \in Y_{k+1}(S)$. It follows that $R_{l_{m+1}}(t_{m+1}) \in M_0$. It means that M_0 obeys G.
Next we must show M_0 doesn't satisfy $\sigma(S)$. Since $R_{j_i}(t_i) \in Y_0(S)$, $1 \leq i \leq k$ and by the definition of M_0, it obtains that $R_{j_i}(t_i) \in M_0$, $1 \leq i \leq k$.
But, $R_{j_{k+1}}(t_{k+1})\theta \notin Y(S)$ for every substitution θ on the y-variables of $\sigma(S)$. It results that M_0 doesn't obey $\sigma(S)$. Since M_0 satisfies $T_0(M, \alpha_L(S, R, X))$ it obtains that M_0 doesn't satisfy $\alpha(R, X)$.

Since M_0 satisfies every element in $T(M, \alpha_L(S, R, X))$, it results that M_0 satisfies each formula in M'. Let $\alpha(R, X)$ be a sentence from $M - M'$ and of the form (1). We have:

$$T(M, \alpha_L(S, R, X)) \not\models \varphi_1(S, R) \text{ or}$$
$$T(M, \alpha_L(S, R, X)) \not\models \psi_1(S, R, X) \text{ or}$$
$$(\exists i)(1 \leq i \leq h) \text{ such that } T(M, \alpha_L(S, R, X)) \not\models \sigma_i(S).$$

Using the same method as for the relation (9), it results that there exists a model M'_1, which satisfies $T(M, \alpha_L(S, R, X))$ and $\alpha(R, X)$. Hence M'_1 satisfies M'. Thus, M'_1 satisfies $M' \cup \{\alpha(R, X)\}$.
Let $(\alpha_i)_{i \in I}$ the family of elements from $M - M'$, where $M - M' \neq \emptyset$. Let us consider $0 \notin I$ and let us denote $I \cup \{0\}$ by I'. Considering α_i instead of $\alpha(R, X)$, we obtain:

$$\text{there exists a model } M_i \text{ which satisfies} \quad T(M, \alpha_L(S, R, X)) \text{ and } \alpha_i, \text{ for every } i \in I \tag{10}$$

Let us consider the direct product of M_0 and M_i, $i \in I$. Let us denote by $\overline{M}$ this direct product, that is

$$\overline{M} = \otimes\langle M_i,\ i \in I'\rangle$$

Since every element in $T(M, \alpha_L(S, R, X))$ is upward faithful with respect to direct products ([4]), it results that $\overline{M}$ satisfies $T(M, \alpha_L(S, R, X))$.

Let us φ_i be an element from left-hand side in α_i, such that M_i obeys $T(M, \alpha_L\ (S, R, X))$ and it doesn't satisfy φ_i, $i \in I$. It results that $T(M, \alpha_L(S, R, X)) \not\models \varphi_i$, $i \in I$.
We have:

$$\overline{M} \text{ doesn't satisfy } \varphi_i, \text{ for every } i \in I. \tag{11}$$

The relation (11) follows from the fact that every φ_i is downward faithful (with respect to direct products).

From (11) it obtains:

$$\overline{M} \text{ satisfies } \alpha_i, \text{ for every } i \in I \tag{12}$$

From $\overline{M}$ satisfies $T(M, \alpha_L(S, R, X))$, it results that

$$\overline{M} \text{ satisfies } M'. \text{ Thus, } \overline{M} \text{ satisfies } M \tag{13}$$

On the other hand, since $\alpha(R, X)$ is downward faithful (with respect to direct products), it obtains that

$$\overline{M} \text{ doesn't satisfy } \alpha(R, X). \tag{14}$$

The relations (14) and (13) contradict the fact that $\alpha(R, X)$ is a logical consequence of M, that is $M \models \alpha(R, X)$. ∎

5 Armstrong models

Let us consider the family of all conditional embedded implicational dependencies on a single relation symbol R, denoted by $\Sigma(R)$.
We show that $\Sigma(R)$ admits Armstrong models. A model is considered as a pair (r, D), where r is a relation over U and D is a set of relations over U.
In the sequel we use the direct product for a family of relations over U, defined by Fagin [4] and the direct product for the models (r_1, D_1) and (r_2, D_2) defined in [7].

Theorem 5.1 *Let $\mathcal{S}$ be a set of sentences. Then a) implies b):*

a) There is an operator $\oplus$ that maps nonempty families of models into models, such that if σ is a sentence in $\mathcal{S}$ and $\langle R_i : i \in I\rangle$ is a nonempty family of models, then σ holds for $\oplus\langle R_i, i \in I\rangle$ iff σ holds for each R_i, $i \in I$.

b) Whenever Σ is a consistent subset of $\mathcal{S}$ and Σ^* is the set of sentences in $\mathcal{S}$ that are logical consequences of Σ, then there is a model (called Armstrong model) that obeys Σ^* and no other sentences in $\mathcal{S}$.

The proof is analogous with that of the Theorem 3.1 [4], considering in $\mathcal{S}$ sentences of the form (1), instead of sentence in first-order logic.

Using the results from [7], it obtains:

Proposition 5.1 *Let $\alpha(R, X)$ be a CEID. Then $\alpha(R, X)$ is faithful with respect to direct products.*

Theorem 5.2 *Let Σ be a set of CEID's and $\Sigma \subseteq \Sigma(R)$. There exists a model $M = (r, D)$ that obeys Σ^* and no other CEID $\alpha(R, X)$ from $\Sigma(R) - \Sigma^*$.*

Proof: Let $(\alpha_i)_{i\in I}$ be the family of all elements from $\Sigma(R) - \Sigma^*$ (whenever $\Sigma(R) - \Sigma^* \neq \emptyset$).

For α_i there is a model M_i, which satisfies Σ and it doesn't satisfy α_i. The direct product $\otimes\langle M_i, i \in I\rangle$ satisfies the conclusion of the theorem.

6 Conclusions

Using second-order logic we defined the family of conditional embedded implicational dependencies. This family contains the class of conditional-functional dependencies, the family of embedded implicational dependencies and the family of conditional implicational dependencies. We gave a characterisation of a conditional embedded implicational dependency implied by a given class in this family.

A method for the construction of an Armstrong model for a given class of this family was given.

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